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On a differential equation in the theory of pseudo-holomorphic functions

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In connection with the representation of pseudo-holomorphic functions of several complex variables A. Koohara [3] obtained the differential equation

$$G_{\bar{t}}=\frac{\bar{K}K_{\bar{t}}}{1-K\bar{K}}G-\frac{\overline{K_t}}{1-K\bar{K}}\bar{G} \tag{1}$$

and asked the following question: Is there a nonconstant function $K(t, \bar{t})$ such that equation (1) is reduced to a differential equation of type

$$v_{\bar{t}}=c\bar{v} \tag{2}$$

where

$$m^2(\log c)_{t\bar{t}}=c\bar{c}\,, \qquad m\in \boldsymbol{N}\,? \tag{3}$$

(We denote by $\boldsymbol{N}$ ($\boldsymbol{R}$) the set of natural (real) numbers and use $\boldsymbol{N}_0=\boldsymbol{N}\cup\{0\}$. The set of integers is denoted by $\boldsymbol{Z}$). For the differential equation (2) an explicit representation of the solutions was derived in [1] by means of differential operators.

In the present paper we prove that such functions $K(t, \bar{t})$ exist and give two examples.

The differential equation $G_{\bar{t}}=aG+b\bar{G}$ is transformed by $G=e^A v$, $A_{\bar{t}}=a$, into a differential equation of type (2) where $c=be^{\bar{A}-A}$. From (3) it follows that we can apply the results derived in [1] if the coefficients a and b satisfy the relation

$$m^2[(\log b)_{t\bar{t}}+\overline{a}_t-a_t]=b\bar{b}\,, \qquad m\in\boldsymbol{N}\,.$$

Therefore we obtain a representation of the solutions of (1) by differential operators if $K(t, \bar{t})$ satisfies the differential equation

$$\begin{aligned} m^2(1-K\bar{K})^2(\log \bar{K}_{\bar{t}})_{t\bar{t}}=K_t\bar{K}_{\bar{t}}-m^2[&2K\bar{K}_{t\bar{t}}(1-K\bar{K})+2K^2\bar{K}\bar{K}_{t\bar{t}}\\ &+K_t\bar{K}_{\bar{t}}+K_{\bar{t}}\bar{K}_t]\,. \end{aligned} \tag{4}$$

A general representation theorem for the solutions of this nonlinear third-order differential equation is not known. However, it is possible to obtain some particular solutions of (4).

If we suppose that $K(t, \bar{t})$ is a real-valued function, then also a_t is real-valued, and equation (4) reduces to

$$m^2(\log b)_{t\bar{t}}=b\bar{b}\,, \qquad b=\frac{K_{\bar{t}}}{K^2-1}\,.$$

Applying [1], Theorem 2, we obtain

$$\frac{K_{\bar{t}}}{K^2-1}=\frac{m\overline{\alpha'}\beta}{(\alpha+\bar{\alpha})\bar{\beta}}$$

where $\alpha(t)$ and $\beta(t)$ are holomorphic functions with $(\alpha+\bar{\alpha})\alpha'\beta\neq 0$. By differentiation with respect to t and multiplication by $-(\alpha+\bar{\alpha})^2\beta\bar{\beta}(m\alpha'\overline{\alpha'})^{-1}$ it follows that

$$\psi=\beta^2-\frac{\beta\beta'(\alpha+\bar{\alpha})}{\alpha'}$$

is a real-valued function. From this and $\psi_{t\bar{t}}=\overline{\psi_{t\bar{t}}}$ we find

$$\beta^2=a\alpha^2+ib\alpha+c\,, \qquad a,\, b,\, c\in\boldsymbol{R}\,, \qquad (a,\, b,\, c)\neq(0,\, 0,\, 0)\,.$$

By integration and suitable normalization we find the following results.

Case 1: $a=b=0$, $c\neq 0$.

$$\beta=\varepsilon_1^{1/2}\,, \qquad \varepsilon_1=\pm 1\,,$$

(5) $$K_1=\varepsilon_1\frac{1+\varepsilon_2\eta^{2m}}{1-\varepsilon_2\eta^{2m}}\,, \qquad \eta=\alpha+\bar{\alpha}\,, \qquad \varepsilon_2=\pm 1\,, \qquad 1-\varepsilon_2\eta^{2m}\neq 0\,.$$

Case 2: $a=0$, $b\neq 0$.

$$\beta=(\varepsilon_1 i\alpha)^{1/2}\,, \qquad \varepsilon_1=\pm 1\,,$$

(6) $$K_2=\varepsilon_1\frac{1+C_1\rho^{2m}}{1-C_1\rho^{2m}}\,, \qquad C_1\in\boldsymbol{R}\,, \qquad C_1\neq 0\,,$$

$$\rho=\frac{i(\alpha-\bar{\alpha})-2(i\alpha)^{1/2}\overline{(i\alpha)^{1/2}}}{\alpha+\bar{\alpha}}\,, \qquad (1-C_1\rho^{2m})\rho\neq 0\,.$$

Case 3: $a\neq 0$.

$$\beta=[\varepsilon_1(\alpha^2+C_2)]^{1/2}\,, \qquad \varepsilon_1=\pm 1\,, \qquad C_2\in\boldsymbol{R}\,,$$

(7) $$K_3=\varepsilon_1\frac{1+\varepsilon_2\sigma^{2m}}{1-\varepsilon_2\sigma^{2m}}\,, \qquad \varepsilon_2=\pm 1\,,$$

$$\sigma=\frac{\alpha\bar{\alpha}-C_2+(\alpha^2+C_2)^{1/2}\overline{(\alpha^2+C_2)^{1/2}}}{\alpha+\bar{\alpha}}\,, \qquad (1-\varepsilon_2\sigma^{2m})\sigma\neq 0\,.$$

Applying [1], Theorem 4, we obtain

THEOREM 1. *Let D be a simply connected domain of the complex plane. $\alpha(t)$ denotes a holomorphic function in D with $(\alpha+\bar{\alpha})\alpha'\neq 0$.*

a) *The real-valued and real-analytic functions K satisfying the differential equation* (4) *in D are given by $K=K_s(t, \bar{t})$, $s=1, 2, 3$, where K_s are defined by* (5)-(7).

b) *Let G_s be a solution of* (1) *with $K=K_s$, $s=1, 2, 3$, defined in D. Then there exists a function $f(t)$ holomorphic in D so that we can represent the solution G_s by*

$$G_s=H_sQ_mf \tag{8}$$

where

$$H_1=\varepsilon_1^{1/2}\frac{1-\varepsilon_2\eta^{2m}}{\eta^m},$$

$$H_2=(\varepsilon_1 i\alpha)^{1/2}\frac{1-C_1\rho^{2m}}{\rho^m},$$

$$H_3=[\varepsilon_1(\alpha^2+C_2)]^{1/2}\frac{1-\varepsilon_2\sigma^{2m}}{\sigma^m}$$

and

$$Q_mf=\sum_{k=0}^{m}\frac{(-1)^{m-k}(2m-1-k)!}{k!(m-k)!\,\eta^{m-k}}[mR^kf-(m-k)\overline{R^kf}], \qquad R=\frac{1}{\alpha'}\frac{\partial}{\partial z}.$$

c) *Conversely,* (8) *represents a solution of* (1) *with $K=K_s$, $s=1, 2, 3$, defined in D for each function $f(t)$ holomorphic in D.*

d) *The particular solutions G_{s1}, G_{s2}, $s=1, 2, 3$, where*

$$\begin{cases} G_{11}=\varepsilon_1^{1/2}(1-\varepsilon_2\eta^{2m}), \\ G_{12}=\dfrac{i\varepsilon_1^{1/2}(1-\varepsilon_2\eta^{2m})}{\eta^{2m}}, \end{cases}$$

$$\begin{cases} G_{21}=\dfrac{(\varepsilon_1 i\alpha)^{1/2}(1-C_1\rho^{2m})\eta^m}{\rho^m}, \\ G_{22}=\dfrac{i(\varepsilon_1 i\alpha)^{1/2}(1-C_1\rho^{2m})}{\rho^m\eta^m}, \end{cases}$$

$$\begin{cases} G_{31}=\dfrac{[\varepsilon_1(\alpha^2+C_2)]^{1/2}(1-\varepsilon_2\sigma^{2m})\eta^m}{\sigma^m}, \\ G_{32}=\dfrac{i[\varepsilon_1(\alpha^2+C_2)]^{1/2}(1-\varepsilon_2\sigma^{2m})}{\sigma^m\eta^m} \end{cases}$$

form a generating pair for (1) *with $K=K_s$ in L. Bers's sense* [2].

If K_0 is a solution of (4), then also $K=K_0e^{i\mu}$, $\mu\in\boldsymbol{R}$, is a solution of this differential equation.

COROLLARY. *The solutions G of* (1) *with $K=K_s e^{i\mu}$, $s=1, 2, 3$, $\mu\in\boldsymbol{R}$, defined in D are given by*

$$G=e^{-\frac{i\mu}{2}}H_sQ_mf, \qquad f(t) \text{ holomorphic in } D.$$

The solutions $K(t, \bar{t})$ of (4) defined in D which form a product of a holomorphic and an antiholomorphic function are given by

$$K=\frac{\gamma^{\lambda+1}}{\bar{\gamma}^{\lambda}}, \qquad \lambda+1=\pm m,$$

where $\gamma(t)$ is holomorphic in D with $\gamma\gamma'(1-\gamma\bar{\gamma})\neq 0$.

In case $\lambda=-1$ we obtain $K=\overline{\gamma(t)}$ and $G=f(t)(1-\gamma\bar{\gamma})^{-1}$, $f(t)$ holomorphic in D.

In case $\lambda\in\boldsymbol{Z}$, $\lambda\neq-1$, a_t is a real-valued function and we have

$$(\lambda+1)^2(\log b)_{t\bar{t}}=b\bar{b}\,.$$

Applying again [1], Theorem 2, we find

$$b=-\frac{(\lambda+1)\bar{\gamma}'\bar{\gamma}^{\lambda}}{(1-\gamma\bar{\gamma})\gamma^{\lambda}}=\frac{m\overline{\alpha'}\beta}{(\alpha+\bar{\alpha})\bar{\beta}}.$$

Setting $\gamma=\dfrac{\alpha+1}{\alpha-1}$ we obtain

$$b=-\frac{(\lambda+1)\overline{\alpha'}}{\alpha+\bar{\alpha}}\frac{(\alpha-1)^{\lambda+1}}{(\alpha+1)^{\lambda}}\frac{(\bar{\alpha}+1)^{\lambda}}{(\bar{\alpha}-1)^{\lambda+1}}$$

and

$$\beta=\frac{i}{\gamma^{\lambda}(\gamma-1)} \quad \text{for} \quad m=\lambda+1, \qquad \lambda\in\boldsymbol{N}_0,$$

$$\beta=\frac{1}{\gamma^{\lambda}(\gamma-1)} \quad \text{for} \quad m=-(\lambda+1), \qquad -\lambda\in\boldsymbol{N}, \qquad \lambda\leqq-2.$$

Using [1], Theorem 4, we obtain

THEOREM 2. *Let D be a simply connected domain in the complex plane. $\gamma(t)$ denotes a holomorphic function in D with $\gamma(1-\gamma\bar{\gamma})\gamma'\neq 0$.*

a) *Let G be a solution of* (1) *with*

$$K=\frac{\gamma^{\lambda+1}}{\bar{\gamma}^{\lambda}}, \qquad \lambda\in\boldsymbol{Z}, \quad \text{defined in } D.$$

Then there exists a function $f(t)$ holomorphic in D so that we can represent the solution G by

$$
(9) \qquad G=\begin{cases} i\left(\dfrac{1-\gamma\bar{\gamma}}{\gamma}\right)^{\lambda} P_{\lambda+1}f & \text{for } \lambda\in \boldsymbol{N}_0, \\ \dfrac{f(t)}{1-\gamma\bar{\gamma}} & \text{for } \lambda=-1, \\ \left(\dfrac{\gamma}{1-\gamma\bar{\gamma}}\right)^{|\lambda|} P_{|\lambda|-1}f & \text{for } -\lambda\in \boldsymbol{N}, \quad \lambda\leqq -2, \end{cases}
$$

with

$$
P_m f=\sum_{k=0}^{m} \frac{(-1)^{m-k}(2m-1-k)!\,(\gamma-1)^{m-k-1}(\bar{\gamma}-1)^{m-k}}{k!\,(m-k)!\,(1-\gamma\bar{\gamma})^{m-k}}\left[mS^k f-(m-k)\overline{S^k f}\right],
$$

$$
m\in \boldsymbol{N}, \qquad S=\frac{(\gamma-1)^2}{\gamma'}\frac{\partial}{\partial z}.
$$

b) *Conversely,* (9) *represents a solution of* (1) *with* $K=\gamma^{\lambda+1}\bar{\gamma}^{-\lambda}$ *defined in* D *for each function* $f(t)$ *holomorphic in* D.

c) *The particular solutions*

$$
G_1=\frac{(\gamma-1)^{\lambda}(\bar{\gamma}-1)^{\lambda+1}}{\gamma^{\lambda}(1-\gamma\bar{\gamma})}, \qquad G_2=\frac{i(1-\gamma\bar{\gamma})^{2\lambda+1}}{\gamma^{\lambda}(\gamma-1)^{\lambda+2}(\bar{\gamma}-1)^{\lambda+1}}, \qquad \lambda\in \boldsymbol{Z},
$$

form a generating pair for (1) *in L. Bers's sense.*

References

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