J. Math. Soc. Japan
Vol. 26, No. 4, 1974

Compact two-transnormal hypersurfaces in a space of constant curvature*)

By Seiki NISHIKAWA

(Received March 12, 1973)
(Revised Nov. 19, 1973)

Introduction.

Let M be a complete Riemannian n-manifold isometrically imbedded into a complete Riemannian $(n+1)$-manifold W. Throughout this paper manifolds are always assumed to be connected and smooth. Furthermore we assume $n\geqq 2$, although some of our results are valid even for $n=1$. For each $x\in M$ there exists, up to parametrization, a unique geodesic τ_x of W which cuts M orthogonally at x. M is called a *transnormal hypersurface* of W if, for each pair $x, y\in M$, the relation $\tau_x\ni y$ implies that $\tau_x=\tau_y$, i.e. if each geodesic of W which cuts M orthogonally at some point cuts M orthogonally at all points of intersection. As is well-known, every surface of constant width in the ordinary Euclidean space has this property ([6]), and it is a model of a transnormal hypersurface.

The order of a transnormal hypersurface, by which the hypersurface is globally characterized, is introduced in the following way. Define an equivalence relation $\sim$ on M by writing $x\sim y$ to mean $y\in\tau_x$. With respect to this relation, take the quotient space $\hat{M}=M/\sim$ and endow $\hat{M}$ with the quotient topology. We call M an *r-transnormal* hypersurface if the natural projection ψ of M onto $\hat{M}$ is an r-fold (topological) covering map. The number r is called the *order of transnormality* of M. It should be remarked that ψ is not always a covering map. However, if W is simply connected and of constant curvature, then ψ is a covering map ([5]).

In [5], we have obtained the following results which determine topological structures of transnormal hypersurfaces.

THEOREM A. *Let M be an n-dimensional transnormal hypersurface of W. Suppose that there exists a point p of M whose cut locus $C(p)$ in W does not intersect M: $C(p)\cap M=\emptyset$. Then the following hold.*

(i) *If M is* 1*-transnormal, then M is homeomorphic to a Euclidean n-space E^n.*

*) This paper was written while the author was at Tokyo Metropolitan University.

(ii) *If M is compact and 2-transnormal, then M is homeomorphic to a Euclidean n-sphere S^n.*

(iii) *If M is compact and $r(<+\infty)$-transnormal, then the Euler characteristic $\chi(M)$ of M is either zero or r.*

The main purpose of this paper is to study differential geometric structures of a compact 2-transnormal hypersurface of a simply connected complete Riemannian manifold of constant curvature (in contrast to Theorem A (ii) which is of topological nature). In fact, we prove the following theorems.

THEOREM B. *Let M be a compact 2-transnormal hypersurface of a Euclidean $(n+1)$-space E^{n+1}.*

(i) *Then, at each point of M, with respect to the inward unit normal, every principal curvature of M is greater than $1/l$, where l is the diameter of M as a subset of E^{n+1}.*

(ii) *Let k be a positive constant. In* (i), *if every principal curvature λ of M satisfies*

$$\lambda \geqq k \qquad (\textit{resp. } 1/l < \lambda \leqq k)$$

at each point of M, then

$$k \leqq 2/l \qquad (\textit{resp. } k \geqq 2/l).$$

(iii) *In* (i), *if every principal curvature λ of M satisfies*

$$\lambda \geqq 2/l \qquad (\textit{or } 1/l < \lambda \leqq 2/l)$$

at each point of M, then M is totally umbilical and hence isometric to a Euclidean n-sphere S^n of radius $l/2$.

THEOREM C. *Let M be a 2-transnormal hypersurface of a Euclidean $(n+1)$-sphere S^{n+1} of radius* 1. *Suppose the diameter l of M as a subset of S^{n+1} satisfies $0<l<\pi$.*

(i) *Then, at each point of M, with respect to the inward unit normal vector (cf.* § 1 *for definition), every principal curvature of M is greater than* $\cot l$.

(ii) *Let k be a constant. In* (i), *if every principal curvature λ of M satisfies*

$$\lambda \geqq k \qquad (\textit{resp. } \cot l < \lambda \leqq k)$$

at each point of M, then

$$k \leqq (1+\cos l)/\sin l \qquad (\textit{resp. } k \geqq (1+\cos l)/\sin l).$$

(iii) *In* (i), *if every principal curvature λ of M satisfies*

$$\lambda \geqq (1+\cos l)/\sin l \qquad (\textit{or } \cot l < \lambda \leqq (1+\cos l)/\sin l)$$

at each point of M, then M is totally umbilical and hence isometric to a Euclidean n-sphere S^n of radius $\sin(l/2)$.

THEOREM D. *Let M be a compact 2-transnormal hypersurface of a hyperbolic $(n+1)$-space H^{n+1} of constant curvature -1.*

(i) *Then, at each point of M, with respect to the inward unit normal vector, every principal curvature of M is greater than* $\coth l$, *where l is the diameter of M as a subset of H^{n+1}.*

(ii) *Let k be a positive constant. In* (i), *if every principal curvature λ of M satisfies*

$$\lambda \geqq k \qquad (\textit{resp.}\ \coth l < \lambda \leqq k)$$

at each point of M, then

$$k \leqq (1+\cosh l)/\sinh l \qquad (\textit{resp.}\ k \geqq (1+\cosh l)/\sinh l)\,.$$

(iii) *In* (i), *if every principal curvature λ of M satisfies*

$$\lambda \geqq (1+\cosh l)/\sinh l \qquad (\textit{or}\ \coth l < \lambda \leqq (1+\cosh l)/\sinh l)$$

at each point of M, then M is totally umbilical and isometric to a Euclidean n-sphere S^n of radius $\sinh(l/2)$.

The proofs of these theorems will be given separately in §§ 2, 3 and 4.

I would like to express my hearty thanks to Professor M. Obata for his constant encouragement during the preparation of this paper.

§ 1. Preliminaries.

This section is devoted to a brief survey of the concepts and formulas used throughout the paper. Let W be a complete Riemannian $(n+1)$-manifold with $n \geqq 2$. We denote by T_xW the tangent space of W at x and by $\langle\, ,\rangle$ the inner product on the tangent space. Let M and P be Riemannian submanifolds of W and τ a geodesic segment perpendicular to M and P at its end points $\tau(0)$ and $\tau(b)$. Denote the Riemannian curvature tensor of W and the second fundamental form of the submanifold under consideration by R and S respectively. Then the second variation of the arc length $l(\tau)$ of τ is given by the formula

$$
\begin{aligned}
(1.1)\qquad l''(0) &= \int_0^b (\langle V', V'\rangle(u) - \langle R(V, \tau_*)\tau_*, V\rangle(u))du + \langle \tau_*, \nabla_v V\rangle]_0^b \\
&= -\int_0^b \langle V'' + R(V, \tau_*)\tau_*, V\rangle(u)du \\
&\quad + \langle S_{\tau_*(b)}V(b) + V'(b), V(b)\rangle - \langle S_{\tau_*(0)}V(0) + V'(0), V(0)\rangle\,,
\end{aligned}
$$

where V is the associated variation vector field along τ whose values are everywhere orthogonal to the tangent vector τ_* of τ, and V' denotes the covariant derivative with respect to τ_* (cf. [1]).

A smooth vector field $Y(t)$ along τ is called a Jacobi field if it satisfies the Jacobi equation

$$Y''+R(Y, \tau_*)\tau_*=0.$$

A Jacobi field arises from the variation of τ whose longitudinal curves are always geodesics. A Jacobi field Y along τ which is perpendicular to τ is said to be an $(M, \tau(0))$-*Jacobi field* when it satisfies the boundary conditions

$$(1.2) \qquad Y(0) \in T_{\tau(0)}M \quad \text{and} \quad S_{\tau_*(0)}Y(0)+Y'(0) \in T_{\tau(0)}M^{\perp},$$

where $\perp$ means orthogonal complement in $T_{\tau(0)}W$. Geometrically, an $(M, \tau(0))$-Jacobi field is precisely the associated vector field of the variation of τ all of whose longitudinal curves are geodesics starting orthogonally from M and parametrized by arc length ([**1**]).

Let e be the restriction of the exponential map of W to the normal bundle $(TM)^{\perp}$ of M in W. Then a focal point of M at x is, by definition, a point $\eta \in T_xM^{\perp}$ at which the differential map of e is singular, and $e(\eta)$ is called a focal point of M along the geodesic $e(t\eta)$, $t>0$. For a given geodesic τ starting orthogonally from M, $\tau(b)$ is known to be a focal point of M along τ if and only if there exists an $(M, \tau(0))$-Jacobi field which vanishes at b. In particular, if W is a Euclidean $(n+1)$-space E^{n+1}, then for a unit normal vector ξ of M at x the point $e(t\xi)=x+t\xi$ is a focal point of M at x if and only if t is a principal radius of curvature of M at x with respect to ξ ([**4**]).

Suppose M is an $r(<+\infty)$-transnormal hypersurface of W and $p \in M$ satisfies the condition $C(p) \cap M=\emptyset$, where $C(p)$ denotes the cut locus of p in W (for the definition of $C(p)$, if necessary, see [**3**]). In the following, unless otherwise mentioned, we always assume that there exists at least one such a point p for each transnormal M. By the *distance function* Λ_p of M we mean the real valued smooth function on M defined by

$$\Lambda_p(x)=d(p, x)^2, \qquad x \in M,$$

where $d(\ ,\ )$ denotes the distance in W. Note that $d(p, x)^2$ is nothing but the square of the length of the unique minimizing geodesic segment $\tau(p, x)$ of W joining p with x. Furthermore, a point $x \in M$ is a critical point of Λ_p if and only if $\tau(p, x)$ is perpendicular to M at x and then at p due to the transnormality of M. It is known that Λ_p *is a Morse function and the number of its critical points coincides with the order* r *of transnormality of* M ([**5**]). Theorem A is an implication of this property together with elementary parts of the Morse theory.

If, in particular, M is compact and 2-transnormal, and W is a simply connected complete Riemannian manifold of constant curvature, then for each $x \in M$ there exists exactly one point $\tilde{x} \in M$ such that the length of the minimizing geodesic segment $\tau(x, \tilde{x})$ joining x with $\tilde{x}$ equals the diameter of M as a subset of W (cf. [**5**]). In this case, $\tau(x, \tilde{x})$ is perpendicular to M at both of its end points. We call $\tilde{x} \in M$ the *antipodal point* of $x \in M$ and the initial vector $\tau_*(0)$ of $\tau(x, \tilde{x})$ the *inward* unit normal vector at x.

In general, a hypersurface M of W is said to be *convex* at $x \in M$ if the second fundamental form S of M is (positive or negative) definite at x, or equivalently if, in a neighborhood of x, x is the only one point of M that lies on the hypersurface of W which is tangent to M at x and is totally geodesic in the neighborhood. M is called a *convex hypersurface* of W if it is convex at every point.

§ 2. Compact 2-transnormal hypersurfaces in a Euclidean space.

First we deal with a compact 2-transnormal hypersurface M of a Euclidean $(n+1)$-space E^{n+1}.

Let $p \in M$ and consider the distance function $\Lambda_p(x) = d(p, x)^2$ on M. Note that the cut locus $C(p)$ of p is empty and then $C(p) \cap M = \emptyset$. At a critical point x of Λ_p, the Hessian H of Λ_p, which is a symmetric bilinear form on T_xM, is given by

$$H(X, Y) = 2\langle (I - lS_\xi)X, Y \rangle, \qquad X, Y \in T_xM,$$

where I denotes the identity transformation and ξ is the unit vector defined by $p = x + l\xi$, $l > 0$ ([4]). It should be remarked that ξ is normal to M and thus l coincides with the diameter of M as a subset of E^{n+1}.

The clue to the proof of Theorem B is the following

LEMMA 1. *If λ is a non-zero principal curvature of M at x with respect to the inward unit normal ξ, then*

$$\tilde{\lambda} = \lambda/(\lambda l - 1)$$

is a principal curvature of M at $\tilde{x}$ with respect to $-\xi$, where $\tilde{x}$ is the antipodal point of x, and l is the diameter of M as a subset of E^{n+1}.

PROOF. Since λ is a non-zero principal curvature of M at x with respect to ξ, the point $x + \lambda^{-1}\xi$ is a focal point of M at x. It is easily seen that each focal point of M at x is also a focal point of M at $\tilde{x}$, because M is a transnormal hypersurface. In fact, we have only to note that each (M, x)-Jacobi field is also an $(M, \tilde{x})$-Jacobi field. Thus $x + \lambda^{-1}\xi$ is a focal point of M at $\tilde{x}$ as well. So there exists a principal curvature $\tilde{\lambda}$ of M at $\tilde{x}$ such that

$$\tilde{x} - \tilde{\lambda}^{-1}\xi = x + \lambda^{-1}\xi .$$

From this equation, we obtain

$$\lambda^{-1} + \tilde{\lambda}^{-1} = l, \tag{2.1}$$

since the length of the vector $\tilde{x} - x$ attains the diameter l of M. Rewriting

(2.1), we get the lemma. Here we note that

$$\lambda l-1>0,$$

which is shown in the proof of Theorem B (i). Q. E. D.

PROOF OF THEOREM B. (i) Choose a point $x\in M$ arbitrarily, and let $\tilde{x}$ be the antipodal point of x. Remark that $\tilde{x}=x+l\xi$ where ξ is the inward unit normal of M at x. Then the Hessian H of the distance function $\Lambda_{\tilde{x}}$ at x is given by

$$H(X, Y)=2\langle(I-lS_{\xi})X, Y\rangle, \qquad X, Y\in T_xM. \tag{2.2}$$

Since M is compact and 2-transnormal, $\Lambda_{\tilde{x}}$ takes its maximum at x, which is a nondegenerate critical point of $\Lambda_{\tilde{x}}$ ([5]). Hence H is negative definite at x, i. e. every eigenvalue of S_{ξ} is greater than $1/l$.

(ii) Let λ be a principal curvature of M at x in (i), and consider the case $\lambda\geqq k$. By Lemma 1, $\tilde{\lambda}=\lambda/(\lambda l-1)$ is a principal curvature of M at $\tilde{x}$. Thus from the assumption we have

$$\frac{\lambda}{\lambda l-1}\geqq k, \tag{2.3}$$

noticing the choice of unit normals in (i). Assume that (ii) is false, i. e. $k>2/l$. Then $\lambda>2/l$, and (2.3) asserts

$$\frac{\lambda}{\lambda l-1}>\frac{2}{l}.$$

This is, however, a contradiction, because the last inequality reduces to $\lambda<2/l$.

The proof for the case $1/l<\lambda\leqq k$ is accomplished in a similar way.

(iii) We prove here only the case $\lambda\geqq 2/l$. The assumption $\lambda\geqq 2/l$ leads to

$$\frac{\lambda}{\lambda l-1}\geqq\frac{2}{l},$$

for the same reason as in the proof of (ii). From these inequalities, we get

$$\lambda=2/l,$$

which shows that M is totally umbilical, and this completes the proof (cf. [3]). Q. E. D.

As a corollary of Theorem B (i), we obtain

PROPOSITION 1. *Let M be a compact 2-transnormal hypersurface of a Euclidean $(n+1)$-space E^{n+1}. Then the following hold.*

(i) *M is a convex hypersurface of E^{n+1}, and then M has positive sectional curvature everywhere.*

(ii) *M is diffeomorphic to a Euclidean n-sphere S^n.*

(iii) *The total curvature of M is* 2.

PROOF. (i) is a direct consequence of Theorem B (i). From (i) we have (ii) as well as (iii). See, for example, [3].

§ 3. 2-transnormal hypersurfaces in a sphere.

In this section we investigate the case where M is a 2-transnormal hypersurface of a Euclidean $(n+1)$-sphere S^{n+1} of radius 1. Note that such M must be closed in S^{n+1} and in consequence compact ([5]). Suppose that the diameter l of M as a subset of S^{n+1} is less than π, then the cut locus $C(p)$ of $p\in M$ in S^{n+1} does not intersect $M: C(p)\cap M=\emptyset$. Unless otherwise stated, this assumption on the diameter is always made throughout the rest of this section.

Fix a point $p\in M$ arbitrarily and consider the distance function $\Lambda_p(x)=d(p, x)^2$ on M. Let $x\in M$ be a critical point of Λ_p and $\tau(p, x)$ the minimizing geodesic segment in S^{n+1} joining p with x. Recall that $\tau(p, x)$ is perpendicular to M at x as well as at p, and then the length of $\tau(p, x)$ equals the diameter l of M. The Hessian H of Λ_p at x is given by

$$H(X, Y)=2l\langle(\cot l\cdot I-S_{-\tau_*(l)})X, Y\rangle\,, \qquad X, Y\in T_xM\,. \tag{3.1}$$

This formula can be derived from the second variation formula (1.1). In fact, the calculation of the Hessian of Λ_p corresponds to the second variation of the square of the length of $\tau(p, x)$ all of whose longitudinal curves are minimizing geodesics. On the other hand, it is well-known that on a unit sphere S^{n+1} every Jacobi field $Y(t)$ along a geodesic $\tau(t)$ parametrized by arc length is written as

$$Y(t)=A(t)\sin t+B(t)\cos t\,, \tag{3.2}$$

where $A(t)$ and $B(t)$ are parallel vector fields along $\tau(t)$. In our case, the Jacobi field under consideration may be expressed in a more simplified form

$$Y(t)=A(t)\sin t\,,$$

where $A(t)$ is a parallel vector field along $\tau(p, x)$ satisfying the condition $A(l)\in T_xM$, since p, one of the end points, is fixed under the variation of $\tau(p, x)$. From these facts, after a simple computation, we get the formula (3.1).

The bulk of the proof of Theorem C lies in the following

LEMMA 2. *Let $x\in M$ and $\tilde{x}$ be the antipodal point of x. Let τ be the minimizing geodesic in S^{n+1} joining x with $\tilde{x}$. Suppose that λ is a principal curvature of M at x with respect to $\tau_*(0)$. Then*

$$\tilde{\lambda}=(\sin l+\lambda\cos l)/(\lambda\sin l-\cos l)$$

is a principal curvature of M at $\tilde{x}$ with respect to $-\tau_*(l)$, where l is the diameter of M as a subset of S^{n+1}.

PROOF. Let $Y(t)=A(t)\sin t+B(t)\cos t$ be an (M, x)-Jacobi field along $\tau(t)$, $0\leqq t\leqq l$, such that the parallel vector fields $A(t)$ and $B(t)$ satisfy the following conditions:

$$A(0)\in T_xM,\ A(l)\in T_{\tilde{x}}M;\ B(0)\in T_xM,\ B(l)\in T_{\tilde{x}}M;\ \text{and}$$

$B(0)$ is a principal vector corresponding to λ, i.e.

$$S_{\tau_*(0)}B(0)=\lambda B(0).$$

The existence of such $Y(t)$ is obvious. From the very definition of an (M, x)-Jacobi field, $Y(t)$ satisfies the boundary condition

$$S_{\tau_*(0)}Y(0)+Y'(0)\in T_xM^{\perp}.$$

This means that

$$S_{\tau_*(0)}B(0)+A(0)\in T_xM\cap T_xM^{\perp}=\{0\}.$$

Therefore $A(0)=-\lambda B(0)$, because $B(0)$ is a principal vector corresponding to λ. Consequently, we have

$$Y(t)=(\cos t-\lambda\sin t)B(t).$$

Since M is a transnormal hypersurface, every (M, x)-Jacobi field is also an $(M, \tilde{x})$-Jacobi field. Thus, the above $Y(t)$ must satisfy the following boundary condition as well:

$$S_{\tau_*(l)}Y(l)+Y'(l)\in T_{\tilde{x}}M^{\perp}.$$

From this it follows that

$$S_{-\tau_*(l)}(\lambda\sin l-\cos l)B(l)=(\sin l+\lambda\cos l)B(l).$$

As is shown in the proof of Theorem C (i),

$$\lambda\sin l-\cos l>0,$$

and thus the lemma is proved. Q.E.D.

Now, we turn to

PROOF OF THEOREM C. (i) Choose a point $x\in M$ arbitrarily, and let $\tilde{x}$ be the antipodal point of x. Let τ be the minimizing geodesic joining x with $\tilde{x}$. Then the Hessian H of the distance function $\Lambda_{\tilde{x}}$ at x is given by

$$H(X, Y)=2l\langle(\cot l\cdot I-S_{\tau_*(0)})X, Y\rangle,\qquad X, Y\in T_xM.$$

By the same argument as in the proof of Theorem B (i), we can conclude that every eigenvalue of $S_{\tau_*(0)}$ is greater than $\cot l$.

(ii) Let λ be a principal curvature of M at x in (i). We need only to consider the case $\lambda\geqq k$, because the other case can be proved in parallel with this one.

By Lemma 2 together with the assumption, we have

$$\frac{\sin l+\lambda\cos l}{\lambda\sin l-\cos l}\geqq k,$$

noticing the choice of unit normal vectors in (i). Suppose that (ii) is not valid, i.e. $k>(1+\cos l)/\sin l$. Then we get

$$\lambda>\frac{1+\cos l}{\sin l} \quad \text{and} \quad \frac{\sin l+\lambda\cos l}{\lambda\sin l-\cos l}>\frac{1+\cos l}{\sin l}.$$

However these inequalities contradict each other, because the last one reduces to

$$\lambda<(1+\cos l)/\sin l.$$

(iii) We have only to see that the assumption consequently yields

$$\lambda=(1+\cos l)/\sin l,$$

but it is straightforward. This equality completes the proof. Q.E.D.

As a corollary of Theorem C (i), we get

PROPOSITION 2. *Let M be a 2-transnormal hypersurface of a Euclidean $(n+1)$-sphere S^{n+1} of radius 1. Suppose the diameter l of M as a subset of S^{n+1} is less than $\pi/2$*[1]*. Then*

(i) *M is a convex hypersurface of S^{n+1}, and hence every sectional curvature of M is greater than 1, and*

(ii) *M is diffeomorphic to a Euclidean n-sphere S^n.*

PROOF. By Theorem 1.1 of [2], (i) implies (ii), whereas (i) is obtained from Theorem C (i) because $l<\pi/2$.

§ 4. Compact 2-transnormal hypersurfaces in a hyperbolic space.

Finally we study a compact 2-transnormal hypersurface M of a hyperbolic $(n+1)$-space H^{n+1} of constant curvature -1. But, as one may immediately realize, the proof of Theorem D is quite similar to that of Theorem C as well as Theorem B. So, we describe here only the matters which are worth mentioning.

Let $p\in M$ be a fixed point and consider the distance function $\Lambda_p(x)=d(p,x)^2$ on M. The cut locus $C(p)$ is empty due to the non-positiveness of the sectional curvature of H^{n+1}. At a critical point x, the Hessian H of Λ_p is given by

$$H(X,Y)=2l\langle(\coth l\cdot I-S_{-\tau_*(l)})X,Y\rangle, \qquad X,Y\in T_xM,$$

1) As to the case $l>\pi/2$, see § 5, 2°.

where τ is the minimizing geodesic joining p with x, and l denotes the diameter of M as a subset of H^{n+1}. This formula can be obtained from the second variation formula (1.1) and the fact that, in H^{n+1} of constant curvature -1, every Jacobi field $Y(t)$ along a geodesic $\tau(t)$ parametrized by arc length is written as

$$Y(t)=A(t)\sinh t+B(t)\cosh t\,,$$

where $A(t)$ and $B(t)$ are parallel vector fields along $\tau(t)$.

The role played by Lemma 2 is replaced with the following

LEMMA 3. *Let $x\in M$ and $\tilde{x}$ be the antipodal point of x. Let τ be the minimizing geodesic in H^{n+1} joining x with $\tilde{x}$. Suppose that λ is a principal curvature of M at x with respect to $\tau_*(0)$. Then*

$$\tilde{\lambda}=(\lambda\cosh l-\sinh l)/(\lambda\sinh l-\cosh l)$$

is a principal curvature of M at $\tilde{x}$ with respect to $-\tau_*(l)$, where l is the diameter of M as a subset of H^{n+1}.

We can prove this lemma by the same method as that of Lemma 2 with a slight modification. In the light of Lemma 3, the proof of Theorem D is now straightforward, and so we omit it. The following proposition is obtained as a corollary of Theorem D (i).

PROPOSITION 3. *Let M be a compact 2-transnormal hypersurface of a hyperbolic $(n+1)$-space H^{n+1} of constant curvature -1.*

(i) *Then, M is a convex hypersurface of H^{n+1}, and moreover has positive sectional curvature everywhere, and*

(ii) *M is diffeomorphic to a Euclidean n-sphere S^n.*

Here we remark that (ii) is an implication of (i). See, for example, [2].

Q. E. D.

§5. Concluding remarks.

1°. As for the order of transnormality, we have proved in [5] the following theorem which states that 1- and 2-transnormal hypersurfaces cover a rather wide class of transnormal hypersurfaces.

THEOREM E. *Let M be an $r(<+\infty)$-transnormal hypersurface of W. Suppose W is simply connected and has non-positive sectional curvature everywhere. Then r is either 1 or 2.*

2°. With regard to 2-transnormal hypersurfaces in a unit sphere S^{n+1}, it can be observed without difficulty that there exists an example which is not convex and has a diameter $l>\pi/2$. But, for a diameter $l<\pi/2$, we have Proposition 2 which assures the convexity of M.

References

[1] R. Bishop and R. Crittenden, Geometry of manifolds, Academic Press, New York, 1964.

[2] M.P. do Carmo and F.W. Warner, Rigidity and convexity of hypersurfaces in spheres, J. Differential Geometry, **4** (1970), 133-144.

[3] S. Kobayashi and K. Nomizu, Foundations of differential geometry, vol. II, Interscience, New York, 1969.

[4] J. Milnor, Morse theory, Ann. of Math. Studies, No. 51, Princeton University Press, 1963.

[5] S. Nishikawa, Transnormal hypersurfaces—Generalized constant width for Riemannian manifolds—, Tôhoku Math. J., **25** (1973), 451-459.

[6] I.M. Yaglom and V.G. Boltyanskiĭ, Convex figures, translation by P.J. Kelly and L.F. Walton, Holt, Rinehart and Winston, New York, 1961.

Seiki NISHIKAWA

Department of Mathematics
Faculty of Science
Tôhoku University
Katahira, Sendai
Japan