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On the ε-entropy of stable processes

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§ 1. Introduction and main result.

In recent years many researches are presented about "*rate distortion theory*" or "*ε-entropy*" or "*information rate*" in certain branches of information theory. They are largely motivated by practical problems—for example, data compression or coding of signals. The notion of ε-entropy is originally due to C. Shannon and is closely connected with his fundamental theorem. We can regard the ε-entropy of a stochastic process as a characteristic quantity of the process—a certain index of complexity of the process in view of finite-dimensional approximation.

It will be important to carry out estimates of ε-entropy for basic stochastic processes. For Gaussian processes Pinsker showed how to estimate the ε-entropy [**5**]. Many other researches are concentrated on the discussions about Gaussian cases. On the other hand, we know few estimates for non-Caussian processes: for a diffusion process [**1**] and for a process of jumping type with discrete state space [**3**].

In the present paper we give an estimate for stable processes.

Let $(\Omega, \mathcal{B}, P)$ be a probability space. For random variables (stochastic processes) ξ, η, ζ etc., whose state spaces might be different measurable spaces, Kolmogorov defined the amount of information $I(\xi, \eta)$ and the average conditional information $EI(\xi, \eta|\zeta)$. Though in the definition of $EI(\xi, \eta|\zeta)$ E is simply a symbol and has no meaning of expectation, in our cases we may consider it equal to the expectation of information between ξ and η under the condition with respect to ζ. We list up several properties of them which we shall use in the sequel without mention. We omit assumptions necessary for the formulas since we shall deal with only the cases when the assumptions are satisfied. We refer readers to [**4**] for assumptions and terminologies which we do not define here.

a) $I(\xi, \eta)=0$ if and only if ξ and η are independent.

b) If (ξ_1, η_1) and (ξ_2, η_2) are independent, then

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$$I((\xi_1, \xi_2), (\eta_1, \eta_2)) = I(\xi_1, \eta_1) + I(\xi_2, \eta_2).$$

c) If f is a measurable mapping from the state space of η to a measurable space, then

$$I(\xi, f(\eta)) \leqq I(\xi, \eta),$$

$$EI(\xi, f(\eta) | \zeta) \leqq EI(\xi, \eta | \zeta).$$

Especially if f is a one to one bimeasurable mapping, then equality holds in the above formulas.

d) If $\tilde{\eta}$ is subordinate to η, in particular, if $\tilde{\eta} = f(\eta)$ as in c), then

$$EI(\xi, \eta | \zeta) = EI(\xi, \tilde{\eta} | \zeta) + EI(\xi, \eta | (\zeta, \tilde{\eta})).$$

Definition. Let $\xi = \{\xi(t); t \in [0, T)\}$ be a real-valued stochastic process defined on the probability space $(R^{[0,T)}, \mathcal{B}_\xi, P_\xi)$, where $\mathcal{B}_\xi$ is the smallest Borel field on $R^{[0,T)}$ generated by cylinder sets and P_ξ is the probability distribution of ξ on $R^{[0,T)}$. In the present paper we call $H_\varepsilon^{(1)}(\xi)$ "ε-entropy of ξ";

$$H_\varepsilon^{(1)}(\xi) = \inf \{I(\xi, \eta); \eta \in W_\varepsilon^{(1)}(\xi)\},$$

$$W_\varepsilon^{(1)}(\xi) = \left\{ \begin{array}{l} \eta = \{\eta(t); t \in [0, T)\}; \ \eta \text{ is a stochastic process on } (R^{[0,T)}, \mathcal{B}_\xi) \\ \qquad \text{and satisfies } E\int_0^T |\xi(t) - \eta(t)| dt \leqq \varepsilon \end{array} \right\},$$

where expectation E is with respect to the joint distribution $P_{\xi\eta}$ of (ξ, η) on $(R^{[0,T)} \times R^{[0,T)}, \mathcal{B}_\xi \times \mathcal{B}_\xi)$.

The amount of information and hence ε-entropy are essentially concerned with the joint distribution of two random variables on the product space of their state spaces. $W_\varepsilon^{(1)}(\xi)$ might be considered as the set of joint measures, which have the marginal distribution P_ξ, on the product space of the sample space of ξ.

We are interested in the speed of growth of $H_\varepsilon^{(1)}(\xi)$ when ε tends to 0.

Our main result is the following.

Theorem. *Let $\xi = \{\xi(t); t \in [0, T)\}$ be a stable process (symmetric or asymmetric or one-sided) with exponent $1 < \alpha < 2$. If $\varepsilon > 0$ is sufficiently small, an asymptotic estimate of $H_\varepsilon^{(1)}(\xi)$ is given as follows:*

$$H_\varepsilon^{(1)}(\xi) \asymp \varepsilon^{-\alpha}$$

that is $0 < \varliminf_{\varepsilon \to 0} \dfrac{H_\varepsilon^{(1)}(\xi)}{\varepsilon^{-\alpha}} \leqq \varlimsup_{\varepsilon \to 0} \dfrac{H_\varepsilon^{(1)}(\xi)}{\varepsilon^{-\alpha}} < \infty.$

In § 2 and § 3 we give preparatory upper estimates of ε-entropy for a certain process with stationary independent increments. An upper estimate for symmetric stable processes is given in § 4, where we apply the results of § 3 and the technique once applied for diffusion processes in [1]. § 5 contains a lower estimate for symmetric stable processes and an estimate for the process considered in § 2. Though the latter is quite an unsatisfactory one, the result may be interesting by itself.

For simplicity of argument we suppose $\xi(0)=0$ a. a. for all the processes under consideration.

§ 2. An upper estimate of ε-entropy for a stationary independent increments process with finite number of jumps.

In the present section we suppose $\xi=\{\xi(t)\,;\ t\in[0, T)\}$ is such a process with stationary independent increments as:

$$Ee^{i\theta\xi(t)}=\exp\left\{t\int_{R\setminus\{0\}}(e^{i\theta u}-1)n(du)\right\},$$

where the Lévy measure $n(du)$ satisfies the following conditions:

(i) $\beta=n(R\setminus\{0\})<\infty$,

(ii) $0<\int_{R\setminus\{0\}}|u|\,n(du)<\infty$,

(iii) $n(du)$ has a density function $n(u)$ and $-\int_{R\setminus\{0\}}n(u)\log n(u)du>-\infty$*).

PROPOSITION 1. *Let ξ be a stochastic process as above. Then for sufficiently small $\varepsilon>0$ its ε-entropy is estimated from above by*

$$H_\varepsilon^{(1)}(\xi)\leqq\beta T\log\frac{9E|\xi_1|T^2}{4e\varepsilon^2}+\beta T\,h(\xi_1)+O(1)\,,$$

where ξ_1 is a random variable of jumping width whose probability density function is $\frac{n(u)}{\beta}$ and $h(\xi_1)$ is the differential entropy of ξ_1:

$$h(\xi_1)=-\int_{R\setminus\{0\}}\frac{n(u)}{\beta}\log\frac{n(u)}{\beta}\,du\,.$$

We prepare several lemmas. Lemma 1 is a well-known fact, easily verified from the characteristic function of $\xi(t)$ [**6**].

LEMMA 1. *There holds*

$$\xi(t)=\xi_1+\xi_2+\cdots+\xi_{N_t}$$

in the sense that the processes of the both sides have the same distribution in $R^{[0,T)}$. Here N_t is a Poisson process with parameter β. $\xi_1, \xi_2, \cdots$ are independent of $N=\{N_t\}$ and mutually independent identically distributed with the density function $\frac{n(u)}{\beta}$.

LEMMA 2 (Linikov [**3**]). *Let $N=\{N_t\,;\ t\in[0, T)\}$ be a Poisson process with parameter β. If we define*

*) The logarithm will be taken to the base e.

$$H_\varepsilon^{(0)}(N|N_T)=\inf\{EI(N, M|N_T)\,;\ M\in W_\varepsilon^{(0)}(N)\}$$

$$W_\varepsilon^{(0)}(N)=\left\{\begin{array}{l} M=\{M(t)\,;\ t\in[0, T)\}\,;\ M \text{ is a } 0,1,2,\cdots \text{-valued} \\ \qquad\qquad\qquad\qquad \text{process and } E\int_0^T \chi_{N_t\neq M_t}dt\leqq\varepsilon \end{array}\right\}$$

($\chi_{N_t\neq M_t}$ *is the indicator function of* $\{N_t\neq M_t\}$ *in* $[0,T)\times R^{[0,T)}$),

then for sufficiently small $\varepsilon>0$

$$H_\varepsilon^{(0)}(N|N_T)\leqq\beta T\log\frac{\beta T}{2e\varepsilon}+\beta T\log T-E(\log N_T\,!)+o(1)\,.$$

PROOF. Since the above estimate is immediately calculated from that of (16) in [**3**], details shall be omitted. But for later use we will reproduce here the construction of $M=\{M_t\}\in W_\varepsilon^{(0)}(N)$, for which we try to calculate $EI(N, M\,|\,N_T)$, and related probabilities with slight modifications of notations.

Let $\tau_1, \tau_2, \cdots, \tau_{N_T}$ be jumping times of $\{N_t\}$ in the time interval $[0, T)$ and put

$$\varDelta=\min_{0\leqq k\leqq N_T}(\tau_{k+1}-\tau_k)\qquad(\tau_0\equiv 0,\ \tau_{N_T+1}\equiv T)\,.$$

We divide the probability space $R^{[0,T)}$ into the events

$$\begin{array}{ll} A_1=\{N_T>n_0\} & B_1=\{\varDelta>2\delta_1\} \\ A_2=\{N_T\leqq n_0\} & B_2=\{2\delta_2<\varDelta\leqq 2\delta_1\} \\ & B_3=\{\varDelta\leqq 2\delta_2\}\,, \end{array}$$

where n_0, δ_1, δ_2 are chosen as follows by fixed $a>0$, c, d, m:

$$n_0=[\varepsilon^{-c}],\ \frac{1}{1+a}<c<1\,;\ \delta_1=\varepsilon^d,\ c<d<1\,;\ \delta_2=\varepsilon^m,\ m>1^{*)}\,,$$

and let

$$C_0=A_1\cup(A_2\cap B_3)\,,\quad C_1=A_2\cap B_1\,,\quad C_2=A_2\cap B_2\,.$$

Now construct a partner process $M=\{M_t\}$ as below.

(i) If $C_0\ni\omega$, put $M_t(\omega)\equiv 0$.

(ii) If $C_1\cup C_2\ni\omega$, $\{M_t\}$ shall be a process which increases with jump 1 and $M_0=N_0=0$, $M_T=N_T$. And its conditional density function of jumping times $\sigma^{(n)}=(\sigma_1, \cdots, \sigma_n)$ under the condition $\{N_T=n, \tau_1=t_1, \cdots, \tau_n=t_n\}$ is given by $p_i(s_1, \cdots, s_n\,|\,t_1, \cdots, t_n)$ according to $\omega\in C_i$ $(i=1, 2)$;

$$p_i(s|t)=p_i(s_1, \cdots, s_n|t_1, \cdots, t_n)=\left\{\begin{array}{l} a_i\exp\{-\gamma\sum_{k=1}^n|s_k-t_k|\} \\ \qquad \text{if } s=(s_1, \cdots, s_n) \text{ belongs to the cube} \\ \qquad \text{of length } 2\delta_i \text{ with center } t=(t_1,\cdots,t_n), \\ 0 \qquad \text{for other } s=(s_1, \cdots, s_n)\,, \end{array}\right.$$

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*) We can choose $b=1$ in this case in [**3**].

where

$$a_i=[2\gamma^{-1}(1-e^{-\gamma\delta_i})]^{-n} \quad \text{and} \quad \gamma=\frac{E(N_T)}{\varepsilon}\frac{1+\psi(\varepsilon)}{1-\phi(\varepsilon)},$$

$$\psi(\varepsilon)=\frac{1}{E(N_T)}(E(N_T^{1+a}))^{\frac{1}{1+a}}O(\varepsilon^{\frac{ad}{1+a}}),$$

$$\phi(\varepsilon)=T[E(N_T^{1+a})\varepsilon^{(1+a)c-1}+O(\varepsilon^{m-1})].$$

For such $M=\{M_t\}$, we have

$$P(C_0)\leqq\frac{\varepsilon}{T}\phi(\varepsilon) \quad (P(A_1)<E(N_T^{1+a})\varepsilon^{(1+a)c}, \quad P(A_2\cap B_3)=O(\varepsilon^m)),$$

$$E\Big(\int_0^T \chi_{M_t\neq N_t}dt \mid C_1\Big)\leqq\frac{\varepsilon(1-\phi(\varepsilon))}{(1+\psi(\varepsilon))P(C_1)},$$

$$E\Big(\int_0^T \chi_{M_t\neq N_t}dt \mid C_2\Big)\leqq\frac{\varepsilon(1-\phi(\varepsilon))\psi(\varepsilon)}{(1+\psi(\varepsilon))P(C_2)}.$$

From these inequalities it is easy to verify $M\in W_\varepsilon^{(0)}(N)$ i.e. $E\int_0^T \chi_{M_t\neq N_t}\,dt\leqq\varepsilon$.

REMARK 1. As is easily seen from the definition of $p_i(s_1,\cdots,s_n\mid t_1,\cdots,t_n)$, for almost all $\omega\in C_1\cup C_2$

$$|N_t-M_t|=0 \text{ or } 1 \qquad \text{for every } t\in[0,T).$$

LEMMA 3 (Linikov [2]). *Let $\tilde{\xi}=(\tilde{\xi}_1,\cdots,\tilde{\xi}_n)$ be an n-dimensional random variable. Define ε-entropy of $\tilde{\xi}$ by*

$$H_\varepsilon^{(1)}(\tilde{\xi})=\inf\{I(\tilde{\xi},\tilde{\eta})\,;\ \tilde{\eta}\in W_\varepsilon^{(1)}(\tilde{\xi})\}$$

$$W_\varepsilon^{(1)}(\tilde{\xi})=\{\tilde{\eta}=(\tilde{\eta}_1,\cdots,\tilde{\eta}_n)\,;\ \frac{1}{n}E\sum_{k=1}^n|\tilde{\xi}_k-\tilde{\eta}_k|\leqq\varepsilon\}.$$

If $\tilde{\xi}$ has the density function $p(x_1,\cdots,x_n)$ and its differential entropy $h(\tilde{\xi})$:

$$h(\tilde{\xi}_1,\cdots,\tilde{\xi}_n)=-\int_{R^n}p(x_1,\cdots,x_n)\log p(x_1,\cdots,x_n)dx_1\cdots dx_n$$

is not negatively infinite, then for sufficiently small $\varepsilon>0$

$$H_\varepsilon^{(1)}(\tilde{\xi})\leqq n\log\frac{1}{\varepsilon}+h(\tilde{\xi})-n\log 2e+O(1),$$

$$H_\varepsilon^{(1)}(\tilde{\xi})\geqq n\log\frac{1}{\varepsilon}+h(\tilde{\xi})-n\log 2e.$$

REMARK 2. The estimates of Lemma 3 are directly deduced from the procedures of the proof of the main theorem in [2]. Moreover we note the fact that the upper estimate is actually evaluated with such $\tilde{\eta}=(\tilde{\eta}_1,\cdots,\tilde{\eta}_n)$ as each $\tilde{\eta}_k$ satisfies $E|\tilde{\xi}_k-\tilde{\eta}_k|=\varepsilon$ $(k=1,2,\cdots,n)$. Unfortunately the relevant formula of Corollary 2 in [2] is inaccurate. In fact even in the case when

$\alpha=2$, $\beta=\frac{1}{2}$ (in notations of [2]) it contradicts the Kolmogorov's well-known formula.

PROOF OF PROPOSITION 1. Let $\xi=\{\xi(t)\}$ be written as in Lemma 1. Put

$$\tilde{\xi}_0=0,\ \tilde{\xi}_1=\xi_1,\ \tilde{\xi}_2=\xi_1+\xi_2,\ \cdots.$$

We will construct a nice process $\eta=\{\eta(t)\,;\,t\in[0,T)\}\in W_\varepsilon^{(1)}(\xi)$ which may make $I(\xi,\eta)$ as small as possible and then evaluate $I(\xi,\eta)$. We modify the technique of the proof of Lemma 2. Take a small $\varepsilon_1>0$ and consider two cases according as $\beta T\leqq 1$ or $\beta T>1$.

Case 1. If $\beta T\leqq 1$, we construct $M=\{M_t\}$ as in Lemma 2 letting

$$\gamma=\frac{\beta T}{\varepsilon_1}\frac{1+\phi(\varepsilon_1)}{1-\phi(\varepsilon_1)}.$$

Case 2. If $\beta T>1$, we construct $M=\{M_t\}$ quite similarly as in Lemma 2 letting

$$\gamma=\frac{\beta T}{\varepsilon_1}\frac{1+\phi(\varepsilon_1)}{1-\beta T\phi(\varepsilon_1)}.$$

Besides, M should be constructed independently of $\tilde{\xi}^{(n)}=(\tilde{\xi}_1,\tilde{\xi}_2,\cdots,\tilde{\xi}_n)$ under the condition $N_T=n$. Then in both cases $E\int_0^T\chi_{M_t\neq N_t}dt\leqq\varepsilon_1$ is satisfied and M gives the upper estimates of Lemma 2 (ε is substituded by ε_1).

We define $\eta=\{\eta(t,\omega)\}$ as follows:

$$\eta(t,\omega)=\begin{cases}0 & \text{if } \omega\in C_0\\ \tilde{\eta}_{M_t}^{(M_T)} & \text{if } \omega\in C_1\cup C_2\,,\end{cases}$$

where $\tilde{\eta}_0^{(n)}\equiv 0$ and $\tilde{\eta}^{(n)}=(\tilde{\eta}_1^{(n)},\cdots,\tilde{\eta}_n^{(n)})$, $n=1,2,\cdots$ are random variables constructed independently of $\tau^{(n)}=(\tau_1,\cdots,\tau_n)$ and $\sigma^{(n)}=(\sigma_1,\cdots,\sigma_n)$ under the condition $M_T=n$ (accordingly $N_T=n$), and which gives the upper estimate of $H_{\varepsilon_2}^{(1)}(\tilde{\xi}^{(n)})$ in Lemma 3 for sufficiently small $\varepsilon_2>0$. From Remark 2

$$E(|\tilde{\xi}_k-\tilde{\eta}_k^{(n)}|\,|\,M_T=n)=\varepsilon_2\qquad(k=1,\cdots,n\,;\ n=1,2,\cdots). \tag{1}$$

First we prove the constructed process $\eta=\{\eta(t)\}$ belongs to $W_\varepsilon^{(1)}(\xi)$ if we choose appropriately ε_1, ε_2.

(i) To prove $\eta\in W_\varepsilon^{(1)}(\xi)$:

$$J=E\int_0^T|\xi(t)-\eta(t)|\,dt=\sum_{i=0}^{2}E\Big\{\int_0^T|\xi(t)-\eta(t)|\,dt\,;\ C_i\Big\}.$$

Denote by J_i the term corresponding to the suffix i. Then

$$J_0=E\Big\{\int_0^T|\xi(t)|\,dt\,;\,C_0\Big\}=J_{01}+J_{02}\,,$$

$$J_{01}=E\left\{\int_0^T|\xi(t)|\,dt\,;\,A_1\right\}=\int_0^T E(|\tilde{\xi}_{N_t}|\,;\,A_1)dt$$

$$\leqq\int_0^T E(|\xi_1|+\cdots+|\xi_{N_t}|\,;\,A_1)dt$$

$$=\int_0^T E(N_T E|\xi_1|\,;\,A_1)dt$$

$$=\beta T^2 E|\xi_1|\left(P(A_1)+\frac{(\beta T)^{n_0}}{n_0\,!}e^{-\beta T}\right),$$

$$J_{02}=E\left\{\int_0^T|\xi(t)|\,dt\,;\,A_2\cap B_3\right\}=\int_0^T E(|\tilde{\xi}_{N_t}|\,;\,A_2\cap B_3)dt$$

$$\leqq n_0 TE|\xi_1|P(A_2\cap B_3)\,.$$

Taking into consideration the relevant probabilities in Lemma 2, we get

$$J_0\leqq\begin{cases} E|\xi_1|\varepsilon_1[\phi(\varepsilon_1)+O(\varepsilon_1^{m-c-1})] & \text{Case } 1\,, \\ E|\xi_1|\beta T\varepsilon_1[\phi(\varepsilon_1)+O(\varepsilon_1^{m-c-1})] & \text{Case } 2\,. \end{cases}$$

On the other hand

$$J_1=J_{11}+J_{12}\,,$$

$$J_{11}=E\left\{\int_0^T|\xi(t)-\eta(t)|\chi_{N_t=M_t}dt\,;\,C_1\right\}$$

$$=E\left\{\int_0^T|\tilde{\xi}_{N_t}-\tilde{\eta}_{N_t}^{(NT)}|\chi_{N_t=M_t}dt\,;\,C_1\right\}$$

$$\leqq E\left\{\sum_{k=1}^{N_T}\int_{\tau_k}^{\tau_{k+1}}|\tilde{\xi}_k-\tilde{\eta}_k^{(NT)}|\,dt\,;\,C_1\right\}$$

$$=E\left\{\sum_{k=1}^{N_T}|\tilde{\xi}_k-\tilde{\eta}_k^{(NT)}|(\tau_{k+1}-\tau_k)\,;\,C_1\right\}$$

$$=\varepsilon_2 E(T-\tau_1\,;\,C_1\}$$

$$\leqq\varepsilon_2 TP(C_1)\,,$$

$$J_{12}=E\left\{\int_0^T|\xi(t)-\eta(t)|\chi_{N_t\neq M_t}dt\,;\,C_1\right\}$$

$$\leqq E\left\{\int_0^T(|\tilde{\xi}_{N_t}-\tilde{\xi}_{M_t}|+|\tilde{\xi}_{M_t}-\tilde{\eta}_{M_t}^{(NT)}|)\chi_{N_t\neq M_t}dt\,;\,C_1\right\}$$

$$\leqq E\left[\int_0^T\{|\xi_{N_t}|\chi_{N_t>M_t}+|\xi_{N_t+1}|\chi_{N_t<M_t}+|\tilde{\xi}_{M_t}-\tilde{\eta}_{M_t}^{(NT)}|\chi_{N_t\neq M_t}\}\,dt\,;\,C_1\right]$$

holds by Remark 1. And by the relation (1)

$$J_{12} \leqq \int_0^T (E|\xi_1|+\varepsilon_2) E\{\chi_{N_t \neq M_t}\,;\,C_1\}\,dt$$
$$=(E|\xi_1|+\varepsilon_2) E\Big\{\int_0^T \chi_{N_t \neq M_t} dt\,;\,C_1\Big\}.$$

Hence we get

$$J_1 \leqq \begin{cases} \varepsilon_2 TP(C_1)+(E|\xi_1|+\varepsilon_2)\dfrac{\varepsilon_1(1-\phi(\varepsilon_1))}{1+\phi(\varepsilon_1)} & \text{Case 1}, \\ \varepsilon_2 TP(C_1)+(E|\xi_1|+\varepsilon_2)\dfrac{\varepsilon_1(1-\beta T\phi(\varepsilon_1))}{1+\phi(\varepsilon_1)} & \text{Case 2}. \end{cases}$$

Quite analogously

$$J_2 \leqq \begin{cases} \varepsilon_2 TP(C_2)+(E|\xi_1|+\varepsilon_2)\dfrac{\varepsilon_1(1-\phi(\varepsilon_1))}{1+\phi(\varepsilon_1)}\phi(\varepsilon_1) & \text{Case 1}, \\ \varepsilon_2 TP(C_2)+(E|\xi_1|+\varepsilon_2)\dfrac{\varepsilon_1(1-\beta T\phi(\varepsilon_1))}{1+\phi(\varepsilon_1)}\phi(\varepsilon_1) & \text{Case 2}. \end{cases}$$

Finally in both cases

$$J=J_0+J_1+J_2$$
$$\leqq \varepsilon_2 T+(E|\xi_1|+\varepsilon_2)\varepsilon_1+\beta TE|\xi_1|O(\varepsilon_1^{m-c}).$$

If we choose

$$\varepsilon_1=\frac{\varepsilon}{3E|\xi_1|}, \qquad \varepsilon_2=\frac{\varepsilon}{3T}, \tag{2}$$

we may think $J\leqq\varepsilon$ for sufficiently small $\varepsilon>0$ (where we choose m so that $m-c>1$).

(ii) Estimation of $I(\xi, \eta)$.

Take ε_1, ε_2 as in (2). If we apply the formula d) twice repeatedly, then we have

$$I(\xi, \eta)=I(N_T, \eta)+EI(\{N_t\}, \eta\,|\,N_T)+EI(\xi, \eta\,|\,N_T, \{N_t\})$$
$$=I_1+I_2+I_3.$$
$$I_1 \leqq H(N_T)=I(N_T, N_T)=-E\Big[\log\Big\{\frac{(\beta T)^{N_T}}{N_T\,!}e^{-\beta T}\Big\}\Big]$$
$$=\beta T\log\frac{e}{\beta T}+E(\log N_T\,!).$$
$$I_2=EI(\{N_t\}, \{M_t\}\,|\,N_T)+EI(\{N_t\}, \{\eta(t)\}\,|\,N_T, \{M_t\})$$
$$=EI(\{N_t\}, \{M_t\}\,|\,N_T)$$
$$\leqq \beta T\log\frac{\beta T}{2e\varepsilon_1}+\beta T\log T-E(\log N_T\,!)+o(1) \quad \text{(cf. Lemma 2)},$$

for the second term of I_2 is equal to $EI((\tau_1, \cdots, \tau_{N_T}), (\tilde{\eta}_1^{(MT)}, \cdots, \tilde{\eta}_{M_T}^{(MT)})\,|\,N_T, \{M_t\})$ and vanishes as long as $(\tau_1, \cdots, \tau_{N_T})$ and $(\tilde{\eta}_1^{(MT)}, \cdots, \tilde{\eta}_{M_T}^{(MT)})$ are independent given that N_T, $\{M_t\}$ are fixed.

$$I_3 = EI((\tilde{\xi}_1, \cdots, \tilde{\xi}_{N_T}), (M_T, \sigma_1, \cdots, \sigma_{M_T}, \tilde{\eta}_1^{(M_T)}, \cdots \tilde{\eta}_{M_T}^{(M_T)}) \mid N_T, \tau_1, \cdots, \tau_{N_T})$$

$$= EI((\tilde{\xi}_1, \cdots, \tilde{\xi}_{N_T}), (M_T, \sigma_1, \cdots, \sigma_{M_T}) \mid N_T, \tau_1, \cdots, \tau_{N_T})$$

$$+ EI((\tilde{\xi}_1, \cdots, \tilde{\xi}_{N_T}), (\tilde{\eta}_1^{(M_T)}, \cdots, \tilde{\eta}_{M_T}^{(M_T)}) \mid N_T, \tau_1, \cdots, \tau_{N_T}, M_T, \sigma_1, \cdots, \sigma_{M_T}).$$

The first term of I_3 vanishes since the relevant variables are conditionally independent under fixed N_T, $(\tau_1, \cdots, \tau_{N_T})$. Apply to the second term the formula:

$$EI(\xi, \eta \mid \zeta, \nu) = EI(\xi, \eta \mid \zeta)$$

if (ξ, η) and ν are conditionally independent given ζ*). Then

$$I_3 = EI((\tilde{\xi}_1, \cdots, \tilde{\xi}_{N_T}), (\tilde{\eta}_1^{(M_T)}, \cdots, \tilde{\eta}_{M_T}^{(M_T)}) \mid N_T, M_T)$$

$$= \sum_{n=0}^{n_0} I((\tilde{\xi}_1, \cdots, \tilde{\xi}_{N_T}), (\tilde{\eta}_1^{(M_T)}, \cdots, \tilde{\eta}_{M_T}^{(M_T)}) \mid N_T = n, N_T = M_T) P(N_T = n)$$

$$(EI((\cdots), (\cdots) \mid \chi_{C_0}) = 0 \quad \text{since} \quad \eta(t) \equiv 0 \text{ on } C_0)$$

$$= \sum_{n=0}^{n_0} \Big(n \log \frac{1}{\varepsilon_2} + h(\tilde{\xi}_1, \cdots, \tilde{\xi}_n) - n \log 2e + O_n(1)\Big) P(N_T = n)$$

from Lemma 3. In the above $O_n(1)$ means the error term $O(1)$ which might depend on each dimension n. Since $h(\tilde{\xi}_1, \cdots, \tilde{\xi}_n) = n\, h(\xi_1)$ from the later Remark 3, we obtain

$$I_3 \leqq E\Big(N_T \log \frac{1}{\varepsilon_2} + N_T h(\xi_1) - N_T \log 2e\Big) + \operatorname*{Max}_{n \leqq n_0} O_n(1)$$

$$= \beta T \log \frac{1}{\varepsilon_2} + \beta T h(\xi_1) - \beta T \log 2e + O(1).$$

Summing up I_1, I_2, I_3 and substituting $\varepsilon_1 = \frac{\varepsilon}{3E|\xi_1|}$, $\varepsilon_2 = \frac{\varepsilon}{3T}$ of (2) we obtain

$$I \leqq \beta T \log \frac{T}{4e\varepsilon_1\varepsilon_2} + \beta T h(\xi_1) + O(1)$$

$$= \beta T \log \frac{9E|\xi_1| T^2}{4e\varepsilon^2} + \beta T h(\xi_1) + O(1). \qquad \text{q. e. d.}$$

REMARK 3. We have used an equality

$$h(\tilde{\xi}_1, \cdots, \tilde{\xi}_n) = n\, h(\xi_1).$$

This is easily seen since the density function of $(\tilde{\xi}_1, \cdots, \tilde{\xi}_n)$ is

$$p(x_1, \cdots, x_n) = p(x_1) p(x_2 - x_1) \cdots p(x_n - x_1 - \cdots - x_{n-1}),$$

where $p(x)$ is the density function of ξ_1.

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*) It is easy to verify the formula from the Dobrushin's formula (p. 45 [4]): $EI((\xi, \eta), \nu \mid \zeta) + EI(\xi, \eta \mid \zeta) = EI(\eta, (\xi, \nu) \mid \zeta) + EI(\xi, \nu \mid \zeta)$.

§ 3. An upper estimate of ε-entropy for a stationary independent increments process with small jumps.

In the present section we investigate a process of the same type as that of § 2. We use the same notations as in § 2. We are concerned with a stationary independent increments process $\xi=\{\xi(t); t\in[0, T)\}$ which satisfies the conditions (i), (ii), (iii) of § 2 and

$$E|\xi_1|=L\varepsilon \qquad (L>1 \text{ is a constant}), \tag{3}$$

where ε is a positive small parameter which coincides with ε of the ε-entropy $H_\varepsilon^{(1)}(\xi)$. Under these circumstances the proof of Lemma 2 fails when ε tends to 0. Hence we must find an another estimate. An upper estimate of $H_\varepsilon^{(1)}(\xi)$ for such ξ serves our purpose in future more effectively than the estimate of Proposition 1. The following Proposition 2, which will be used in § 4 under the present situations, holds true for the process under consideration.

PROPOSITION 2. *Let* $\xi=\{\xi(t); t\in[0, T)\}$ *be a stationary independent increments process which satisfies* (i), (ii), (iii) *in* § 2 *and* (3). *Then the ε-entropy of* ξ *is estimated from above by*

$$H_\varepsilon^{(1)}(\xi)\leqq K(T, L)\beta T+\beta T\log(\beta T+2)/\beta T+O(1),$$

where $K(T, L)=2(1+\sqrt{T_0L})\log 4T_0L+\log e^2(L+1/4T_0)$ *and* $T_0=\mathrm{Max}\{T, 1\}$.

In order to prove Proposition 2 we need the following two lemmas. Lemma 4 is a partial version of Lemma 3 for the n-dimensional jumping location $\tilde{\xi}^{(n)}$ of ξ when the jumping width is subject to (3).

LEMMA 4. *Let* $\tilde{\xi}^{(n)}=(\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_n)$, $\tilde{\xi}_k=\xi_1+\xi_2+\cdots+\xi_k$ $(k=1, 2, \cdots, n)$ *be given, where each of* $\xi_1, \xi_2, \cdots, \xi_n$ *is distributed with an identical density function. If we assume* $E|\xi_1|=L\varepsilon$ $(L>1)$, *then*

$$H_\varepsilon^{(1)}(\tilde{\xi}^{(n)})\leqq n\log L+O(1). \tag{4}$$

PROOF. (4) is deduced from Lemma 3, Remark 3 and the inequality:

$$h(\xi_1)\leqq -\log(1/2L\varepsilon)+\log e,$$

which is valid for such ξ_1 that satisfies $E|\xi_1|=L\varepsilon$.

We prove an inequality about the jumping time variable $\tau^{(N_T)}=(\tau_1, \tau_2, \cdots, \tau_{N_T})$ of ξ.

LEMMA 5. *There holds*

$$E\Big[\log\prod_{k=1}^{N_T}(\tau_{k+1}-\tau_k)\Big]\geqq E\Big[N_T\log\frac{T}{e(N_T+1)}\Big]. \tag{5}$$

PROOF. For each k $(1\leqq k\leqq n)$

$$E[\log(\tau_{k+1}-\tau_k)\,|\,N_T=n]=\int_{D_T^n}\log(t_{k+1}-t_k)\cdot\frac{n!}{T^n}dt$$

$$=\frac{n!}{T^n}\int_0^T dt_n\int_0^{t_n}dt_{n-1}\cdots\int_0^{t_{k+2}}dt_{k+1}\int_0^{t_{k+1}}\frac{t_k^{k-1}}{(k-1)!}\log(t_{k+1}-t_k)dt_k\,,$$

where

$$D_T^n=\{t=(t_1,t_2,\cdots,t_n)\,;\ 0\leqq t_1<t_2<\cdots<t_n<T\}.$$

If we apply successively integral formulas

$$\int_0^a(a-s)^{k-1}\log s\,ds=\frac{a^k}{k}\Big(\log a-1-\frac{1}{2}-\cdots-\frac{1}{k}\Big),$$

$$\int_0^a s^k\log s\,ds=\frac{a^{k+1}}{k+1}\Big(\log a-\frac{1}{k+1}\Big),$$

we see that

$$E[\log(\tau_{k+1}-\tau_k)\,|\,N_T=n]$$

$$=\frac{n!}{T^n}\frac{T^n}{n!}\Big(\log T-1-\frac{1}{2}-\cdots-\frac{1}{n}\Big)\geqq\log T-\log(n+1)-1\,.$$

It is trivial to obtain (5) from this inequality.

We are ready to prove Proposition 2.

PROOF OF PROPOSITION 2. For the process ξ we construct a partner process $\eta=\{\eta(t)\,;\ t\in[0,T)\}\in W_\varepsilon^{(1)}(\xi)$ for which we try an estimate $I(\xi,\eta)$, as follows. We will construct η to be a process of jumping type. η is expressed with the number of jumps M_T in $[0,T)$, the jumping time variable $\sigma^{(M_T)}=(\sigma_1,\sigma_2,\cdots,\sigma_{M_T})$ and its location variable $\tilde{\eta}^{(M_T)}=(\tilde{\eta}_1^{(M_T)},\tilde{\eta}_2^{(M_T)},\cdots,\tilde{\eta}_{M_T}^{(M_T)})$.

Let us fix an integer $n_0=[(1+2\sqrt{T_0L})\beta T]+1$.

First we define $M_T=N_T$ when $N_T\leqq n_0$ and $M_T=0$ when $N_T>n_0$.

Second we define the jumping time variable $\sigma^{(n)}=(\sigma_1,\sigma_2,\cdots,\sigma_n)$ given that $N_T=n$, $\tau^{(n)}=t$, $t=(t_1,t_2,\cdots,t_n)$ by the conditional density function

(6) $\quad p(s\,|\,t)=p(s_1,\cdots,s_n\,|\,t_1,\cdots,t_n)$

$$=\begin{cases}\dfrac{(L+1/4T_0)^nT_0^n}{\prod_{k=1}^n(t_{k+1}-t_k)} & \text{if}\quad s\in\prod_{k=1}^n\Big[t_k,\,t_k+\dfrac{t_{k+1}-t_k}{(L+1/4T_0)T_0}\Big)\\[2ex] 0 & \text{for other } s\,,\end{cases}$$

where $T_0=\max\{T,1\}$ and we put $t_{n+1}\equiv T$. Notice the probability that $0\leqq\tau_1<\sigma_1<\tau_2<\sigma_2<\cdots<\tau_n<\sigma_n<T$ is equal to one since $(L+1/4T_0)T_0>1$.

Third we define the location variable $\tilde{\eta}^{(n)}=(\tilde{\eta}_1^{(n)},\tilde{\eta}_2^{(n)},\cdots,\tilde{\eta}_n^{(n)})$, which shall be conditionally independent of $\tau^{(n)}$ under the condition $N_T=n$ so that $\tilde{\eta}^{(n)}$ may give the upper estimate for $H_{\varepsilon/4T_0}^{(1)}(\tilde{\xi}^{(n)})$ in Lemma 4.

Finally we define $\eta=\{\eta(t)\}$ given that $M_T=n$ and $\sigma^{(n)}=s$ by

$$\eta(t)=\tilde{\eta}_k^{(n)} \qquad \text{for} \quad t\in[s_k, s_{k+1}) \quad (k=0,1,\cdots,n),$$

where we put $\tilde{\eta}_0^{(n)}\equiv 0$ and $s_{n+1}\equiv T$.

Mutual information between $\tau^{(N_T)}$ and $\sigma^{(N_T)}$ and that between $\tilde{\xi}^{(N_T)}$ and $\tilde{\eta}^{(N_T)}$ under the condition $N_T=n\leqq n_0$ are estimated from above as follows:

$$I(\tau^{(N_T)}, \sigma^{(N_T)}\,|\,N_T=n)\leqq n\log(n+1)+n\log eT_0(L+1/4T_0)-\log n! \tag{7}$$

$$I(\tilde{\xi}^{(N_T)}, \tilde{\eta}^{(N_T)}\,|\,N_T=n)\leqq n_0\log 4T_0L+O(1). \tag{8}$$

In the first place we will prove (7). Since $\sigma^{(n)}$ has a density function,

$$I(\tau^{(n)}, \sigma^{(n)}\,|\,N_T=n)=h(\sigma^{(n)})-Eh(\sigma^{(n)}\,|\,\tau^{(n)}).$$

As is well known, the maximum of differential entropy for the class of density functions on a bounded domain is attained by the uniform distribution; that is

$$h(\sigma^{(n)})\leqq h(\tau^{(n)})=\log\frac{T^n}{n!}.$$

On the other hand

$$\begin{aligned}-Eh(\sigma^{(n)}\,|\,\tau^{(n)})&=n\log T_0(L+1/4T_0)-E\Big[\sum_{k=1}^{n}\log(\tau_{k+1}-\tau_k)\,|\,N_T=n\Big]\\ &\leqq n\log T_0(L+1/4T_0)-n\log\frac{T}{e(n+1)}\end{aligned}$$

by (6) and Lemma 5. Thus (7) is proved from the above inequalities.

Next in order to verify (8), it needs to note that L should be substituted by $4T_0L$ in (4) as well as ε is substituted by $\varepsilon/4T_0$. This is because $E|\xi_1|=L\varepsilon=4T_0L\cdot\dfrac{\varepsilon}{4T_0}$. Since we may consider that $(\tilde{\xi}^{(n)}, \tilde{\eta}^{(n)})$ and $(\tilde{\xi}^{(n)}, \tilde{\eta}_1^{(n+1)},\cdots,\tilde{\eta}_n^{(n+1)})$ have the same distribution (cf. Remark 2), we have $I(\tilde{\xi}^{(N_T)}, \tilde{\eta}^{(N_T)}\,|\,N_T=n)\leqq I(\tilde{\xi}^{(n_0)}, \tilde{\eta}^{(n_0)})$. (8) is obtained from this inequality and (4).

There remains only to follow the steps in § 2.

(i) To prove $\eta\in W_\varepsilon^{(1)}(\xi)$.

$$\begin{aligned}&E\Big(\int_0^T|\xi(t)-\eta(t)|\,dt\,|\,N_T=n,\ \tau^{(n)}=t,\ \sigma^{(n)}=s\Big)\\ &=E\Big[\sum_{k=1}^{n}\{|\tilde{\xi}_k-\tilde{\eta}_{k-1}^{(n)}|(s_k-t_k)+|\tilde{\xi}_k-\tilde{\eta}_k^{(n)}|(t_{k+1}-s_k)\}\,|\,N_T=n,\ \tau^{(n)}=t,\ \sigma^{(n)}=s\Big]\\ &\leqq\sum_{k=1}^{n}\{(L+1/4T_0)\varepsilon(s_k-t_k)+\varepsilon(t_{k+1}-t_k)/4T_0\}\end{aligned}$$

since $E|\tilde{\xi}_k-\tilde{\eta}_k^{(n)}|=\varepsilon/4T_0$ and $E|\tilde{\xi}_k-\tilde{\eta}_{k-1}^{(n)}|\leqq E|\tilde{\xi}_{k-1}-\tilde{\eta}_{k-1}^{(n)}|+E|\xi_1|\leqq(L+1/4T_0)\varepsilon$. As is easily calculated,

$$E(\tau_{k+1}-\tau_k\,|\,N_T=n)=\frac{T}{n+1}\,,$$

$$E(\sigma_k-\tau_k\,|\,N_T=n)=\frac{1}{2(L+1/4T_0)T_0}E(\tau_{k+1}-\tau_k\,|\,N_T=n)$$

$$=\frac{1}{2(L+1/4T_0)T_0}\,\frac{T}{n+1}\,.$$

Then we have for $n\leqq n_0$

$$E\Big(\int_0^T|\xi(t)-\eta(t)|\,dt\,|\,N_T=n\Big)$$

$$\leqq\sum_{k=1}^{n}\Big(\frac{\varepsilon}{4T_0}\cdot\frac{T}{n+1}+\frac{\varepsilon}{4T_0}\cdot\frac{T}{n+1}\Big)\leqq\frac{3}{4}\varepsilon\,.$$

On the other hand

$$E\Big(\int_0^T|\xi(t)|\,dt\,;\;N_T>n_0\Big)\leqq TE|\xi_1|E(N_T\,;\;N_T>n_0)$$

$$=TE|\xi_1|E(N_T)P(N_T\geqq n_0)\leqq\varepsilon/4$$

because of Chebyshev's inequality

$$P(N_T\geqq n_0)\leqq P(|N_T-\beta T|\geqq 2\sqrt{T_0L}\,\beta T)\leqq 1/4T_0L\beta T\,.$$

Hence it is proved that $E\int_0^T|\xi(t)-\eta(t)|\,dt\leqq\varepsilon.$

(ii) Estimation of $I(\xi,\eta)$.

Quite analogously to § 2 we have

$$I(\xi,\eta)=I(N_T,\eta)+EI(\tau^{(N_T)},\eta\,|\,N_T)+EI(\xi,\eta\,|\,N_T,\tau^{(N_T)})$$

$$=I_1+I_2+I_3\,,$$

where

$$I_1=I(N_T,\eta)\leqq H(N_T)=E(N_T)\log\frac{e}{E(N_T)}+E[\log N_T\,!]$$

$$=\beta T\log\frac{e}{\beta T}+E[\log N_T\,!]\,,$$

$$I_2=EI(\tau^{(N_T)},\eta\,|\,N_T)=EI(\tau^{(N_T)},\sigma^{(N_T)}\,|\,N_T)+EI(\tau^{(N_T)},\eta\,|\,N_T,\sigma^{(N_T)})$$

$$=EI(\tau^{(N_T)},\sigma^{(N_T)}\,|\,N_T)$$

$$\leqq E[N_T\log(N_T+1)]+E(N_T)\log eT_0(L+1/4T_0)-E[\log N_T\,!]$$

$$\leqq\beta T\log(\beta T+2)+\beta T\log eT_0(L+1/4T_0)-E[\log N_T\,!]$$

by (7) and Jensen's inequality for the concave function $\log(x+2)$:

$$E[N_T\log(N_T+1)]=\beta TE\log(N_T+2)\leqq\beta T\log[E(N_T)+2]\,,$$

and

$$
\begin{aligned}
I_3 &= EI(\xi, \eta \mid N_T, \tau^{(N_T)}) = EI(\tilde{\xi}^{(N_T)}, (\sigma^{(N_T)}, \tilde{\eta}^{(N_T)}) \mid N_T, \tau^{(N_T)}) \\
&= EI(\tilde{\xi}^{(N_T)}, \sigma^{(N_T)} \mid N_T, \tau^{(N_T)}) + EI(\tilde{\xi}^{(N_T)}, \tilde{\eta}^{(N_T)} \mid N_T, \tau^{(N_T)}, \sigma^{(N_T)}) \\
&= EI(\tilde{\xi}^{(N_T)}, \tilde{\eta}^{(N_T)} \mid N_T) \\
&\leqq n_0 \log 4T_0 L + O(1) \leqq 2(1+\sqrt{T_0 L})\beta T \log 4T_0 L + O(1)
\end{aligned}
$$

from (8). Summing up I_1, I_2 and I_3, we obtain the upper estimate of $H_\varepsilon^{(1)}(\xi)$. Proposition 2 is proved.

§ 4. An upper estimate of ε-entropy for stable processes.

Throughout the rest of the paper let $\xi = \{\xi(t) \,;\, t \in [0, T)\}$ be a symmetric stable process with exponent α $(1 < \alpha < 2)$ whose characteristic function is

$$
Ee^{i\theta\xi(t)} = \exp\left\{t\int_{R\setminus\{0\}} (e^{i\theta u} - 1 - i\theta u)\frac{du}{|u|^{1+\alpha}}\right\}.
$$

Fix a small positive number δ and define mutually independent processes $\xi^{(\delta)} = \{\xi^{(\delta)}(t)\}$, $\zeta^{(\delta)} = \{\zeta^{(\delta)}(t)\}$ with the characteristic functions

$$
Ee^{i\theta\xi^{(\delta)}(t)} = \exp\left\{t\int_{|u|\geqq\delta} (e^{i\theta u} - 1)\frac{du}{|u|^{1+\alpha}}\right\},
$$

$$
Ee^{i\theta\zeta^{(\delta)}(t)} = \exp\left\{t\int_{|u|<\delta} (e^{i\theta u} - 1 - i\theta u)\frac{du}{|u|^{1+\alpha}}\right\}.
$$

Then

$$
\xi(t) = \xi^{(\delta)}(t) + \zeta^{(\delta)}(t)
$$

in the sense that both sides have the same distribution in $R^{[0,T)}$.

As for $\xi^{(\delta)}$ the conditions in § 2 are satisfied. We calculate the quantities related to § 2.

LEMMA 6. *$\xi^{(\delta)} = \{\xi^{(\delta)}(t) \,;\, t \in [0, T)\}$ is a stochastic process of the same type as in § 2 and the density function $p(u)$ of jumping width $\xi_1^{(\delta)}$ is*

$$
p(u) = \begin{cases} 0 & |u| < \delta \\ \dfrac{1}{\beta |u|^{1+\alpha}} & |u| \geqq \delta\,, \end{cases}
$$

where

$$
\beta = \beta(\delta) = \frac{2}{\alpha}\delta^{-\alpha}.
$$

And we have

$$
E|\xi_1^{(\delta)}| = \frac{2}{\alpha-1}\left(\frac{2}{\alpha}\right)^{\frac{1-\alpha}{\alpha}}\beta^{-\frac{1}{\alpha}} = \frac{\alpha}{\alpha-1}\delta\,,
$$

$$h(\xi_1^{(\delta)}) = -\frac{1}{\alpha}\log\beta + \frac{1+\alpha}{\alpha}\left(\log\frac{2}{\alpha}+1\right).$$

As for $\zeta^{(\delta)} = \{\zeta^{(\delta)}(t);\ t\in[0, T)\}$ there holds the following Proposition 3 about the upper estimate of $H_\varepsilon^{(1)}(\zeta^{(\delta)})$.

PROPOSITION 3. *For sufficiently small $\varepsilon > 0$*

$$\begin{aligned} H_\varepsilon^{(1)}(\zeta^{(\delta)}) &\leqq H_{\varepsilon/\sqrt{T}}^{(2)}(\zeta^{(\delta)}) \\ &\leqq \frac{4}{2-\alpha}\frac{T^3}{\pi^2}\delta^{2-\alpha}\cdot\varepsilon^{-2} + o(\varepsilon^{-2}), \end{aligned}$$

where $H_\varepsilon^{(2)}(\zeta^{(\delta)})$ is defined by

$$H_\varepsilon^{(2)}(\zeta^{(\delta)}) = \inf_{\eta=\{\eta(t)\}}\left\{I(\zeta^{(\delta)}, \eta);\ E\int_0^T|\zeta^{(\delta)}(t)-\eta(t)|^2dt \leqq \varepsilon^2\right\}.$$

As will be verified below in Lemma 7, $\zeta^{(\delta)}$ is itself a mean continuous process. For the proof of Proposition 3 we can repeat the discussion developed in [1], where it is shown how to estimate $H_\varepsilon^{(2)}(\cdot)$ for a mean continuous process. We shall check in the following Lemma 7, 8, 9 and 10 the each steps of calculation applying that method. For details we refer readers to [1]. From now on we denote $\zeta^{(\delta)}$ with ζ for brevity of notation as long as confusion does not occur.

LEMMA 7. *The covariance function $r(t, s)$ of $\zeta = \{\zeta(t)\}$ is a continuous function. The j-th eigenvalue λ_j of the integral equation:*

$$\int_0^T r(t, s)\phi(s)ds = \lambda\phi(t) \tag{9}$$

is

$$\lambda_j = \frac{2\delta^{2-\alpha}}{2-\alpha}\cdot\frac{T^2}{\pi^2}\frac{1}{\left(j-\frac{1}{2}\right)^2} = Cj^{-2}+o(j^{-2}) \qquad (j=1, 2, \cdots),$$

$$C = \frac{2\delta^{2-\alpha}}{2-\alpha}\frac{T^2}{\pi^2}.$$

PROOF. In fact if $t > s$,

$$\begin{aligned} r(t, s) &= E[\zeta(t)\zeta(s)] = E[(\zeta(t)-\zeta(s))\zeta(s)] + E[\zeta(s)^2] \\ &= E[\zeta(s)^2] = -\frac{d^2}{d\theta^2}E[e^{i\theta\zeta(s)}]\Big|_{\theta=0} = \frac{2\delta^{2-\alpha}}{2-\alpha}s. \end{aligned}$$

The lemma is easily proved.

LEMMA 8. *Let $\phi_j(t)$ be the j-th eigenfunction of* (9). *Define*

$$\zeta_j(\omega) = \int_0^T\zeta(t)\phi_j(t)dt \qquad (j=1, 2, \cdots).$$

Then $E|\zeta_j|^2 = \lambda_j$. Finite-dimensional random variable $(\zeta_1, \zeta_2, \cdots, \zeta_m)$ has a bounded continuous density function for every m.

PROOF. We have

$$\zeta_j(\omega)=\int_0^T \zeta(t)\phi_j(t)dt=\int_0^T f_j(s)d\zeta(s)\,,$$

where $f_j(s)=\int_s^T \phi_j(u)du$. We investigate the characteristic function of $(\zeta_1, \zeta_2, \cdots, \zeta_m)$

$$\begin{aligned}\psi(\theta_1, \theta_2, \cdots, \theta_m)&=E\exp\Big[i\sum_{j=1}^m \theta_j\zeta_j\Big]\\&=E\exp\Big[i\sum_{j=1}^m \theta_j\int_0^T f_j(s)d\zeta(s)\Big].\end{aligned}$$

The characteristic function of a stochastic integral of this type is found in [6].

$$\psi(\theta_1, \cdots, \theta_m)=\exp\Big[\int_0^T\int_{|x|<\delta}(e^{ix\sum_{j=1}^m \theta_j f_j(t)}-1-ix\sum_{j=1}^m \theta_j f_j(t))\frac{dtdx}{|x|^{\alpha+1}}\Big].$$

In order to obtain the conclusion of Lemma 8 it is sufficient to prove $\psi(\theta_1, \cdots, \theta_m)\in L^1(R^m)$. We will prove that in the obvious inequality

$$|\psi(\theta_1, \cdots, \theta_m)|\leqq\exp\Big[-\int_0^T\int_{|x|<\delta}(1-\cos x\sum_{j=1}^m \theta_j f_j(t))\frac{dtdx}{|x|^{\alpha+1}}\Big]$$

the right-hand side is summable outside a compact set.

We note that

(i) there exists such a positive constant $K(\delta)$ that

$$1-\cos z\geqq K(\delta)z^2 \qquad \text{for every } |z|<\delta\,,$$

(ii) $\phi_j(t)=\sqrt{\frac{2}{T}}\sin\left(j-\frac{1}{2}\right)\frac{\pi t}{T}$,

$$f_j(t)=a_j\cos\left(j-\frac{1}{2}\right)\frac{\pi t}{T}\,, \qquad a_j=\sqrt{\frac{2\lambda_j}{DT}} \quad \left(D=\frac{2}{2-\alpha}\delta^{2-\alpha}\right),$$

$$\int_0^T f_j(t)f_k(t)dt=0 \quad (j\neq k)\,, \qquad \int_0^T f_j(t)^2dt=\frac{\lambda_j}{D}.$$

Now consider a point $\theta=(\theta_1, \cdots, \theta_m)$, $\|\theta\|>\frac{1}{A\delta}$ where $A=(\sum_{j=1}^\infty a_j^2)^{\frac{1}{2}}$. Since

$$J_\theta=\Big\{x;\ |x|<\frac{\delta}{A\|\theta\|}\Big\}\subset\{x;\ |x|<\delta\}$$

in view of $\delta<1$ and

$$|x\sum_{j=1}^m \theta_j f_j(t)|\leqq|x|A\|\theta\|<\delta \qquad \text{for every } x\in J_\theta\,,$$

we have

$$\int_0^T\int_{|x|<\delta}(1-\cos x\sum_{j=1}^m \theta_j f_j(t))\frac{dtdx}{|x|^{\alpha+1}}$$

$$\geqq \int_0^T \int_{x \in J_\theta} (1-\cos x \sum_{j=1}^m \theta_j f_j(t)) \frac{dtdx}{|x|^{\alpha+1}}$$

$$\geqq \int_0^T \int_{x \in J_\theta} K(\delta) x^2 (\sum_{j=1}^m \theta_j f_j(t))^2 \frac{dtdx}{|x|^{\alpha+1}}$$

$$= K(\alpha, T, \delta) \sum_{j=1}^m \lambda_j \theta_j^2 / (\sum_{j=1}^m \theta_j^2)^{1-\frac{\alpha}{2}},$$

where $K(\alpha, T, \delta)$ is a positive constant depending on α, T and δ only. In conclusion there holds for $\|\theta\| > \frac{1}{A\delta}$

$$|\phi(\theta_1, \cdots, \theta_m)| \leqq \exp\left\{-K(\alpha, T, \delta) \frac{\sum_{j=1}^m \lambda_j \theta_j^2}{\sum_{j=1}^m \theta_j^2} (\sum_{j=1}^m \theta_j^2)^{\frac{\alpha}{2}}\right\},$$

the right-hand side of which is obviously a summable function in the set $\left\{\theta ; \|\theta\| > \frac{1}{A\delta}\right\}$. q. e. d.

LEMMA 9.

$$H_\varepsilon^{(2)}(\zeta) \leqq H_{\tilde{\varepsilon}}^{(2)}((\zeta_1, \cdots, \zeta_m))$$

where $\tilde{\varepsilon}^2 = \varepsilon^2 - \sum_{j=m+1}^\infty \lambda_j > 0$ *and* $H_\varepsilon^{(2)}((\zeta_1, \cdots, \zeta_m))$ *is defined by*

$$H_\varepsilon^{(2)}((\zeta_1, \cdots, \zeta_m)) = \inf_{(\eta_1, \cdots, \eta_m)} \{I((\zeta_1, \cdots, \zeta_m), (\eta_1, \cdots, \eta_m)) ; E \sum_{j=1}^m |\zeta_j - \eta_j|^2 \leqq \varepsilon^2\}.$$

LEMMA 10. *For sufficiently small* $\varepsilon > 0$

$$H_\varepsilon^{(2)}((\zeta_1, \cdots, \zeta_m)) \leqq \frac{m}{2} \log \frac{1}{\varepsilon^2} + \frac{m}{2} \log m - m \log \{2\pi e (1-\varepsilon^2 / \sum_{j=1}^m \lambda_j)\}^{\frac{1}{2}}$$
$$+ h((\zeta_1, \cdots, \zeta_m)) + o(1).$$

Generally there holds for the differential entropy

$$h((\zeta_1, \cdots, \zeta_m)) \leqq \log \prod_{j=1}^m (2\pi e \lambda_j)^{\frac{1}{2}}.$$

For proof of Lemma 9 and Lemma 10 we refer readers to [**1**]. The existence of differential entropy $h((\zeta_1, \cdots, \zeta_m))$ is assured by Lemma 8. Combining the above lemmas we will prove Proposition 3.

PROOF OF PROPOSITION 3. The first inequality is obvious from the definitions of $H_\varepsilon^{(1)}$ and $H_\varepsilon^{(2)}$. Take a sufficiently small $\varepsilon^2 \leqq \sum_{j=1}^\infty \lambda_j$. Combining Lemma 9 and 10, we have for each m such that $\varepsilon^2/T - \sum_{j=m+1}^\infty \lambda_j = \tilde{\varepsilon}^2 > 0$

$$H_{\varepsilon}^{(1)}(\zeta^{(\delta)}) \leqq H_{\varepsilon/\sqrt{T}}^{(2)}(\zeta^{(\delta)}) \leqq H_{\tilde{\varepsilon}}^{(2)}((\zeta_1^{(\delta)}, \cdots, \zeta_m^{(\delta)}))$$
$$\leqq \frac{1}{2}\log \prod_{j=1}^{m} \frac{\lambda_j}{\tilde{\varepsilon}^2/m} - m\log\Big(1-\tilde{\varepsilon}^2/\sum_{j=1}^{m}\lambda_j\Big)^{\frac{1}{2}} + o(1).$$

If we choose m and $\tilde{\varepsilon}$ so that $\tilde{\varepsilon}^2/m \sim \lambda_m$, the relations $\tilde{\varepsilon}^2 = \varepsilon^2/T - \sum_{j=m+1}^{\infty}\lambda_j$, $\lambda_j = Cj^{-2} + o(j^{-2})$ (cf. Lemma 7) and $\sum_{j=m+1}^{\infty} j^{-2} \sim m^{-1}$ determine $m = \Big[\frac{2CT}{\varepsilon^2}\Big]$. Finally by using Stirling's formula $m! \sim \sqrt{2\pi}\, e^{-m} m^{m+\frac{1}{2}}$ we have

$$H_{\varepsilon}^{(1)}(\zeta^{(\delta)}) \leqq m + o(m) = 2CT\varepsilon^{-2} + o(\varepsilon^{-2})$$
$$= \frac{4}{2-\alpha}\frac{T^3}{\pi^2}\delta^{2-\alpha}\varepsilon^{-2} + o(\varepsilon^{-2}).$$

Now we proceed to the main proposition of this section.

PROPOSITION 4. *Let* $\xi = \{\xi(t)\,;\, t \in [0, T)\}$ *be a symmetric stable process with exponent* $1 < \alpha < 2$. *Then*

$$H_{\varepsilon}^{(1)}(\xi) \leqq \bar{C}(\alpha, T)\varepsilon^{-\alpha} + o(\varepsilon^{-\alpha}),$$

where

$$\bar{C}(\alpha, T) = \frac{2^{\alpha+2}}{2-\alpha}\frac{T^3}{\pi^2} + \frac{2^{\alpha+1}}{\alpha}TK\Big(T, \frac{\alpha}{\alpha-1}\Big)$$

and $K(\cdot, \cdot)$ *is the constant in Proposition* 2.

PROOF. We put $\delta = \varepsilon/2$ and decompose $\xi(t)$ into the sum of two independent processes just as indicated in the beginning of this section:

$$\xi(t) = \xi^{(\delta)}(t) + \zeta^{(\delta)}(t).$$

First let us investigate an upper estimate of $H_{\varepsilon/2}^{(1)}(\xi^{(\delta)})$. $\xi^{(\delta)}$ satisfies the conditions (i), (ii), (iii) of § 2 and

$$E|\xi_1^{(\delta)}| = \frac{\alpha}{\alpha-1}\delta = L\frac{\varepsilon}{2} \qquad \Big(L = \frac{\alpha}{\alpha-1} > 1\Big)$$

from Lemma 6. We apply Proposition 2 of § 3 replacing ε by $\varepsilon/2$. In the present case

$$L = \frac{\alpha}{\alpha-1}, \qquad \beta = \frac{2}{\alpha}\Big(\frac{\varepsilon}{2}\Big)^{-\alpha} \qquad \text{(cf. Lemma 6)}.$$

Even if β and ε are related in this way, all the discussions in § 3 go well, because it has no effect on the argument of § 3 whether the expectation of number of jumps βT is varying with ε. Also for such L and β as above, Proposition 2 holds just as it does.

$$\text{(10)} \qquad I(\xi^{(\delta)}, \eta_1) \leqq K\Big(T, \frac{\alpha}{\alpha-1}\Big)\beta T + \beta T\log(\beta T+2)/\beta T + O(1)$$
$$= \frac{2^{\alpha+1}}{\alpha}TK\Big(T, \frac{\alpha}{\alpha-1}\Big)\cdot\varepsilon^{-\alpha} + O(1),$$

where η_1 is the process which belongs to $W_{\varepsilon/2}^{(1)}(\xi^{(\delta)})$ and gives the upper estimate corresponding to Proposition 2.

Next we investigate an upper estimate of $H_{\varepsilon/2}^{(1)}(\zeta^{(\delta)})$. We will apply Proposition 3 substituting ε by $\varepsilon/2$. Even when δ is given by the relation $\delta=\varepsilon/2$, all the discussions in Proposition 3 go well; i.e. for the process η_2 which belongs to $W_{\varepsilon/2}^{(1)}(\zeta^{(\delta)})$ and gives the estimate of Proposition 3,

$$I(\zeta^{(\delta)}, \eta_2)=m+o(m)=\frac{2^{\alpha+2}}{2-\alpha}\frac{T^3}{\pi^2}\varepsilon^{-\alpha}+o(\varepsilon^{-\alpha}), \tag{11}$$

if we take $m=\left[\frac{2CT}{(\varepsilon/2)^2}\right]$ in the proof of Proposition 3 with $C=\frac{2}{2-\alpha}\left(\frac{\varepsilon}{2}\right)^{2-\alpha}\frac{T^2}{\pi^2}$ (Lemma 7).

It is obvious that the process $\eta=\{\eta(t)\}$:

$$\eta(t)=\eta_1(t)+\eta_2(t)$$

belongs to $W_\varepsilon^{(1)}(\xi)$ for $\eta_1\in W_{\varepsilon/2}^{(1)}(\xi^{(\delta)})$ and $\eta_2\in W_{\varepsilon/2}^{(1)}(\zeta^{(\delta)})$. The mapping of path functions

$$\{\xi(t)\}\longrightarrow(\{\xi^{(\delta)}(t)\},\ \{\zeta^{(\delta)}(t)\})$$

is a one-to-one bimeasurable mapping, since $\xi^{(\delta)}(\cdot)$ is a step function whose jump widths are greater than δ and $\zeta^{(\delta)}(\cdot)$ changes its value with jumps less than δ. Hence by the property c) of §1

$$\begin{aligned}H_\varepsilon^{(1)}(\xi)&\leqq I(\xi,\eta)=I((\xi^{(\delta)},\zeta^{(\delta)}),\eta)\\&\leqq I((\xi^{(\delta)},\zeta^{(\delta)}),(\eta_1,\eta_2)).\end{aligned}$$

As is easily seen from the constructions of η_1 and η_2 in the preceding sections, $(\xi^{(\delta)},\eta_1)$ and $(\zeta^{(\delta)},\eta_2)$ are mutually independent. If we apply the property b) of §1 to the last quantity, it follows that

$$H_\varepsilon^{(1)}(\xi)\leqq I(\xi^{(\delta)},\eta_1)+I(\zeta^{(\delta)},\eta_2). \tag{12}$$

Finally we obtain from (10), (11) and (12)

$$H_\varepsilon^{(1)}(\xi)\leqq\bar{C}(\alpha,T)\varepsilon^{-\alpha}+o(\varepsilon^{-\alpha}).$$

REMARK 4. Though we have proved $\overline{\lim}_{\varepsilon\to0}\varepsilon^\alpha H_\varepsilon^{(1)}(\xi)\leqq\bar{C}(\alpha,T)$, the bound $\bar{C}(\alpha,T)$ is not necessarily satisfactory. In fact $\bar{C}(\alpha,T)$ tends to infinity when $\alpha\to1$ or $\alpha\to2$. This is because we have treated with equal weight large jumping part $\xi^{(\delta)}$ and small jumping part $\zeta^{(\delta)}$, and as a consequence overestimated the information about the part of small probability when α is close to 1 or 2. It is desired to give a more critical coefficient of $\varepsilon^{-\alpha}$ for $H_\varepsilon^{(1)}(\xi)$.

We will here explain why we must have established Proposition 2 in addition to Proposition 1.

We succeeded in the present estimation by decomposing $\xi=\xi^{(\delta)}+\zeta^{(\delta)}$, where we chose $\delta=\delta(\varepsilon)$ so small that $E|\xi_1^{(\delta)}|\asymp\varepsilon$. On the other hand, even if we improve the discussions of § 2 to apply to the present proof, we can show at most that $\overline{\lim_{\varepsilon\to 0}}\frac{\varepsilon^\alpha}{\log 1/\varepsilon}H_\varepsilon^{(1)}(\xi)<\infty$. This is due to the fact that $E|\xi_1|$ must be a constant in the situation of § 2.

§ 5. A lower estimate of ε-entropy for stable processes.

We give a lower estimate of $H_\varepsilon^{(1)}(\xi)$ for symmetric stable processes. The problem will be reduced to a lower estimate of ε-entropy for a finite-dimensional random variable (cf. Lemma 3).

PROPOSITION 5.

$$H_\varepsilon^{(1)}(\xi)\geqq \underline{C}(\alpha, T)\varepsilon^{-\alpha}+O(1),$$

where

$$\underline{C}(\alpha, T)=\frac{T^{\alpha+1}}{(8e^2)^\alpha}\frac{1+\alpha}{\alpha}\left(\log\frac{2}{\alpha}+1\right).$$

The following Lemma 11 gives a first step to obtain the estimate of Proposition 5. Let be given $\varepsilon>0$ and fix $\delta>\delta'$, $\delta'=\varepsilon^{\frac{4}{2-\alpha}}$. We use the notations $\xi^{(\delta)}$, $\zeta^{(\delta)}$ etc. in § 2, § 4 and omit the suffix δ when confusion does not occur. We denote with $\xi^{(\delta',\delta)}=\{\xi^{(\delta',\delta)}(t)\,;\,t\in[0, T)\}$ the stationary independent increments process with the characteristic function

$$Ee^{i\theta\xi^{(\delta',\delta)}(t)}=\exp\left\{t\int_{\delta'\leqq|u|<\delta}(e^{i\theta u}-1)\frac{du}{|u|^{1+\alpha}}\right\}. \tag{13}$$

LEMMA 11. *For small $\varepsilon>0$ there holds an equality*

$$H_\varepsilon^{(1)}(\xi)\geqq\inf\{EI(\xi^{(\delta)}, \tilde{\eta}\,|\,\xi^{(\delta',\delta)})\,;\,\tilde{\eta}\in W_{2\varepsilon}^{(1)}(\xi^{(\delta)})\}.$$

PROOF. Since for $\eta\in W_\varepsilon^{(1)}(\xi)$

$$E\int_0^T|\xi^{(\delta')}(t)-\eta(t)|\,dt\leqq E\int_0^T|\xi(t)-\eta(t)|\,dt+E\int_0^T|\zeta^{(\delta')}(t)|\,dt$$

$$\leqq\varepsilon+\int_0^T\{E|\zeta^{(\delta')}(t)|^2\}^{\frac{1}{2}}dt\leqq\varepsilon+\left(\frac{2}{2-\alpha}\right)^{\frac{1}{2}}(\delta')^{\frac{2-\alpha}{2}}T^{\frac{3}{2}}\leqq 2\varepsilon$$

(cf. Proof of Lemma 7)

holds for small $\varepsilon>0$, we have a series of inequalities from c) and d) in § 1

$$\begin{aligned}H_\varepsilon^{(1)}(\xi)&=\inf\{I(\xi,\eta)\,;\,\eta\in W_\varepsilon^{(1)}(\xi)\}\\&=\inf\{I((\xi^{(\delta)},\xi^{(\delta',\delta)},\zeta^{(\delta')}),\eta)\,;\,\eta\in W_\varepsilon^{(1)}(\xi)\}\\&\geqq\inf\{I((\xi^{(\delta)},\xi^{(\delta',\delta)}),\eta)\,;\,\eta\in W_{2\varepsilon}^{(1)}(\xi^{(\delta')})\}\end{aligned}$$

$$\geqq \inf \{EI(\xi^{(\delta)}, \eta \mid \xi^{(\delta',\delta)}) ; \eta \in W_{2\varepsilon}^{(1)}(\xi^{(\delta')})\}$$
$$\geqq \inf \{EI(\xi^{(\delta)}, \tilde{\eta} \mid \xi^{(\delta',\delta)}) ; \tilde{\eta} \in W_{2\varepsilon}^{(1)}(\xi^{(\delta)})\} ,$$

where we defined a new process $\tilde{\eta} = \{\tilde{\eta}(t)\}$ for $\eta \in W_{2\varepsilon}^{(1)}(\xi^{(\delta')})$ by

$$\tilde{\eta}(t) = \eta(t) - \xi^{(\delta',\delta)}(t) ,$$

which is subordinate to $\eta = \{\eta(t)\} \in W_{2\varepsilon}^{(1)}(\xi^{(\delta')})$ given $\xi^{(\delta',\delta)}$ and belongs to $W_{2\varepsilon}^{(1)}(\xi^{(\delta)})$.

PROOF OF PROPOSITION 5. From Lemma 11

$$H_\varepsilon^{(1)}(\xi) \geqq \inf \{EI(\xi^{(\delta)}, \tilde{\eta} \mid \xi^{(\delta',\delta)}) ; \tilde{\eta} \in W_{2\varepsilon}^{(1)}(\xi^{(\delta)})\}$$
$$\geqq \inf \{EI((\tilde{\xi}_1, \cdots, \tilde{\xi}_{N_T}), \tilde{\eta} \mid N_T, \tau_1, \cdots, \tau_{N_T}, \xi^{(\delta',\delta)}) ; \tilde{\eta} \in W_{2\varepsilon}^{(1)}(\xi^{(\delta)})\} .$$

Since $\xi^{(\delta',\delta)}$ is a process with a finite number of jumps in $[0, T)$ like that of Lemma 1, it is possible to express both the average conditional information and the restriction $\tilde{\eta} \in W_{2\varepsilon}^{(1)}(\xi^{(\delta)})$ with $p_n(t)$ and $q_m(s, x)$, where

$$p_n(t) = \text{the density of } \tau^{(n)} \text{ restricted on the set } \{N_T = n\}$$
$$= \frac{(\beta T)^n}{n!} e^{-\beta T} \frac{n!}{T^n} \qquad \left(\beta = \int_{|u| \geqq \delta} \frac{du}{|u|^{1+\alpha}}\right)$$
$$t \in D_T^n = \{t = (t_1, \cdots, t_n) ; 0 \leqq t_1 < \cdots < t_n < T\}$$

and

$$q_m(s, x) = \text{the density of}$$
$$P(N_T^{(\delta',\delta)} = m, \tau^{(\delta',\delta)} \leqq s, \xi_1^{(\delta',\delta)} \leqq x_1, \cdots, \xi_m^{(\delta',\delta)} \leqq x_m)$$
$$x = (x_1, \cdots, x_m) \in R^m , \qquad s = (s_1, \cdots, s_m) \in D_T^m ,$$

where $N_T^{(\delta',\delta)}$, $\tau_k^{(\delta',\delta)}$ and $\xi_k^{(\delta',\delta)}$ denote respectively the number of jumps, k-th jumping time and k-th jumping width of $\xi^{(\delta',\delta)}$. Then it is easy to see

$$H_\varepsilon^{(1)}(\xi) \geqq \inf \Big[E \inf_{\tilde{\eta}} \Big\{ I((\tilde{\xi}_1, \cdots, \tilde{\xi}_n), \tilde{\eta} \mid (\sharp)) ;$$
$$\int_0^T E(|\xi^{(\delta)}(t) - \tilde{\eta}(t)| \mid (\sharp)) dt \leqq \varepsilon_{nm}(t, s, x) \Big\} \Big] ,$$

where $(\sharp)$ is the condition

$$N_T = n,\ N_T^{(\delta',\delta)} = m,\ \tau = t,\ \tau^{(\delta',\delta)} = s,\ \xi_1^{(\delta',\delta)} = x_1, \cdots, \xi_m^{(\delta',\delta)} = x_m$$

and the outer infimum is taken over all non-negative functions $\{\varepsilon_{nm}(t, s, x) ; n, m = 0, 1, 2, \cdots\}$ that satisfy

$$\sum_{n,m=0}^{\infty} \int_{R^m} \Big\{ \int_{D_T^n \times D_T^m} \varepsilon_{nm}(t, s, x) p_n(t) q_m(s, x) dt ds \Big\} dx \leqq 2\varepsilon . \tag{14}$$

First let us investigate the inner infimum. We omit for simplicity of notations the condition (#) (n, m, t, s, x are fixed). If $\tilde{\eta}$ belongs to the range where the inner infimum is taken,

$$\begin{aligned}\varepsilon_{nm}(t, s, x) &\geqq E\int_0^T |\xi^{(\delta)}(t)-\tilde{\eta}(t)|\,dt \\ &= E\sum_{l=0}^{n}\int_{t_l}^{t_{l+1}} |\tilde{\xi}_l-\tilde{\eta}(t)|\,dt \qquad (\tilde{\xi}_0=0,\ t_0=0,\ t_{n+1}=T) \\ &\geqq \sum_{l=1}^{n}(t_{l+1}-t_l)\inf_t \{E|\tilde{\xi}_l-\tilde{\eta}(t)|\ ;\ t_l\leqq t<t_{l+1}\}\,.\end{aligned}$$

Hence we conclude that there exist $s_l\in[t_l, t_{l+1})$, $l=1, 2, \cdots, n$, for which

$$\frac{1}{n}E\sum_{l=1}^{n}|\bar{\xi}_l-\bar{\eta}(s_l)|\leqq 2\varepsilon_{nm}(t, s, x)$$

holds, where we put

$$\begin{aligned}\bar{\xi}_l &= n(t_{l+1}-t_l)\tilde{\xi}_l\,, \\ \bar{\eta}(s_l) &= n(t_{l+1}-t_l)\tilde{\eta}(s_l)\,.\end{aligned} \tag{15}$$

Since $\bar{\eta}(s_1), \cdots, \bar{\eta}(s_n)$ are subordinate to $\tilde{\eta}$, the inner infimum is greater than

$$\begin{aligned}\inf&\Big\{I((\bar{\xi}_1, \cdots, \bar{\xi}_n), (\bar{\eta}(s_1), \cdots, \bar{\eta}(s_n)))\ ;\ \frac{1}{n}E\sum_{l=1}^{n}|\bar{\xi}_l-\bar{\eta}(s_l)|\leqq 2\varepsilon_{nm}(t, s, x)\Big\} \\ &= H^{(1)}_{2\varepsilon_{nm}(t,s,x)}(\bar{\xi}_1, \cdots, \bar{\xi}_n) \\ &\geqq n\log\frac{1}{2\varepsilon_{nm}(t, s, x)}+h(\bar{\xi}_1, \cdots, \bar{\xi}_n)-n\log 2e\end{aligned}$$

from Lemma 3. The last quantity equals

$$F(n, t, \varepsilon_{nm}(t, s, x))=n\log\frac{n}{4e\varepsilon_{nm}(t, s, x)}+nh(\xi_1^{(\delta)})+\log\prod_{l=1}^{n}(t_{l+1}-t_l)$$

since

$$h(\bar{\xi}_1, \cdots, \bar{\xi}_n)=h(\tilde{\xi}_1, \cdots, \tilde{\xi}_n)+\log n^n\prod_{l=1}^{n}(t_{l+1}-t_l)$$

by change of variables (15) and $h(\tilde{\xi}_1, \cdots, \tilde{\xi}_n)=nh(\xi_1^{(\delta)})$ by Remark 3 of § 2.

Next we investigate the outer infimum. The problem is to minimize

$$\sum_{n,m=0}^{\infty}\int_{R^m}\Big\{\int_{D_T^n\times D_T^m}F(n, t, \varepsilon_{nm}(t, s, x))p_n(t)q_m(s, x)\,dt\,ds\Big\}dx$$

under the condition (14). As is easily seen from the functional F we may suppose that the restriction (14) is given by an equality. By usual variational method we see that

$$\varepsilon_{nm}(t, s, x)=\frac{2n\varepsilon}{\beta T}$$

gives the minimum, so that

(16)
$$H_\varepsilon^{(1)}(\xi) \geqq \sum_{n,m=0}^{\infty} \int_{R^m} \left\{ \int_{D_T^n \times D_T^m} F(n, t, \varepsilon_{nm}(t, s, x)) p_n(t) q_m(s, x) dt ds \right\} dx$$
$$\geqq \beta T \log \frac{\beta T}{8e\varepsilon} + \beta T h(\xi_1^{(\delta)}) + E[\log \prod_{l=1}^{N_T} (\tau_{l+1} - \tau_l)] .$$

About the last term of (16) holds the following inequality

(17)
$$E\left[\log \prod_{l=1}^{N_T} (\tau_{l+1} - \tau_l)\right] \geqq \beta T \log \frac{T}{e(\beta T + 2)} .$$

This is because

$$E\left[\log \prod_{l=1}^{N_T} (\tau_{l+1} - \tau_l)\right] \geqq E\left[N_T \log \frac{T}{e(N_T + 1)}\right]$$

from Lemma 5 in § 3 and

$$E[N_T \log (N_T + 1)] = \beta T E[\log (N_T + 2)] \leqq \beta T \log (\beta T + 2)$$

from Jensen's inequality for a concave function.

If we substitute the third term of (16) by (17) and the second term $h(\xi_1^{(\delta)})$ of (16) by the expression in Lemma 6, there holds

$$H_\varepsilon^{(1)}(\xi) \geqq \beta T \log \frac{T}{8e^2\varepsilon} - \frac{\beta T}{\alpha} \log \beta + \beta T \frac{1+\alpha}{\alpha} \left(\log \frac{2}{\alpha} + 1\right) + \beta T \log \frac{\beta T}{\beta T + 2} .$$

Here we will choose $\beta = \beta(\delta)$ so that the first and the second term cancel each other, that is

$$\beta = \left(\frac{T}{8e^2}\right)^\alpha \varepsilon^{-\alpha} \qquad (\delta \asymp \varepsilon \text{ for such } \beta) .$$

Then we obtain

$$H_\varepsilon^{(1)}(\xi) \geqq \frac{T^{\alpha+1}}{(8e^2)^\alpha} \frac{1+\alpha}{\alpha} \left(\log \frac{2}{\alpha} + 1\right) \varepsilon^{-\alpha} - \frac{T}{(8e^2)^\alpha} \varepsilon^{-\alpha} \log \left(1 + \frac{2^{3\alpha+1} e^{2\alpha}}{T^{\alpha+1}} \varepsilon^\alpha\right)$$
$$= \underline{C}(\alpha, T) \varepsilon^{-\alpha} + O(1) . \qquad \text{q. e. d.}$$

We have proved the main Theorem.

More generally for asymmetric or one-sided stable processes we can obtain the asymptotic estimate of order $\varepsilon^{-\alpha}$ quite similarly. The proof needs almost no change. The crucial point is concerned with the order of growth of Lévy measure near the origin——we can choose $\beta \asymp \varepsilon^{-\alpha}$ by favor of the relations $E|\xi_1^{(\delta)}| \asymp \beta^{-\frac{1}{\alpha}}$, $h(\xi_1^{(\delta)}) \asymp -\frac{1}{\alpha} \log \beta$ in Proposition 4 and Proposition 5.

We will prove the following Proposition 6 for the process considered in § 2.

PROPOSITION 6. *Let* $\xi = \{\xi(t) ; t \in [0, T)\}$ *be the process in Proposition* 1, § 2. *Then*

$$\beta T \leqq \varliminf_{\varepsilon \to 0} H_\varepsilon^{(1)}(\xi) / \log \frac{1}{\varepsilon} \leqq \varlimsup_{\varepsilon \to 0} H_\varepsilon^{(1)}(\xi) / \log \frac{1}{\varepsilon} \leqq 2\beta T .$$

PROOF. $\overline{\lim_{\varepsilon \to 0}} H_\varepsilon^{(1)}(\xi)/\log\frac{1}{\varepsilon} \leqq 2\beta T$ is obvious from the estimate of Proposition 1.

In order to prove $\beta T \leqq \underline{\lim}_{\varepsilon \to 0} H_\varepsilon^{(1)}(\xi)/\log\frac{1}{\varepsilon}$ we note that

$$H_\varepsilon^{(1)}(\xi) = \inf\{I(\xi, \eta)\,;\ \eta \in W_\varepsilon^{(1)}(\xi)\}$$
$$\geqq \inf\{EI((\tilde{\xi}_1, \cdots, \tilde{\xi}_{N_T}),\ \eta \mid N_T, \tau_1, \cdots, \tau_{N_T})\,;\ \eta \in W_\varepsilon^{(1)}(\xi)\}$$

and modify the proof of Proposition 5. That is

$$H_\varepsilon^{(1)}(\xi) \geqq \inf\Big[E\inf\Big\{I((\tilde{\xi}_1, \cdots, \tilde{\xi}_n), \eta \mid (\sharp))\,;$$
$$\int_0^T E(|\xi(t)-\eta(t)| \mid (\sharp))dt \leqq \varepsilon_n(t)\Big\}\Big],$$

where $(\sharp)$ is the condition $N_T = n$, $\tau^{(n)} = t$ and the outer infimum is taken over all non-negative functions $\{\varepsilon_n(t)\,;\ n = 0, 1, \cdots\}$ that satisfy

$$\sum_{n=0}^{\infty}\int_{D_T^n} \varepsilon_n(t)p_n(t)dt \leqq \varepsilon\,.$$

Here $p_n(t)$ is the density of $\tau^{(n)}$ restricted on the set $\{N_T = n\}$. Succeeding arguments are quite analogous and in conclusion we see that

$$H_\varepsilon^{(1)}(\xi) \geqq \beta T\log\frac{T}{4e\varepsilon} + \beta Th(\xi_1) + E\Big[\log\prod_{l=1}^{N_T}(\tau_{l+1}-\tau_l)\Big]$$
$$= \beta T\log\frac{1}{\varepsilon} + O(1)\,.$$

Thus Proposition is proved.

REMARK 5. As is suggested by the results about the ε-entropy in function spaces, also the ε-entropy of a stochastic process has a kind of character of dimension. Lemma 3, § 2 states that

$$\lim_{\varepsilon \to 0} H_\varepsilon^{(1)}(\tilde{\xi})/\log\frac{1}{\varepsilon} = n \tag{18}$$

for an n-dimensional random variable $\tilde{\xi}$.

If we simply compare the result of Proposition 6 and (18), we may well expect that the process ξ of Proposition 6 has a finite-dimensional character. In fact the process ξ can be completely described by N_T and $(\xi_1, \cdots, \xi_{N_T})$ and $(\tau_1, \cdots, \tau_{N_T})$. The dimension of those random variables is $2N_T$ in total ——$2\beta T$ in the mean (we excluding the random variable N_T itself, which is of different character: i. e. a discrete-valued random variable). Hence in consideration of (18) it is expected that $\lim_{\varepsilon \to 0} H_\varepsilon^{(1)}(\xi)/\log\frac{1}{\varepsilon} = 2\beta T$. However, since we have neglected about the information contained in $(\tau_1, \cdots, \tau_{N_T})$ in the lower estimation of Proposition 6 (also in the proof of Proposition 5), we have necessarily attained to an unsatisfactory estimate.

As it is shown in the estimate of the main Theorem that

$$\lim_{\varepsilon\to 0} \log H_\varepsilon^{(1)}(\xi)/\log\frac{1}{\varepsilon}=\alpha\,,$$

stable process ξ has certainly a higher dimensional character than the above process. This estimate is, in a sense, a generalization of

$$\lim_{\varepsilon\to 0} \log H_\varepsilon^{(2)}(B)/\log\frac{1}{\varepsilon}=2$$

for one-dimensional Brownian motion process $B=\{B(t)\}$ [5].

References

[1] K. Kazi, On the ε-entropy of diffusion processes, Ann. Inst. Stat. Math., 21 (1969), 347–356.

[2] Ю. Н. Линьков, Вычисление ε-энтропии случайных величин при малых ε, Провлемы Передачи Информации, 1 (1965), 18–26.

[3] Ю. Н. Линьков, Эпсилон-энтропия случаиных процессов с непрерывным временем и дискретным фазовым пространством, Провлемы Передачи Информации, 7 (1971), 16–25.

[4] M. S. Pinsker, Information and information stability of random variables and processes, Holden-Day, 1964 (English translation).

[5] М. С. Пинскер, Гауссовские источники, Проблемы Передачи Информации, 14 (1963), 59–100.

[6] А. В. Скороход, Случайные процессы с независимыми приращениями, Наука, Москва, 1964.

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