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Pseudo-umbilical submanifolds of a Riemannian manifold of constant curvature, II

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§ 0. Introduction.

Let M^n be an n-dimensional manifold immersed in an $(n+p)$-dimensional Riemannian manifold R^{n+p}. Let $\boldsymbol{h}$ be the second fundamental form and $\boldsymbol{H}$ the mean curvature vector of this immersion. If there exists a function λ on M^n such that

$$\langle \boldsymbol{h}(X, Y), \boldsymbol{H}\rangle = \lambda\langle X, Y\rangle \tag{0.1}$$

for all tangent vector fields X, Y on M^n, then M^n is called a pseudo-umbilical submanifold of R^{n+p}.

In this part of this series of papers, firstly, we obtained an integral inequality on mean curvature for flat surfaces in higher dimensional euclidean space and proved that the equality sign holds only when the surfaces are pseudo-umbilical in the euclidean space. Secondly, we proved two characterization theorems for pseudo-umbilical submanifolds in a higher dimensional sphere. Lastly, we obtained a necessary and sufficient condition for a product manifold to be a pseudo-umbilical submanifold.

§ 1. Preliminaries.

Let M^n be an n-dimensional manifold immersed in an $(n+p)$-dimensional Riemannian manifold R^{n+p}. We choose a local field of orthonormal frames $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+p}$ in R^{n+p} such that, restricted to M^n, the vectors $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ are tangent to M^n (and consequently, $\boldsymbol{e}_{n+1}, \cdots, \boldsymbol{e}_{n+p}$ are normal to M^n). We shall make use of the following convention on the ranges of indices:

$$1 \leqq i, j, k, \cdots \leqq n; \qquad n+1 \leqq r, s, t, \cdots \leqq n+p;$$
$$1 \leqq A, B, C, \cdots \leqq n+p$$

unless otherwise stated. With respect to the frame field of R^{n+p} chosen above,

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let $\omega_1, \cdots, \omega_{n+p}$ be the field of dual frames. Then the structure equations of R^{n+p} are given by

(1.1) $$d\omega_A = \sum \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

(1.2) $$d\omega_{AB} = \sum \omega_{AC} \wedge \omega_{CB} + \Phi_{AB}, \quad \Phi_{AB} = (1/2) \sum K_{ABCD} \omega_C \wedge \omega_D,$$

(1.3) $$K_{ABCD} + K_{ABDC} = 0.$$

We restrict these forms to M^n. Then $\omega_r = 0$. Since $0 = d\omega_r = \sum \omega_{ri} \wedge \omega_i$, by Cartan's lemma we may write

(1.4) $$\omega_{ir} = \sum h^r_{ij} \omega_j, \qquad h^r_{ij} = h^r_{ji}.$$

From these formulas, we obtain

(1.5) $$d\omega_i = \sum \omega_{ij} \wedge \omega_j,$$

(1.6) $$d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \qquad \Omega_{ij} = (1/2) \sum R_{ijkl} \omega_k \wedge \omega_l,$$

(1.7) $$R_{ijkl} = K_{ijkl} - \sum (h^r_{ik} h^r_{jl} - h^r_{il} h^r_{jk}),$$

(1.8) $$d\omega_{rt} = \sum \omega_{rs} \wedge \omega_{st} + \Omega_{rt}, \qquad \Omega_{rt} = (1/2) \sum R_{rtkl} \omega_k \wedge \omega_l,$$

(1.9) $$R_{rtkl} = K_{rtkl} - \sum (h^r_{ik} h^t_{il} - h^r_{il} h^t_{ik}).$$

We call $\boldsymbol{h} = \sum h^r_{ij} \omega_i \omega_j \boldsymbol{e}_r$ the second fundamental form and $K_N = \sum (R_{rtkl})^2$ the scalar normal curvature [1]. The mean curvature vector $\boldsymbol{H}$ is given by $(1/n) \sum_r (\sum_i h^r_{ii}) \boldsymbol{e}_r$. If the mean curvature vector $\boldsymbol{H} = 0$ identically, then M^n is called a minimal submanifold.

We take exterior differentiation of (1.4) and define h^r_{ijk} by

(1.10) $$\sum h^r_{ijk} \omega_k = dh^r_{ij} - \sum h^r_{il} \omega_{lj} - \sum h^r_{lj} \omega_{il} + \sum h^t_{ij} \omega_{tr}.$$

Then h^r_{ijk} is the covariant derivative of h^r_{ij} and we have

(1.11) $$h^r_{ijk} - h^r_{ikj} = K_{irkj} = -K_{irjk}.$$

For any unit normal vector $\boldsymbol{e}$ at x in M^n, there corresponds a symmetric transformation $A(\boldsymbol{e})$ of the tangent space T_x at x into itself which is given by $\langle A(\boldsymbol{e})(\boldsymbol{X}), \boldsymbol{Y} \rangle = \langle \boldsymbol{e}, \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{Y}) \rangle$, for all tangent vectors $\boldsymbol{X}, \boldsymbol{Y}$ at x. We call $A(\boldsymbol{e})$ the second fundamental form at $\boldsymbol{e}$.

§2. Flat surfaces in E^{2+p}.

Let M be a surface immersed in a euclidean $(2+p)$-space E^{2+p}, and let $T_x^{\perp}$ denote the normal space of M in E^{2+p} at x. We define a linear mapping γ from $T_x^{\perp}$ into the space of all symmetric matrices of order 2 by

$$\gamma\left(\sum_{r=3}^{2+p} v_r \boldsymbol{e}_r\right) = \sum_{r=3}^{2+p} v_r A(\boldsymbol{e}_r).$$

Let O_x denote the kernel of γ. Then we have $\dim O_x \geqq p-3$. We define the *N-index* of M at x by

$$N\text{-index}_x = p - \dim O_x .$$

In general, for a surface in E^{2+p}, we have N-index $\leqq 3$. A surface in E^{2+p} with N-index $\leqq 2$ everywhere is not necessarily contained in a 4-dimensional linear subspace of E^{2+p}.

The following theorems are the main results of this section.

THEOREM 2.1. *Let M be a compact flat surface immersed in a euclidean $(2+p)$-space E^{2+p}. If the N-index of M is $\leqq 2$ everywhere, then we have*

$$\int_M \langle \boldsymbol{H}, \boldsymbol{H} \rangle dV \geqq 2\pi^2 , \tag{2.1}$$

where dV denotes the area element of M. The equality sign of (2.1) *holds only when M is a pseudo-umbilical surface in E^{2+p} with zero scalar normal curvature.*

PROOF. Let S_x denote the $(p-1)$-sphere of all unit normal vectors of M in E^{2+p} at x. For any unit normal vector $\boldsymbol{e}$ at x, the Lipschitz-Killing curvature, $K(x, \boldsymbol{e})$, is defined as the determinant of the second fundamental form at $\boldsymbol{e}$, that is, $K(x, \boldsymbol{e}) = \det A(\boldsymbol{e})$. Put $U = \{x \in M : N\text{-index}_x \geqq 1\}$. It is easy to see that U is an open subset of M. If $N\text{-index}_x < 1$, then we have $K(x, \boldsymbol{e}) = 0$ for all $\boldsymbol{e}$ in S_x. In the following, we choose the local frame fields in such a way that $\boldsymbol{e}_3$, $\boldsymbol{e}_4$ in N_x if $N\text{-index}_x = 2$ and $\boldsymbol{e}_3$ in N_x if $N\text{-index}_x = 1$, where N_x denotes the subspace of the normal space given by

$$T_x^{\perp} = N_x \oplus O_x , \qquad N_x \perp O_x .$$

Then we have

$$h_{ij}^r = 0 , \qquad \text{for } r > 4 . \tag{2.2}$$

Thus, by (2.2), the Lipschitz-Killing curvature $K(x, \boldsymbol{e})$ with $\boldsymbol{e} = \sum_{r=3}^{2+p} \cos\theta_r \boldsymbol{e}_r$ is given by

$$K(x, \boldsymbol{e}) = \{(\cos\theta_3) h_{11}^3 + (\cos\theta_4) h_{11}^4\}\{(\cos\theta_3) h_{22}^3 + (\cos\theta_4) h_{22}^4\}$$
$$- \{(\cos\theta_3) h_{12}^3 + (\cos\theta_4) h_{12}^4\}^2 .$$

The right hand side is a quadratic form on $\cos\theta_r$. Hence, by choosing a suitable cross-section with $\boldsymbol{e}_3$, $\boldsymbol{e}_4$ in N_x for points x with $N\text{-index}_x = 2$, we may write

$$K(x, \boldsymbol{e}) = \lambda(x) \cos^2\theta_3 + \mu(x) \cos^2\theta_4 , \qquad \lambda = -\mu \geqq 0 . \tag{2.3}$$

Thus by (2.3) we see that the total curvature $K^*(x)$ satisfies

$$K^*(x) \overset{\text{def}}{=} \int_{S_x} |K(x, \boldsymbol{e})| d\sigma \tag{2.4}$$
$$= \lambda \int_{S_x} |\cos^2 \theta_3 - \cos^2 \theta_4| d\sigma ,$$

where $d\sigma$ denotes the volume element of S_x. On the other hand, by a formula of spherical integration, we have

$$\int_{S_x} |\cos^2 \theta_3 - \cos^2 \theta_4| d\sigma = 2c_{p+1}/\pi^2 , \tag{2.5}$$

where c_{p+1} denotes the area of unit $(p+1)$-sphere. Hence by substituting (2.5) into (2.4), we see that

$$\lambda(x) = K^*(x)\pi^2/2c_{p+1} . \tag{2.6}$$

Since M is flat and compact, by a well-known inequality of Chern-Lashof [**3**], we have

$$\int_M K^*(x) dV \geqq 4c_{p+1} . \tag{2.7}$$

Combining (2.6) and (2.7) we obtain

$$\int_M \lambda(x) dV \geqq 2\pi^2 . \tag{2.8}$$

On the other hand, if we choose $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ in the principal directions of $\boldsymbol{e}_4$, then we have $h^4_{12} = 0$. Hence, by the flatness of M, we obtain

$$\begin{aligned} 4\langle \boldsymbol{H}, \boldsymbol{H} \rangle &= (h^3_{11} + h^3_{22})^2 + (h^4_{11} + h^4_{22})^2 \\ &= (h^3_{11})^2 + (h^3_{22})^2 + 2(h^3_{12})^2 + (h^4_{11})^2 + (h^4_{22})^2 \\ &\geqq 4\lambda + 4(h^3_{12})^2 \\ &\geqq 4\lambda . \end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.8) we obtain (2.1). Now, suppose that the inequality of (2.1) is actually an equality, then the inequalities of (2.9) are actually equalities. Hence, we have

$$h^3_{11} = h^3_{22} , \qquad h^4_{11} = -h^4_{22} , \qquad h^3_{12} = h^4_{12} = 0 .$$

This shows that M is a pseudo-umbilical surface of E^{2+p} and the scalar normal curvature vanishes. This completes the proof of the theorem.

COROLLARY 2.2. *Let M be a compact flat surface immersed in a euclidean 4-space. Then we have*

$$\int_M \langle \boldsymbol{H}, \boldsymbol{H} \rangle dV \geqq 2\pi^2 .$$

This corollary follows immediately from Theorem 2.1. If M is orientable, then it has been proved in [2].

THEOREM 2.3. *Let M be a compact flat surface immersed in a euclidean $(2+p)$-space E^{2+p} with zero scalar normal curvature. Then we have*

$$\int_M \langle \boldsymbol{H}, \boldsymbol{H}\rangle dV \geqq 2\pi^2 .$$

PROOF. If the scalar normal curvature K_N vanishes, then the N-index of M is <3 everywhere. This implies the theorem.

§ 3. Pseudo-umbilical submanifolds in space forms.

Let $\boldsymbol{u}$ be a normal vector field on M^n and we choose our local frame fields in such a way that $\boldsymbol{e}=\boldsymbol{e}_{n+1}$ is given by

$$\boldsymbol{u}=|\boldsymbol{u}|\boldsymbol{e}, \qquad \boldsymbol{u}=\langle \boldsymbol{u}, \boldsymbol{u}\rangle^{1/2}, \tag{3.1}$$

where $\langle , \rangle$ denotes the scalar product in R^{n+p}. We define a normal vector field $\boldsymbol{a}(\boldsymbol{u})$ by

$$\boldsymbol{a}(\boldsymbol{u})=\frac{1}{n}|\boldsymbol{u}|\sum_{r=n+2}^{n+p}\mathrm{Tr}\,(A(\boldsymbol{e})A(\boldsymbol{e}_r))\boldsymbol{e}_r . \tag{3.2}$$

Then $\boldsymbol{a}(\boldsymbol{u})$ is a well-defined normal vector at each point and it is continuous on M^n. We call $\boldsymbol{a}(\boldsymbol{u})$ the *allied vector field* of $\boldsymbol{u}$. For example, if M^n is contained in a hypersphere S^{m-1} of a euclidean m-space E^m and $\boldsymbol{u}$ is the unit outer hypersphere normal in E^m along M^n, then the allied vector field $\boldsymbol{a}(\boldsymbol{u})$ of $\boldsymbol{u}$ is nothing but the mean curvature vector of M^n in S^{m-1}.

The allied vector field of the mean curvature vector $\boldsymbol{H}$ is a well-defined normal vector field perpendicular to $\boldsymbol{H}$. We call it the *allied mean curvature vector*. If the allied mean curvature vector $\boldsymbol{a}(\boldsymbol{H})=0$ identically, then M^n is called an $\mathcal{A}$-submanifold of R^{n+p}. It is easy to see that minimal submanifolds, pseudo-umbilical submanifolds and hypersurfaces are $\mathcal{A}$-submanifolds. There are $\mathcal{A}$-submanifolds which are not one of the submanifolds we just mentioned (see § 4).

THEOREM 3.1. *Let M^n be an $\mathcal{A}$-submanifold of a euclidean hypersphere S^{m-1} in E^m. Then M^n is a pseudo-umbilical submanifold of S^{m-1} if and only if M^n is an $\mathcal{A}$-submanifold of E^m.*

PROOF. Let M^n be an $\mathcal{A}$-submanifold of S^{m-1}. Then the allied mean curvature vector $\boldsymbol{a}(\boldsymbol{H})$ vanishes, i. e.

$$\boldsymbol{a}(\boldsymbol{H})=\frac{1}{n}|\boldsymbol{H}|\sum_{r=n+2}^{m-1}\mathrm{Tr}\,(A(\boldsymbol{e}_{n+1})A(\boldsymbol{e}_r))\boldsymbol{e}_r=0 , \tag{3.3}$$

where we have chosen $\boldsymbol{H}=|\boldsymbol{H}|\boldsymbol{e}_{n+1}$. If $\boldsymbol{H}=0$ at a point x in M^n, then M^n is pseudo-umbilical at x in S^{m-1} and the allied mean curvature vector of M^n in

E^m vanishes at x. Therefore, it suffices to show the theorem for the points in M^n where the mean curvature vector $\boldsymbol{H}$ of M^n in S^{m-1} is nonzero. In the latter case, we have

$$\operatorname{Tr}(A(\boldsymbol{e}_{n+1})A(\boldsymbol{e}_r))=0, \qquad \text{for } r=n+2, \cdots, m-1. \tag{3.4}$$

Now, suppose that M^n is an $\mathcal{A}$-submanifold of E^m and $\bar{\boldsymbol{H}}$ is the mean curvature vector of M^n in E^m. Then, without loss of generality, we may assume that S^{m-1} is the unit hypersphere of E^m centered at the origin and $\boldsymbol{X}$ is its position vector field. Then we have $\bar{\boldsymbol{H}}=\boldsymbol{H}-\boldsymbol{X}$. Hence, if we put

$$\bar{\boldsymbol{e}}_{n+1}=\bar{\boldsymbol{H}}/|\bar{\boldsymbol{H}}|, \qquad \bar{\boldsymbol{e}}_{n+2}=\boldsymbol{e}_{n+2}, \cdots, \bar{\boldsymbol{e}}_{m-1}=\boldsymbol{e}_{m-1},$$
$$\bar{\boldsymbol{e}}_m=b(\boldsymbol{H}+\langle\boldsymbol{H},\boldsymbol{H}\rangle\boldsymbol{X}),$$

with $b=(\langle\boldsymbol{H},\boldsymbol{H}\rangle+\langle\boldsymbol{H},\boldsymbol{H}\rangle^2)^{-1/2}$. Then the second fundamental form $(\bar{h}^r_{ij})$ of M^n in E^m are given by

$$\begin{aligned} \bar{h}^{n+1}_{ij}&=\{|\boldsymbol{H}|h^{n+1}_{ij}+\delta_{ij}\}/|\bar{\boldsymbol{H}}|, \\ \bar{h}^r_{ij}&=h^r_{ij}, \qquad r=n+2, \cdots, m-1, \\ \bar{h}^m_{ij}&=b\{|\boldsymbol{H}|h^{n+1}_{ij}-\langle\boldsymbol{H},\boldsymbol{H}\rangle\delta_{ij}\}, \end{aligned} \tag{3.5}$$

where δ_{ij} are the Kronecker deltas. The condition of $\mathcal{A}$-submanifold for M^n in E^m implies

$$\sum \bar{h}^{n+1}_{ij}\bar{h}^r_{ij}=0, \qquad \text{for } r=n+2, \cdots, m. \tag{3.6}$$

From (3.4), (3.5) and (3.6) we obtain

$$n\langle\boldsymbol{H},\boldsymbol{H}\rangle=\sum(h^{n+1}_{ij})^2. \tag{3.7}$$

From this we can easily verify that M^n is a pseudo-umbilical submanifold of S^{m-1}. The converse is trivial. This completes the proof of the theorem.

Let $\boldsymbol{u}$ be a normal vector field of M^n in R^m and $\tilde{\nabla}$ denote the covariant differentiation of R^m. Then we may decompose $\tilde{\nabla}\boldsymbol{u}$ into two components $\nabla\boldsymbol{u}$ and $D\boldsymbol{u}$, where $\nabla\boldsymbol{u}$ is the tangential component and $D\boldsymbol{u}$ the normal component. Then D defines a connection in the normal bundle. If $D\boldsymbol{u}=0$, then $\boldsymbol{u}$ is said to be *parallel.*

THEOREM 3.2. *Let M^n be a compact, non-minimal, $\mathcal{A}$-submanifold of a euclidean $(m-1)$-sphere S^{m-1}. If the mean curvature vector $\boldsymbol{H}$ of M^n in S^{m-1} is parallel, then M^n is pseudo-umbilical in S^{m-1} when and only when the Gauss image of $\boldsymbol{H}/|\boldsymbol{H}|$ lies in an open hemisphere of S^{m-1}.*

PROOF. Since M^n is non-minimal and the mean curvature vector $\boldsymbol{H}$ is parallel, then mean curvature, $|\boldsymbol{H}|$, is a nonzero constant. Let $\boldsymbol{e}=\boldsymbol{e}_{n+1}$ be the unit normal vector field given by $\boldsymbol{H}=|\boldsymbol{H}|\boldsymbol{e}$. Then $\boldsymbol{e}$ is parallel. Without loss of generality, we may assume that S^{m-1} is the unit hypersphere of E^m

centered at the origin with the position vector field X. By (1.4), we may write

$$de = de_{n+1} = \sum \omega_{n+1\, i} e_i = -\sum h_{ij}^{n+1} \omega_j e_i . \tag{3.8}$$

Applying the Hodge star operator * on both sides of (3.8), we obtain

$$*de = \sum (-1)^j h_{ij}^{n+1} \omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge \omega_{j+1} \wedge \cdots \wedge \omega_n e_i . \tag{3.9}$$

Taking exterior differentiation of (3.9) we obtain

$$-d*de = \Big\{ \sum h_{ijj}^{n+1} e_i - \sum h_{ii}^{n+1} X + \sum_{r=n+1}^{m-1} h_{ij}^{n+1} h_{ij}^{r} e_r \Big\} dV . \tag{3.10}$$

Since $e = e_{n+1}$ is parallel and S^{m-1} is of constant curvature 1, we obtain from (1.10) and (1.11) that

$$\sum h_{ijk}^{n+1} \omega_k = \sum h_{ij;k}^{n+1} \omega_k - \sum h_{il}^{n+1} \omega_{jl} - \sum h_{lj}^{n+1} \omega_{il} , \tag{3.11}$$

$$h_{ijk}^{n+1} = h_{ikj}^{n+1} , \tag{3.12}$$

where $h_{ij;k}^{n+1}$ are given by $dh_{ij}^{n+1} = \sum h_{ij;k}^{n+1} \omega_k$. Therefore, by the assumption that M^n is an $\mathcal{A}$-submanifold, and $H = |H| e$, we see that the Laplacian Δe of e is given by

$$\Delta e = X \operatorname{Tr} A(e) - e \operatorname{Tr} ((A(e))^2) . \tag{3.13}$$

Similarly, by a direct computation, we have

$$\Delta X = e \operatorname{Tr} A(e) - nX . \tag{3.14}$$

Combining (3.13) and (3.14) we obtain

$$\begin{aligned} \Delta(ne + X \operatorname{Tr} A(e)) &= -\{ n \operatorname{Tr} ((A(e))^2) - (\operatorname{Tr} A(e))^2 \} e \\ &= -\sum_{i<j} (k_i(e) - k_j(e))^2 e , \end{aligned} \tag{3.15}$$

where $k_1(e), \cdots, k_n(e)$ are the eigenvalues of $A(e)$. Therefore, if the Gauss image of e lies in an open hemisphere of S^{m-1}, then there exists a constant vector c such that $\langle e, c \rangle > 0$. Hence, by taking scalar product of c with both sides of (3.15), we obtain

$$\Delta \langle ne + X \operatorname{Tr} A(e), c \rangle \leqq 0 .$$

Therefore, Hopf's lemma implies that $k_1(e) = \cdots = k_n(e)$. This shows that M^n is a pseudo-umbilical submanifold of S^{m-1}. Conversely, if M^n is pseudo-umbilical in S^{m-1}, then by the parallelism of the mean curvature vector H, we see that M^n is a minimal submanifold of a small $(m-2)$-sphere of S^{m-1}. This implies that the Gauss image of $e = H/|H|$ lies in an open hemisphere of S^{m-1}. This completes the proof of the theorem.

REMARK 3.1. For the hypersurfaces of an m-sphere, see de Giorgi [4] and Nomizu-Smyth [5].

§ 4. Product submanifolds.

Let M^{n_i} $(i=1, 2)$ be n_i-dimensional submanifolds of m_i-dimensional Riemannian manifolds R^{m_i} with nowhere zero mean curvature vector $\boldsymbol{H}_i$. The main purpose of this section is to derive a necessary and sufficient condition for the product manifold $M^{n_1}\times M^{n_2}$ to be a pseudo-umbilical submanifold of $R^{m_1}\times R^{m_2}$. In fact we have the following more general result:

PROPOSITION 4.1. *The product manifold $M^{n_1}\times M^{n_2}$ is an $\mathcal{A}$-submanifold of $R^{m_1}\times R^{m_2}$ when and only when M^{n_1} and M^{n_2} are $\mathcal{A}$-submanifolds of R^{m_1} and R^{m_2} respectively, and the second fundamental forms at $\boldsymbol{\eta}_i=\boldsymbol{H}_i/|\boldsymbol{H}_i|$ of M^{n_i} in R^{m_i} satisfy* $\operatorname{Tr}((A(\boldsymbol{\eta}_1))^2)=\operatorname{Tr}((A(\boldsymbol{\eta}_2))^2)$.

PROOF. We choose the local frame fields $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{m_1+m_2}$ in $R^{m_1}\times R^{m_2}$ such that, restricted to $M^{n_1}\times M^{n_2}$, $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n_1}$ are tangent to M^{n_1}, $\boldsymbol{e}_{n_1+1}, \cdots, \boldsymbol{e}_{n_1+n_2}$ are tangent to M^{n_2}, $\boldsymbol{e}_{n_1+n_2+1}, \cdots, \boldsymbol{e}_{m_1+n_2}$ are normal to M^{n_1} in R^{m_1}, and $\boldsymbol{e}_{m_1+n_2+1}, \cdots, \boldsymbol{e}_{m_1+m_2}$ are normal to M^{n_2} in R^{m_2}. Moreover, we assume that $\boldsymbol{e}_{n_1+n_2+1}=\boldsymbol{\eta}_1$ and $\boldsymbol{e}_{m_1+n_2+1}=\boldsymbol{\eta}_2$. Then, by a straightforward computation, we see that the mean curvature vector $\bar{\boldsymbol{H}}$ of the product manifold $M^{n_1}\times M^{n_2}$ is given by

$$\bar{\boldsymbol{H}}=(1/n)(n_1\boldsymbol{H}_1+n_2\boldsymbol{H}_2), \qquad n=n_1+n_2. \tag{4.1}$$

In the following, we put

$$\begin{cases} \bar{\boldsymbol{e}}_i=\boldsymbol{e}_i, \qquad i=1, \cdots, n, \\ \bar{\boldsymbol{e}}_{n+1}=\bar{\boldsymbol{H}}/|\bar{\boldsymbol{H}}|, \\ \bar{\boldsymbol{e}}_{m_1+n_2+1}=\boldsymbol{Y}/|\boldsymbol{Y}|, \qquad \boldsymbol{Y}=n_2\langle\boldsymbol{H}_2, \boldsymbol{H}_2\rangle\boldsymbol{H}_1-n_1\langle\boldsymbol{H}_1, \boldsymbol{H}_1\rangle\boldsymbol{H}_2, \\ \bar{\boldsymbol{e}}_r=\boldsymbol{e}_r, \qquad r\neq n+1, \quad m_1+n_2+1, \quad r>n. \end{cases} \tag{4.2}$$

Then the second fundamental forms, $A(\boldsymbol{e}_r)$; $r=n+1, \cdots, m_1+m_2$, of the product manifold are given by

$$\begin{aligned} & A(\bar{\boldsymbol{e}}_{n+1})=c_1\begin{pmatrix} n_1|\boldsymbol{H}_1|A(\boldsymbol{e}_{n+1}) & 0 \\ 0 & n_2|\boldsymbol{H}_2|A(\boldsymbol{e}_{m_1+n_2+1})\end{pmatrix} \\ & A(\bar{\boldsymbol{e}}_r)=\begin{pmatrix} A(\boldsymbol{e}_r) & 0 \\ 0 & 0\end{pmatrix}, \qquad r=n+2, \cdots, m_1+n_2, \\ & A(\bar{\boldsymbol{e}}_{m_1+n_2+1})=c_2\begin{pmatrix} n_2\langle\boldsymbol{H}_2, \boldsymbol{H}_2\rangle|\boldsymbol{H}_1|A(\boldsymbol{e}_{n+1}) & 0 \\ 0 & -n_1\langle\boldsymbol{H}_1, \boldsymbol{H}_1\rangle|\boldsymbol{H}_2|A(\boldsymbol{e}_{m_1+n_2+1})\end{pmatrix} \\ & A(\bar{\boldsymbol{e}}_t)=\begin{pmatrix} 0 & 0 \\ 0 & A(\boldsymbol{e}_t)\end{pmatrix}, \qquad t=m_1+n_2+2, \cdots, m_1+m_2, \end{aligned} \tag{4.3}$$

where c_1, and c_2 are nonzero, $A(\boldsymbol{e}_r)$; $r=n+1, \cdots, m_1+n_2$ are second fundamental forms at e_r for M^{n_1} in R^{m_1} and $A(e_t)$, $t=m_1+n_2+1, \cdots, m_1+m_2$, are the corresponding matrices for M^{n_2} in R^{m_2}. From these formulas we see that the product manifold is an $\mathcal{A}$-submanifold when and only when

$$\mathrm{Tr}\,((A(e_{n+1}))^2)=\mathrm{Tr}\,((A(e_{m_1+n_2+1}))^2),$$

$$\mathrm{Tr}\,(A(e_{n+1})A(e_r))=0, \qquad r=n+2, \cdots, m_1+n_2,$$

$$\mathrm{Tr}\,(A(e_{m_1+n_2+1})(A(e_t)))=0, \qquad t=m_1+n_2+2, \cdots, m_1+m_2.$$

These formulas show that the product submanifold is an $\mathcal{A}$-submanifold when and only when M^{n_1} and M^{n_2} are $\mathcal{A}$-submanifolds of R^{m_1} and R^{m_2} respectively and $\mathrm{Tr}\,((A(\eta_1))^2)=\mathrm{Tr}\,((A(\eta_2))^2)$. This proves the proposition.

THEOREM 4.2. *The product manifold $M^{n_1}\times M^{n_2}$ is a pseudo-umbilical submanifold of $R^{m_1}\times R^{m_2}$ when and only when M^{n_1} and M^{n_2} are pseudo-umbilical submanifolds of R^{m_1} and R^{m_2} respectively and $n_1\langle \boldsymbol{H}_1, \boldsymbol{H}_1\rangle=n_2\langle \boldsymbol{H}_2, \boldsymbol{H}_2\rangle$.*

This theorem follows immediately from (4.3).

From Proposition 4.1, we may construct some non-trivial examples of $\mathcal{A}$-submanifolds. For examples, we have

EXAMPLE 4.1. Let M^4 be a 4-dimensional submanifold of the euclidean 7-sphere $S^7(\sqrt{2})$ with radius $\sqrt{2}$ given by

$(a\cos u,\ a\sin u,\ b\cos v,\ b\sin v,\ c\cos w,\ c\sin w,\ d\cos y,\ d\sin y)$,

$$a^2+b^2=c^2+d^2=1, \qquad \left(\frac{a}{b}\right)^2+\left(\frac{b}{a}\right)^2=\left(\frac{c}{d}\right)^2+\left(\frac{d}{c}\right)^2.$$

Then M^4 is an $\mathcal{A}$-submanifold of $S^7(\sqrt{2})$ such that the mean curvature vector is parallel in the normal bundle. It is easy to verify that M^4 is not a pseudo-umbilical submanifold of $S^7(\sqrt{2})$. This example is interesting in view of Theorem 3.2.

EXAMPLE 4.2. Let M^n be a hypersurface of a Riemannian manifold such that 1) the mean curvature vector is nowhere zero and 2) the length of the second fundamental form is constant. Then $M^n\times M^n$ is an $\mathcal{A}$-submanifold.

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