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Fractional powers of operators, III
Negative powers

By Hikosaburo KOMATSU

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This is a continuation of the author's work on fractional powers of operators A in a Banach space X whose resolvent $(\lambda+A)^{-1}$ exists for $\lambda>0$ and satisfies $\|\lambda(\lambda+A)^{-1}\| \leqq M<\infty$, $0<\lambda<\infty$. This part deals with fractional powers A^{α} of exponent α with negative real part, and their relationship with interpolation spaces of X and the range $R(A^{m})$. A unified discussion of mean ergodic theorems is also given, which may be regarded as the theory of A^{-0}.

We use the same notations as in [2] and [3] throughout this paper. In particular, A stands for a closed linear operator in a Banach space X such that $(0, \infty)$ is contained in the resolvent set of $-A$ and that

$$\|\lambda(\lambda+A)^{-1}\| \leqq M, \qquad 0<\lambda<\infty .$$

Such an operator A will be called *non-negative.* The negatives of the infinitesimal generators of bounded continuous semi-groups are non-negative.

When we discussed the basic properties of fractional powers A^{α}, $\alpha \in \boldsymbol{C}$, of non-negative operators A in [2], the following generalization of abelian ergodic theorem played an important role:

$(\lambda+A)^{-1}x$ has the asymptotic expansion

$$(\lambda+A)^{-1}x=\lambda^{-1}x_0-\lambda^{-2}x_1+\cdots+(-1)^{n}\lambda^{-n-1}x_n+o(\lambda^{-n-1}) \quad \text{as } \lambda\to\infty$$

$$(=\lambda^{-1}x_h+x_{-1}-\lambda x_{-2}+\cdots+(-1)^{n-1}\lambda^{n-1}x_{-n}+o(\lambda^{n-1}) \quad \text{as } \lambda\to 0)$$

if and only if $x \in D(A^{n})$ and $A^{n}x \in \overline{D(A)}$ ($x=x_h+A^{n}x_{-n}$ with $x_h \in N(A)$ and $x_{-n} \in D(A^{n}) \cap \overline{R(A)}$, respectively).

Also important were the subspaces D^{σ} (and R^{σ}) of X composed of elements x for which the remainder in the above expansion has the order $O(\lambda^{-\sigma-1})$ ($O(\lambda^{\sigma-1})$ and $x_h=0$, respectively).

In view of the above theorem, D^{σ} seems to give an interpolation space. Actually we proved in [3] that D^{σ} coincides with the mean interpolation space $S(\infty, \sigma/m, X; \infty, \sigma/m-1, D(A^{m}))$ of Lions-Peetre [6] if σ is not an integer and m is an integer greater then $\sigma>0$. We also obtained a related characterization of elements x in the interpolation space $D_p^{\sigma}=S(p, \sigma/m, X; p, \sigma/m-1, D(A^{m}))$ for $1 \leqq p \leqq \infty$ or $p=\infty-$ in terms of $(\lambda+A)^{-m}x$. ($(\lambda+A)^{-m}x$ is more convenient to include the case where σ is an integer.) When $-A$ generates a bounded continuous or analytic semi-group T_t, we had another characterization in terms of $(1-T_t)^m x$ or $A^m T_t x$. (Note that the equivalence of these characterizations has applications to the theory of approximation. See [2] and [9].)

The main purpose of this paper is to give analogous results for the mean interpolation spaces of X and $R(A^m)$ and discuss fractional powers A^α in the case where $\mathrm{Re}\,\alpha<0$. Techniques are similar to the ones used in [3].

In § 1 we prove various mean ergodic theorems, i.e., the abelian ergodic theorem for non-negative operators, the Cesàro ergodic theorem for bounded continuous semi-groups, and the simple ergodic theorem for bounded analytic semi-groups.

In § 2 we introduce the spaces $R_p^\sigma(A)$ and investigate their fundamental properties. $R_p^\sigma(A)$, $\sigma>0$, $1\leqq p\leqq\infty$, is defined to be the space of all $x\in X$ such that $\lambda^{-\sigma+m}(\lambda+A)^{-m}x\in L^p(X)$ with integer $m>\sigma$, where $L^p(X)$ is the L^p space of X-valued functions relative to the measure $d\lambda/\lambda$.

Fractional powers A_-^α for $\mathrm{Re}\,\alpha<0$ is defined in § 3 for non-negative operators A and their integral representation is given.

It is shown in § 4 that $R_p^\sigma(A)$ coincides with the interpolation space $S(p, \sigma/m, X; p, \sigma/m-1, R(A^m))$ of X and the range $R(A^m)$. Further, we discuss the relationship among the spaces R_p^σ, the domain $D(A_-^\alpha)$ and the range $R(A_+^{-\alpha})$.

The cases in which $-A$ generates a bounded continuous semi-group and a bounded analytic semi-group are treated in §§ 5 and 6 respectively. We introduce spaces Q_p^σ and P_p^σ respectively and then show their coincidence with R_p^σ. Another integral representation of negative powers is given in each of the cases.

Some of the results of this paper have been obtained by K. Masuda [7] independently. He gives also interesting applications.

§ 1. Mean ergodic theorems.

THEOREM 1.1. *Let A be a non-negative operator. Then $N(A)+\overline{R(A)}$ is a closed subspace of X and decomposed as the direct sum of Banach spaces $N(A)$ and $\overline{R(A)}$. If $x=x_0+x_1$ with $x_0\in N(A)$ and $x_1\in\overline{R(A)}$, then $(\lambda(\lambda+A^{-1})^m x$ converges strongly to x_0 as $\lambda\to 0$ for any positive integer m. Conversely, if $(\lambda_j(\lambda_j+A)^{-1})^m x$ converges weakly for a positive integer m and for a sequence $\lambda_j\to 0$, then x belongs to $N(A)+\overline{R(A)}$.*

If X is reflexive, the subspace $N(A)+\overline{R(A)}$ coincides with X.

PROOF. If $x_0\in N(A)$, clearly $(\lambda(\lambda+A)^{-1})^m x_0=x_0$ for any λ and m. Next let $x_1=Ay\in R(A)$. Then $(\lambda(\lambda+A)^{-1})^m x_1=\lambda(A(\lambda+A)^{-1})(\lambda(\lambda+A)^{-1})^{m-1}y$ converges strongly to zero as $\lambda\to 0$. Since $(\lambda(\lambda+A)^{-1})^m$ is uniformly bounded, the Banach-

Steinhaus theorem shows that $(\lambda(\lambda+A)^{-1})^m x_1$ converges to zero also for $x_1 \in \overline{R(A)}$. Consequently $N(A) \cap \overline{R(A)} = \{0\}$, and if $x = x_0 + x_1$ is in $N(A) + \overline{R(A)}$, $(\lambda(\lambda+A)^{-1})^m x$ tends to x_0 as $\lambda \to 0$.

Suppose that $(\lambda_j(\lambda_j+A)^{-1})^m x$ converges weakly to x_0 as $\lambda_j \to 0$. Then x_0 belongs to $N(A)$. In fact, $(\lambda_j(\lambda_j+A)^{-1})^m x$ is in $D(A)$, and we have $A(\lambda_j(\lambda_j+A)^{-1})^m x = \lambda_j(A(\lambda_j+A)^{-1})(\lambda_j(\lambda_j+A)^{-1})^{m-1} x \to 0$. Since A is closed, it follows that $x_0 \in N(A)$.

Let $x_1 = x - x_0$. Then $(\lambda_j(\lambda_j+A)^{-1})^m x_1$ converges weakly to zero. Since $x_1 = (\lambda_j(\lambda_j+A)^{-1} + A(\lambda_j+A)^{-1})^m x_1 = (\lambda_j(\lambda_j+A)^{-1})^m x_1 + Ay_j$ for some $y_j \in D(A)$, x_1 belongs to $\overline{R(A)}$.

Thus, $(\lambda(\lambda+A)^{-1})^m x$ converges strongly if and only if $x \in N(A) + \overline{R(A)}$. The Banach-Steinhaus argument proves that $N(A) + \overline{R(A)}$ is a closed subspace.

If X is reflexive, we can choose a weakly convergent sequence $(\lambda_j(\lambda_j+A)^{-1})^m x$ for every $x \in X$. This completes the proof.

When $-A$ is the infinitesimal generator of a bounded continuous semi-group $T_t = \exp(-tA)$, we have

$$(\lambda(\lambda+A)^{-1})^m x = (m-1)!^{-1} \int_0^\infty \lambda^m t^{m-1} e^{-\lambda t} T_t x \, dt .$$

Therefore, Theorem 1.1 turns out to be a generalization of the abelian ergodic theorem for bounded continuous semi-groups [1].

To formulate the standard mean ergodic theorem, we introduce the following notations:

$$I_t x = \int_0^t T_s x \, ds , \tag{1.1}$$

$$I_t^{(\sigma)} x = \Gamma(\sigma)^{-1} \int_0^t (t-s)^{\sigma-1} T_s x \, ds \qquad \text{for } \sigma > 0 . \tag{1.2}$$

LEMMA 1.2. *If $\sigma \geqq 1$ and $t > 0$, then $I_t^{(\sigma)} x$ is contained in $D(A)$ for each $x \in X$ and we have*

$$x = T_t x + A I_t x , \qquad \sigma = 1 \tag{1.3}$$

and

$$\Gamma(\sigma)^{-1} t^{\sigma-1} x = I_t^{(\sigma-1)} x + A I_t^{(\sigma)} x , \qquad \sigma > 1 . \tag{1.4}$$

PROOF. (1.3) is well-known. See [1] or [5] for a proof. The latter gives a proof under the most general conditions on T_t and X. Multiplying both sides of (1.3) by $\Gamma(\sigma-1)^{-1}(s-t)^{\sigma-2}$ and integrating them over $(0, s)$, we have (1.3) with t replaced by s. Since A is closed, A commutes with the integration.

LEMMA 1.3. *Let $0 < \sigma < 1$ and $t > 0$. Then $I_t^{(\sigma)} x$ belongs to $C_\infty^\sigma(A)$ for each x and*

$$p_\sigma(I_t^{(\sigma)}x) = \sup_h h^{-\sigma}\| (I-T_h)I_t^{(\sigma)}x \|$$
$$\leqq 2\Gamma(\sigma+1)^{-1}M\| x \| .$$

PROOF. Suppose $h < t$. Then we have

$$\| (I-T_h)I_t^{(\sigma)}x \| = \Gamma(\sigma)^{-1}\left\| \int_0^t (t-s)^{\sigma-1}(T_s - T_{s+h})x\, ds \right\|$$
$$\leqq \Gamma(\sigma)^{-1}M\| x \| \Big(\int_0^h (t-s)^{\sigma-1} d\sigma$$
$$+ \int_h^t ((t-s)^{\sigma-1} - (t+h-s)^{\sigma-1})ds + \int_t^{t+h}(t+h-s)^{\sigma-1}ds \Big)$$
$$= 2\Gamma(\sigma+1)^{-1}M\| x \| h^\sigma .$$

When $h \geqq t$, we have similarly

$$\| (I-T_h)I_t^{(\sigma)}x \| \leqq 2\Gamma(\sigma+1)^{-1}M\| x \| t^\sigma .$$

THEOREM 1.4. *Let $-A$ be the infinitesimal generator of a bounded continuous semi-group T_t and let m be a positive integer. If $x = x_0 + x_1 \in N(A) + \overline{R(A)}$, then $(\Gamma(\sigma+1)t^{-\sigma}I_t^{(\sigma)})^m x$ converges strongly to x_0 as $t \to \infty$. Conversely, if there is a sequence $t_j \to \infty$ such that $(\Gamma(\sigma+1)t_j^{-\sigma}I_{t_j}^{(\sigma)})^m x$ converges weakly, then x belongs to $N(A) + \overline{R(A)}$.*

PROOF. First let $\sigma \geqq 1$. Then $t^{-\sigma}I_t^{(\sigma)}$ and $t^{-\sigma+1}AI_t^{(\sigma)}$ are both uniformly bounded by the definition of $I_t^{(\sigma)}$ and Lemma 1.2. Thus the same argument as in the proof of Theorem 1.1 works.

Next let $0 < \sigma < 1$. It follows from Lemma 1.3 above and Propositions 4.7 and 11.2 of [2] that $I_t^{(\sigma)}x \in D(A_+^{\sigma/2})$ for each $x \in X$ and there exists a constant C such that

$$\| A_+^{\sigma/2}I_t^{(\sigma)}x \| \leqq Ct^{\sigma/2}\| x \| , \qquad t > 0 .$$

Therefore, if $x_1 = Ay \in R(A)$, we have

$$(t^{-\sigma}I_t^{(\sigma)})^m x_1 = (t^{-\sigma}I_t^{(\sigma)})^{m-1}t^{-\sigma}A_+^{\sigma/2}I_t^{(\sigma)}A_+^{1-\sigma/2}y \to 0$$

by the additivity of fractional powers. Since $(t^{-\sigma}I_t^{(\sigma)})^m$ is uniformly bounded, $(t^{-\sigma}I_t^{(\sigma)})^m x_1$ converges strongly to zero for each $x_1 \in \overline{R(A)}$. It is clear that $(\Gamma(\sigma+1)t^{-\sigma}I_t^{(\sigma)})^m x_0 = x_0$ for $x_0 \in N(A)$.

Conversely suppose that $(\Gamma(\sigma+1)t_j^{-\sigma}I_{t_j}^{(\sigma)})^m x$ converges weakly to x_0 as a sequence $t_j \to \infty$. Then we have $A_+^{\sigma/2}(t_j^{-\sigma}I_{t_j}^{(\sigma)})^m x = t_j^{-\sigma/2}(t_j^{-\sigma}I_{t_j}^{(\sigma)})^{m-1}t_j^{-\sigma/2}A_+^{\sigma/2}I_{t_j}^{(\sigma)}x \to 0$. This implies $x_0 \in N(A_+^{\sigma/2})$. Since we have $N(A_+^\alpha) = N(A)$ for $\operatorname{Re}\alpha > 0$ from Theorems 7.1 and 8.1 of [2], x_0 belongs to $N(A)$. To prove that $x_1 = x - x_0$ is in $\overline{R(A)}$, it suffices to note that

$$x_1 = (\Gamma(\sigma+1)t_j^{-\sigma}I_{t_j}^{(\sigma)} + \Gamma(\sigma+1)t_j^{-\sigma}AI_{t_j}^{(\sigma+1)})^m x_1 .$$

This completes the proof.

When $\sigma=m=1$, Theorem 1.4 is exactly the mean ergodic theorem with continuous parameter. Since σ is an arbitrary positive number, we can replace the arithmetic mean by the Cesàro mean of arbitrary order >-1. If T_t is a bounded analytic semi-group, we have a stronger result.

THEOREM 1.5. *Suppose that* $-A$ *generates a bounded analytic semi-group* T_t. *If* $x=x_0+x_1$ *belongs to* $N(A)+\overline{R(A)}$, *then* $T_t x$ *converges strongly to* x_0 *as* $t\to\infty$. *Conversely, if there is a sequence* $t_j\to\infty$ *such that* $T_{t_j}x$ *converges weakly, then* x *belongs to* $N(A)+\overline{R(A)}$.

PROOF. Since tAT_t is uniformly bounded for $t>0$, it follows that $T_t x_1\to 0$ for $x_1\in R(A)$ and hence for $x_1\in\overline{R(A)}$. Clearly we have $T_t x_0=x_0$ for $x_0\in N(A)$.

Conversely suppose that $T_{t_j}x$ converges to x_0 weakly. Since $AT_{t_j}x\to 0$, we have $x_0\in N(A)$. Let $x_1=x-x_0$. Then $T_{t_j}x_1=x_1-AI_{t_j}x_1$ converges weakly to zero. Hence we have $x_1\in\overline{R(A)}$, completing the proof.

An analogous theorem has been given in [4] when the parameter is discrete.

This theorem shows, in particular, that if A is an operator of type $(\omega, M(\theta))$ with $\omega<\pi/2$ in a reflexive Banach space X, then every solution $x(t)$ of the parabolic equation

$$\frac{d}{dt}x(t)=-Ax(t) \tag{1.5}$$

converges strongly to an equilibrium state x_0 which satisfies $Ax_0=0$ as t tends to infinity, for, every weak solution $x(t)$ of (1.5) is written $x(t)=T_{t-t_0}x(t_0)$.

§ 2. Spaces R_p^σ.

As in [3] we denote by $L^p(X)$ the Banach space of all X-valued measurable functions $f(\lambda)$ on $(0,\infty)$ such that

$$\|f\|_{L^p(X)}=\left(\int_0^\infty\|f(\lambda)\|_X^p d\lambda/\lambda\right)^{1/p}<\infty\,, \qquad \text{if } 1\leqq p<\infty\,,$$

$$\|f\|_{L^p(X)}=\sup_{0<\lambda<\infty}\|f(\lambda)\|_X<\infty\,, \qquad \text{if } p=\infty \text{ or } \infty-\,,$$

and

$$f(\lambda)\to 0 \quad\text{as}\quad \lambda\to 0 \quad\text{or}\quad \infty\,, \qquad \text{if } p=\infty-\,.$$

DEFINITION 2.1. Let σ be a positive number, let m be an integer with $0<\sigma<m$, and let $1\leqq p\leqq\infty$. Then, $R_{p,m}^\sigma=R_{p,m}^\sigma(A)$ denotes the space of all $x\in X$ such that $\lambda^{-\sigma}(\lambda(\lambda+A)^{-1})^m x\in L^p(X)$.

It is easy to see that $R_{p,m}^\sigma$ is a Banach space with the norm

$$\|x\|_{R_{p,m}^\sigma}=\|x\|_X+\|\lambda^{-\sigma}(\lambda(\lambda+A)^{-1})^m x\|_{L^p(X)}\,. \tag{2.1}$$

$R^{\sigma}_{\infty,1}$ coincides with R^{σ} in [2], though topologies are different.

As in [2] we denote by A_R the restriction of A to $D(A_R)=D(A)\cap\overline{R(A)}$. Usually we regard A_R as an operator in the Banach space $\overline{R(A)}$. Then A_R is one-to-one and has dense range in $\overline{R(A)}$ by the ergodic Theorem 1.1.

THEOREM 2.2. *Both A_R and A_R^{-1} are non-negative in $\overline{R(A)}$, and we have*

$$R^{\sigma}_{p,m}(A)=R^{\sigma}_{p,m}(A_R)=D^{\sigma}_{p,m}(A_R^{-1}) \tag{2.2}$$

with the same norms.

PROOF. This is essentially the same as Theorem 3.2 of [2]. We note that for every $x\in R^{\sigma}_{p,m}$ there is a sequence $\lambda_j\to 0$ such that $(\lambda_j(\lambda_j+A)^{-1})^m x\to 0$ and hence x belongs to $\overline{R(A)}$ by Theorem 1.1. Also we make use of the following identities:

$$(\lambda+A_R)^{-1}=(\lambda+A)^{-1}|_{\overline{R(A)}}\,, \tag{2.3}$$

$$\lambda(\lambda+A_R)^{-1}=A_R^{-1}(\lambda^{-1}+A_R^{-1})^{-1}\,, \qquad 0<\lambda<\infty\,. \tag{2.4}$$

Consequently the following two propositions and a theorem follow directly from the results in § 1 of [3].

PROPOSITION 2.3. *If integers m and n are greater than σ, then $R^{\sigma}_{p,m}$ and $R^{\sigma}_{p,n}$ are the same spaces with equivalent norms.*

DEFINITION 2.4. $R^{\sigma}_p=R^{\sigma}_p(A)$ is defined to be the Banach space $R^{\sigma}_{p,m}$ with the least integer m greater than σ.

PROPOSITION 2.5. *If $\mu>0$, $A(\mu+A)^{-1}$ is a one-to-one continuous mapping of R^{σ}_p onto $R^{\sigma+1}_p$. If $p\leqq\infty-$, we have for every $x\in R^{\sigma}_p$.*

$$A(\mu+A)^{-1}x\to x \quad (R^{\sigma}_p) \quad as \quad \mu\to 0\,. \tag{2.5}$$

THEOREM 2.6. *We have $R^{\sigma}_p\subset R^{\tau}_p$ if $\sigma>\tau$ or if $\sigma=\tau$ and $p\leqq q$. The injection is continuous, and when $q\leqq\infty-$, R^{σ}_p is dense in R^{τ}_p.*

PROPOSITION 2.7. *If $\mu>0$, then $A(\mu+A)^{-1}$ is non-negative and*

$$R^{\sigma}_p(A)=R^{\sigma}_p(A(\mu+A)^{-1})\,. \tag{2.6}$$

PROOF. It is proved in Proposition 6.2 of [2] that $A(\mu+A)^{-1}$ is a non-negative operator. Note that

$$(A(\mu+A)^{-1})_R=A_R(\mu+A_R)^{-1} \tag{2.7}$$

and that (2.4) implies

$$A_R(\mu+A_R)^{-1}=\mu^{-1}(\mu^{-1}+A_R^{-1})^{-1}\,. \tag{2.8}$$

Thus we have by Proposition 2.5 of [3]

$$\begin{aligned} R^{\sigma}_p(A)&=D^{\sigma}_p(A_R^{-1})=D^{\sigma}_p(\mu^{-1}+A_R^{-1})\\ &=R^{\sigma}_p(A_R(\mu+A_R)^{-1})=R^{\sigma}_p(A(\mu+A)^{-1})\,. \end{aligned}$$

§ 3. Negative powers.

Suppose that a complex number α, a real number σ and an integer m satisfy $0>\operatorname{Re}\alpha\geqq-\sigma>-m$. If $x\in R_1^\sigma$, then the integral

$$(3.1)\qquad A_{-\sigma}^{\alpha}x=\frac{\Gamma(m)}{\Gamma(-\alpha)\Gamma(m+\alpha)}\int_0^\infty\lambda^{\alpha-1}(\lambda(\lambda+A)^{-1})^m x\,d\lambda$$

converges absolutely. Since $x\in\overline{R(A)}$, $(\lambda+A)^{-1}$ in the integrand may be replaced by $(\lambda+A_R)^{-1}$. Thus it follows from (2.4) and the definition of positive powers in [3] that

$$(3.2)\qquad A_{-\sigma}^{\alpha}x=(A_R^{-1})_{\sigma}^{-\alpha}x\,,\qquad x\in R_1^\sigma(A)=D_1^\sigma(A_R^{-1})\,.$$

In particular, $A_{-\sigma}^{\alpha}x$ does not depend on m or on σ as far as $x\in R_1^\sigma$, and the operator $A_{-\sigma}^{\alpha}$ with the domain R_1^σ has the smallest closed extension which does not depend on $\sigma\geqq-\operatorname{Re}\alpha$.

DEFINITION 3.1. The fractional power A_-^α for $\operatorname{Re}\alpha<0$ is defined to be the smallest closed extension of $A_{-\sigma}^{\alpha}$ for a $\sigma\geqq-\operatorname{Re}\alpha$.

THEOREM 3.2. *If* $\operatorname{Re}\alpha<0$, *we have*

$$(3.3)\qquad A_-^\alpha=(A_R)_-^\alpha=(A_R^{-1})_+^{-\alpha}\,,$$

and if α *is a negative integer* $-m$,

$$(3.4)\qquad A_-^\alpha=A_R^{-m}\,.$$

PROOF. The first identity of (3.3) is clear from the definition and Theorem 2.2. The second identity follows from (3.2). Since A_R^{-1} has dense domain in $\overline{R(A)}$, Proposition 2.2 of [3] implies (3.4).

Proposition 4.10 of [2] shows therefore that Definition 3.1 is consistent with the definition of A_-^α in [2].

The following theorem shows that the negative power A_-^α is equal to the integral (3.1) interpreted as an improper integral.

THEOREM 3.3. *Let* $0>\operatorname{Re}\alpha>-m$ *with an integer* m. *If there is a sequence* $\varepsilon_j\to0$ *such that*

$$(3.5)\qquad y=\operatorname*{w-lim}_{j\to\infty}\frac{\Gamma(m)}{\Gamma(-\alpha)\Gamma(m+\alpha)}\int_{\varepsilon_j}^\infty\lambda^{\alpha-1}(\lambda(\lambda+A)^{-1})^m x\,d\lambda$$

exists, then $x\in D(A_-^\alpha)$ *and* $y=A_-^\alpha x$.

If $x\in D(A_-^\alpha)$, *then*

$$(3.6)\qquad A_-^\alpha x=\operatorname*{s-lim}_{\varepsilon\to0}\frac{\Gamma(m)}{\Gamma(-\alpha)\Gamma(m+\alpha)}\int_{\varepsilon}^\infty\lambda^{\alpha-1}(\lambda(\lambda+A)^{-1})^m x\,d\lambda$$

possibly except for the case in which $\operatorname{Im}\alpha\neq0$ *and* $\operatorname{Re}\alpha$ *is an integer.*

PROOF. Suppose that weak limit (3.5) exists. Then

$$-\alpha\varepsilon_j^{-\alpha}\int_{\varepsilon_j}^{\infty}\lambda^{\alpha-1}(\lambda(\lambda+A)^{-1})^m x d\lambda$$

$$=-\alpha\varepsilon_j^{-\alpha}\int_{\varepsilon_j}^{\infty}\lambda^{\alpha-1}(1-A(\lambda+A)^{-1})^m x d\lambda$$

$$=x-y_j\to 0\,,$$

where $y_j\in\overline{R(A)}$. Hence x belongs to $\overline{R(A)}$. Thus employing (2.3) and (2.4), and changing variable by $\mu=\lambda^{-1}$, we obtain

$$y=\text{w-lim}\frac{\Gamma(m)}{\Gamma(-\alpha)\Gamma(m+\alpha)}\int_0^{\varepsilon_j^{-1}}\mu^{-\alpha-1}(A_R^{-1}(\mu+A_R^{-1})^{-1})^m x d\mu\,.$$

Thus Theorem 2.10 of [3] proves that x is in $D((A_R^{-1})_+^{-\alpha})=D(A_-^{\alpha})$ and that $y=A_-^{\alpha}x$.

The converse is proved in the same way.

If $x\in R_p^{\sigma}$, we denote by $q_p^{-\sigma}(x)$ the second term of (2.1) with m the least integer greater than σ.

PROPOSITION 3.4. *If* $0>\operatorname{Re}\alpha>-\sigma$, *there is a constant* $C(\alpha,-\sigma,p)$ *such that*

$$\| A_-^{\alpha}x\|\leqq C(\alpha,-\sigma,p)q_p^{-\sigma}(x)^{-\operatorname{Re}\alpha/\sigma}\|x\|^{(\sigma+\operatorname{Re}\alpha)/\sigma}\,,\qquad x\in R_p^{\sigma}\,. \tag{3.7}$$

Proof is the same as that of Proposition 2.4 of [3].

THEOREM 3.5. *Let* $0>\operatorname{Re}\alpha>-\sigma$. *Then* $x\in R_p^{\sigma}$ *if and only if* $x\in D(A_-^{\alpha})$ *and* $A_-^{\alpha}x=R_p^{\sigma+\operatorname{Re}\alpha}$.

PROOF. Since $R_p^{\sigma}(A)=D_p^{\sigma}(A_R^{-1})$ and $A_-^{\alpha}=(A_R^{-1})^{-\alpha}$, this is reduced to Theorem 2.6 of [3].

In the same way the following two results are derived from Theorem 2.7 and Proposition 2.8 of [3] respectively.

THEOREM 3.6. *If the domain* $D(A_-^{\alpha})$ *contains (is contained in)* $R_p^{-\operatorname{Re}\alpha}$ *for a* $\operatorname{Re}\alpha<0$, *then* $D(A_-^{\alpha})$ *contains (is contained in)* $R_p^{-\operatorname{Re}\alpha}$ *for all* $\operatorname{Re}\alpha<0$.

PROPOSITION 3.7. *For each* $\operatorname{Re}\alpha<0$

$$R_1^{-\operatorname{Re}\alpha}(A)\subset D(A_-^{\alpha})\subset R_{\infty}^{-\operatorname{Re}\alpha}(A)\,. \tag{3.8}$$

As for the additivity of fractional powers we have

$$A_-^{\alpha}A_-^{\beta}=A_-^{\alpha+\beta}\,,\qquad \operatorname{Re}\alpha<0\,,\qquad \operatorname{Re}\beta<0 \tag{3.9}$$

by Theorem 7.1 of [2]. We remark also the following.

PROPOSITION 3.8. *If m is a positive integer, then*

$$R_1^m(A)\subset R(A^m)\subset R_{\infty}^m(A)\,. \tag{3.10}$$

PROOF. The first inclusion is derived from Proposition 2.8 of [3], as we have

$$R_1^m(A) = D_1^m(A_R^{-1}) \subset D((A_R^{-1})^m) = R(A_R^m) \subset R(A^m)\,.$$

To prove the second inclusion let $x = A^m y \in R(A^m)$. Then

$$\begin{aligned}(\lambda(\lambda+A)^{-1})^{m+1}x &= \lambda^m(\lambda(\lambda+A)^{-1})(A(\lambda+A)^{-1})^m y \\ &= O(\lambda^m)\,.\end{aligned}$$

THEOREM 3.9. *Let* $0 > \mathrm{Re}\,\alpha > -m$ *with* m *an integer. Then for each* $x = A^m y \in R(A^m)$ *we have*

$$A_-^\alpha x = A_+^{m+\alpha} y\,. \tag{3.11}$$

PROOF. Since $R(A^m) \subset R_\infty^m(A)$, $x \in R(A^m)$ is contained in $D(A_-^\alpha)$. Substitute $A^m y$ for x in expression (3.6). Then we obtain by (2.1) of [3]

$$\begin{aligned}A_-^\alpha x &= \frac{\Gamma(m)}{\Gamma(m+\alpha)\Gamma(-\alpha)} \int_0^\infty \lambda^{m+\alpha-1}(A(\lambda+A)^{-1})^m y\, d\lambda \\ &= A_+^{m+\alpha} y\,.\end{aligned}$$

Since A_-^α is the smallest closed extension of its restriction to $R(A^m) \supset R_1^m(A)$, we could start with (3.10) for the definition of negative powers.

§ 4. Interpolation spaces of X and $R(A^m)$.

When B is a closed linear operator in X, we define the norm in the range by

$$\| x \|_{R(B)} = \inf_{By=x} \| y \| + \| x \|\,. \tag{4.1}$$

This is equal to the quotient norm in $D(B)/N(B)$ which is identified with $R(B)$. Since $N(B)$ is a closed subspace of the Banach space $D(B)$, $R(B)$ forms a Banach space. If B is one-to-one, the norm in $R(B)$ defined above is the same as that in $D(B^{-1})$.

The integral powers A^m of non-negative operators A are closed, as A have non-void resolvent sets. Thus we can discuss interpolation spaces of X and $R(A^m)$.

In case X is reflexive, X is decomposed as $N(A) + \overline{R(A)}$. Hence it follows that $R(A^m)$, $R(A_R^m)$ and $D((A_R^{-1})^m)$ are the same spaces with the same norms. On the other hand, if X is not reflexive, $R(A^m)$ can be strictly larger than $R(A_R^m)$. The following theorem shows, however, that their interpolation spaces turn out to be the same.

According to Peetre [8] we denote by $(X, Y)_{\theta,p}$ the mean space $S(p, \theta, X; p, \theta-1, Y)$ of Banach spaces X and Y contained in a Hausdorff vector space [6], where $0 < \theta < 1$ and $1 \leqq p \leqq \infty$. We admit also $p = \infty-$.

THEOREM 4.1. *Let m be a positive integer. We have*

(4.2) $$R_p^{\theta m}(A)=(X,\ R(A^m))_{\theta,p}=(\overline{R(A)},\ R(A_R^m))_{\theta,p}$$

with equivalent norms for $0<\theta<1$ *and* $1\leqq p\leqq\infty$ *or* $p=\infty-$.

PROOF. Since we have $R_p^{\theta m}(A)=D_p^{\theta m}(A_R^{-1})$ by Theorem 2.2, Theorem 3.1 of [3] shows that

$$R_p^{\theta m}(A)=(\overline{R(A)},\ D(A_R^{-m}))_{\theta,p}=(\overline{R(A)},\ R(A_R^m))_{\theta,p}\,.$$

The definition of mean spaces trivially implies $(X,\ R(A^m))_{\theta,p}=(\overline{R(A)},\ R(A^m))_{\theta,p}$, and Proposition 3.8 states that $R(A^m)$ is, e.g., of class $K_{1/2}(\overline{R(A)},\ R(A_R^{2m}))$. Thus it follows from the reiteration theorem of Lions-Peetre [6] that $R_p^{\theta m}(A)=(X,\ R(A^m))_{\theta,p}$.

The domain $D(A_-^{-\alpha})$ is naturally connected with the range $R(A_+^\alpha)$.

THEOREM 4.2. *Let* $\mathrm{Re}\,\alpha>0$. *If X is reflexive, we have*

(4.3) $$R_1^{\mathrm{Re}\,\alpha}(A)\subset D(A_-^{-\alpha})=R(A_+^\alpha)\subset R_{\infty-}^{\mathrm{Re}\,\alpha}(A)\,.$$

If $D(A)$ is dense, we have at least

(4.4) $$R_1^{\mathrm{Re}\,\alpha}(A)\subset D(A_-^{-\alpha})\subset R(A_+^\alpha)\subset R_\infty^{\mathrm{Re}\,\alpha}(A)\,.$$

PROOF. Let X be reflexive. Then $D(A)$ is dense and X is decomposed as $N(A)+\overline{R(A)}$. Thus $A_-^{-\alpha}$ coincides with $A_0^{-\alpha}$ of [2], and A_+^α vanishes on $N(A)$ and coincides with A_0^α on $\overline{R(A)}$. In particular, we have $R(A_+^\alpha)=R(A_0^\alpha)$. Since $(A_0^\alpha)^{-1}=A_0^{-\alpha}$ by Theorem 7.3 of [2], we obtain (4.3). Actually (4.3) holds if $D(A)$ is dense and $X=N(A)+\overline{R(A)}$.

Let $D(A)$ be dense. By the same reasoning as above we get $D(A_-^{-\alpha})=D(A_0^{-\alpha})=R(A_0^\alpha)\subset R(A_+^\alpha)$. If $x=A_+^\alpha y\in R(A_+^\alpha)$, then we have by Proposition 2.4 of [3]

$$\begin{aligned}\|(\lambda(\lambda+A)^{-1})^m x\| &= \lambda^m\|A_+^\alpha(\lambda+A)^{-m}y\| \\ &\leqq \lambda^m C\,\|A^m(\lambda+A)^{-m}y\|^{\mathrm{Re}\,\alpha/m}\|(\lambda+A)^{-m}y\|^{1-\mathrm{Re}\,\alpha/m} \\ &= C_1\lambda^{\mathrm{Re}\,\alpha} \qquad \text{for } m>\mathrm{Re}\,\alpha\,.\end{aligned}$$

THEOREM 4.3. *Let A be a non-negative operator of type $(\omega,\ M(\theta))$. Then*

(4.5) $$R_p^\sigma(A_+^\alpha)=R_p^{\alpha\sigma}(A)\,,\qquad 0<\alpha<\pi/\omega\,,\qquad \sigma>0\,.$$

PROOF. If m is an integer greater than σ, we have $R_p^\sigma(A_+^\alpha)=(X,\ R(A_+^{m\alpha}))_{\sigma/m,p}$. Hence (4.4) together with the reiteration theorem gives (4.5).

§ 5. Infinitesimal generators of bounded continuous semi-groups.

In this section we assume that $-A$ is the infinitesimal generator of a bounded continuous semi-group T_t: $T_t=\exp(-tA)$.

DEFINITION 5.1. Let $0<\sigma<m$ with m an integer, and let $1\leqq p\leqq\infty$. We

denote by $Q^{\sigma}_{p,m}=Q^{\sigma}_{p,m}(A)$ the space of all $x\in X$ such that $t^{\sigma}(t^{-1}I_t)^m x\in L^p(X)$.

$Q^{\sigma}_{p,m}$ is a Banach space with the norm

$$\|x\|_{Q^{\sigma}_{p,m}}=\|x\|_X+\|t^{\sigma}(t^{-1}I_t)^m x\|_{L^p(X)}. \tag{5.1}$$

PROPOSITION 5.2. *If $x\in Q^{\sigma}_{p,m}$, then x belongs to $D(A^{\alpha}_{-})$ for $0>\operatorname{Re}\alpha>-\sigma$ and*

$$A^{\alpha}_{-}x=\frac{1}{K_{\alpha+m,m}}\int_0^{\infty}t^{-\alpha-1}(t^{-1}I_t)^m x\,dt, \tag{5.2}$$

where

$$K_{\alpha+m,m}=\int_0^{\infty}t^{-\alpha-m-1}(1-e^{-t})^m dt. \tag{5.3}$$

PROOF. The right-hand side of (5.2) converges absolutely and represents an analytic function of α for $0>\operatorname{Re}\alpha>-\sigma$.

Let $x=A^m y\in R(A^m)$. Then x belongs to $Q^{\sigma}_{\infty,m}$, because $I^m_t x=A^m I^m_t y=(1-T_t)^m y$. Proposition 4.2 of [3] shows

$$\begin{aligned}\frac{1}{K_{\alpha+m,m}}&\int_0^{\infty}t^{-\alpha-1}(t^{-1}I_t)^m x\,dt\\ &=\frac{1}{K_{\alpha+m,m}}\int_0^{\infty}t^{-\alpha-m-1}(1-T_t)^m y\,dt\\ &=A^{\alpha+m}_{+}y.\end{aligned}$$

In view of Theorem 3.9 we obtain (5.2).

Next let $x\in Q^{\sigma}_{p,m}$. It follows from Theorem 1.4 that $x\in\overline{R(A)}$. Hence $x_{\lambda}=(A(\lambda+A)^{-1})^m x\in R(A^m)$ converges strongly to x as λ tends to zero. The integral (5.2) with x replaced by x_{λ} is equal to $A^{\alpha}_{-}x_{\lambda}$ and converges strongly to integral (5.2). Since A^{α}_{-} is closed, x belongs to $D(A^{\alpha}_{-})$ and (5.2) holds.

THEOREM 5.3. *$Q^{\sigma}_{p,m}$ coincides with R^{σ}_{p} with an equivalent norm.*

PROOF. Let $x\in Q^{\sigma}_{p,m}$. Then for each $\lambda>0$ $(A(\lambda+A)^{-1})^m x$ belongs to $Q^{\sigma+m}_{p,2m}$. In fact noting (1.3), we have

$$\begin{aligned}&\|t^{\sigma+m}(t^{-1}I_t)^{2m}(A(\lambda+A)^{-1})^m x\|\\ &\qquad\leqq\|(1-T_t)^m\|\cdot\|(\lambda+A)^{-m}\|\cdot\|t^{\sigma}(t^{-1}I_t)^m x\|.\end{aligned}$$

In particular, $(A(\lambda+A)^{-1})^m x$ is in $D(A^{-m}_{-})=D(A^{-m}_{R})$. Since A is one-to-one on $\overline{R(A)}$ to which $(\lambda+A)^{-m}x$ belongs, it follows from Proposition 5.2 that

$$\begin{aligned}(\lambda+A)^{-m}x&=c\int_0^{\infty}t^{m-1}(t^{-1}I_t)^{2m}(A(\lambda+A)^{-1})^m x\,dt\\ &=c\int_0^{1/\lambda}(A(\lambda+A)^{-1})^m(t^{-1}I_t)^m\cdot t^{m-1}(t^{-1}I_t)^m x\,dt\\ &\quad+c\int_{1/\lambda}^{\infty}(\lambda+A)^{-m}(1-T_t)^m\cdot t^{-1}(t^{-1}I_t)^m x\,dt,\end{aligned}$$

where $c=K_{m,2m}^{-1}$. Hence

$$\|\lambda^{-\sigma}(\lambda(\lambda+A)^{-1})^m x\| \leqq cL^m M^m \lambda^{m-\sigma}\int_0^{1/\lambda} t^{m-\sigma}\|t^\sigma(t^{-1}I_t)^m x\|dt/t$$

$$+cM^m(2M)^m\lambda^{-\sigma}\int_{1/\lambda}^\infty t^{-\sigma}\|t^\sigma(t^{-1}I_t)^m x\|dt/t$$

belongs to $L^p(\boldsymbol{R}^+)$ with measure $d\lambda/\lambda$.

Now let $x\in R_{p,m}^\sigma$. If $t>0$, we have

$$\lambda^{-\sigma-m}(\lambda(\lambda+A)^{-1})^{2m}(1-T_t)^m x$$
$$=(A(\lambda+A)^{-1})^m I_t^m\cdot\lambda^{-\sigma}(\lambda(\lambda+A)^{-1})^m x\,.$$

Therefore, $(1-T_t)^m x$ belongs to $R_{p,2m}^{\sigma+m}$. Since $I_t x=A_R^{-1}(1-T_t)x$ for $x\in\overline{R(A)}$, w obtain from Theorem 3.3

$$I_t^m x=A_-^{-m}(1-T_t)^m x$$
$$=c\int_0^{1/t}(A(\lambda+A)^{-1})^m I_t^m(\lambda(\lambda+A)^{-1})^m x d\lambda/\lambda$$
$$+c\int_{1/t}^\infty(\lambda(\lambda+A)^{-1})^m(1-T_t)^m\lambda^{-m}(\lambda(\lambda+A)^{-1})^m x\, d\lambda/\lambda\,,$$

where $c=\Gamma(2m)/\Gamma(m)^2$. Hence we conclude that $t^\sigma(t^{-1}I_t)^m x\in L^p(X)$ as above.

THEOREM 5.4. *Let $0>\operatorname{Re}\alpha>-m$ with m an integer. If there is a sequence $N_j\to\infty$ such that*

$$y=\underset{j\to\infty}{\text{w-lim}}\frac{1}{K_{\alpha+m,m}}\int_0^{N_j}t^{-\alpha-1}(t^{-1}I_t)^m x\,dt \tag{5.4}$$

exists, then $x\in D(A_-^\alpha)$ and $y=A_-^\alpha x$.

PROOF. We repeat the argument in the proof of [2], Proposition 4.6. First it follows from (1.4) for $\sigma=2$ that

$$-\alpha N_j^\alpha\int_0^{N_j}t^{-\alpha-1}(t^{-1}I_t)^m x\,dt$$
$$=-\alpha N_j^\alpha\int_0^{N_j}t^{-\alpha-1}(1-t^{-1}AI_t^{(2)})^m x\,dt$$
$$=x-y_j\,,$$

where $y_j\in\overline{R(A)}$. Since this converges to zero, x belongs to $\overline{R(A)}$. Hence $x_\mu=(A(\mu+A)^{-1})^m x$ converges to x as μ tends to zero. Since $x_\mu\in R_\infty^m$, we have

$$A_-^\alpha x_\mu=c\int_0^\infty t^{-\alpha-1}(t^{-1}I_t)^m(A(\mu+A)^{-1})^m x\,dt$$
$$=(A(\mu+A)^{-1})^m\ \text{w-lim}\ c\int_0^{N_j}t^{-\alpha-1}(t^{-1}I_t)^m x\,dt$$
$$=(A(\mu+A)^{-1})^m y\,.$$

y is also in $\overline{R(A)}$. Thus the right-hand side converges to y as μ tends to zero. Noting that A_-^{α} is closed, we have $x \in D(A_-^{\alpha})$ and $y = A_-^{\alpha} x$.

Probably the converse statement as in Theorems 3.3 and 6.3 holds good in this case, too.

§ 6. Infinitesimal generators of bounded analytic semi-groups.

In this section we assume that $-A$ generates a bounded analytic semi-group T_t.

DEFINITION 6.1. Let $\sigma > 0$ and let $1 \leqq p \leqq \infty$. We denote by $P_p^\sigma = P_p^\sigma(A)$ the Banach space of all $x \in X$ such that $t^\sigma T_t x \in L^p(X)$. The norm is defined by

$$\| x \|_{P_p^\sigma} = \| x \|_X + \| t^\sigma T_t x \|_{L^p(X)} . \tag{6.1}$$

THEOREM 6.2. *P_p^σ is the same space as R_p^σ with an equivalent norm.*

PROOF. First we prove that $P_p^\sigma \subset Q_{p,m}^\sigma$ for $m > \sigma$. It is easy to see that $(t^{-1}I_t)^m x$ is written

$$(t^{-1}I_t)^m x = \int_0^t K_m(s/t) T_{ms} x \, ds/s , \tag{6.2}$$

and the kernel $K_m(s)$ has properties:

$$K_m(s) \geqq 0 , \qquad \int_0^1 K_m(s) \, ds/s = 1 , \tag{6.3}$$

and

$$K_m(s) = O(s^m) \qquad \text{as } s \to 0 . \tag{6.4}$$

Therefore, if $x \in P_p^\sigma$,

$$t^\sigma (t^{-1}I_t)^m x = \int_0^t (s/t)^{-\sigma} K_m(s/t) s^\sigma T_{ms} x \, ds/s$$

belongs to $L^p(X)$.

We note that we have not employed the fact that T_t is an analytic semi-group. Therefore, the inclusion $P_p^\sigma \subset Q_p^\sigma$ holds for any bounded semi-group T_t.

Next let $x \in R_{p,m}^\sigma$. If $t > 0$, $A^m T_t x$ belongs to $R_{p,2m}^{\sigma+m}$. In fact, we have

$$\begin{aligned} &\lambda^{-\sigma-m} (\lambda(\lambda+A)^{-1})^{2m} A^m T_t x \\ &\qquad = (A(\lambda+A)^{-1})^m T_t \cdot \lambda^{-\sigma} (\lambda(\lambda+A)^{-1})^m x \in L^p(X) . \end{aligned}$$

Since T_t maps $\overline{R(A)}$ into $\overline{R(A)}$, $T_t x$ is in $\overline{R(A)}$. Hence we have by (3.1)

$$\begin{aligned} t^\sigma T_t x &= t^\sigma A_-^{-m} A^m T_t x \\ &= c \int_0^{1/t} (A(\lambda+A)^{-1})^m T_t (t\lambda)^\sigma \lambda^{-\sigma} (\lambda(\lambda+A)^{-1})^m x \, d\lambda/\lambda \\ &\quad + c \int_{1/t}^\infty (\lambda(\lambda+A)^{-1})^m (tAT_{t/m})^m (t\lambda)^{\sigma-m} \lambda^{-\sigma} (\lambda(\lambda+A)^{-1})^m x \, d\lambda/\lambda . \end{aligned}$$

It is easy to see that each of these terms is in $L^p(X)$.

THEOREM 6.3. *Let* $\operatorname{Re}\alpha<0$. *If there is a sequence* $N_j\to\infty$ *such that*

$$y=\operatorname*{w-lim}_{j\to\infty}\frac{1}{\Gamma(-\alpha)}\int_0^{N_j}t^{-\alpha-1}T_tx\,dt\,, \tag{6.5}$$

exists, then $x\in D(A_-^\alpha)$ *and* $y=A_-^\alpha x$. *If* $x\in D(A_-^\alpha)$, *then*

$$A_-^\alpha x=\operatorname*{s-lim}_{N\to\infty}\frac{1}{\Gamma(-\alpha)}\int_0^{N}t^{-\alpha-1}T_tx\,dt\,. \tag{6.6}$$

PROOF. First we prove that (6.6) holds for $x\in P_p^\sigma$ when $\sigma>-\operatorname{Re}\alpha$. If $x\in P_p^\sigma$, the right-hand side of (6.6) converges absolutely and represents a continuous operator from P_p^σ to X. When an $x\in P_p^\sigma$ is fixed, it is clearly an analytic function of α for $0>\operatorname{Re}\alpha>-\sigma$. Thus it is sufficient to prove (6.6), say, in the case where $x\in P_\infty^1$ and $0>\operatorname{Re}\alpha>-1$. In this case we have

$$\begin{aligned}A_-^\alpha x&=\frac{1}{\Gamma(-\alpha)\Gamma(1+\alpha)}\int_0^\infty\lambda^\alpha d\lambda\int_0^\infty e^{-\lambda t}T_tx\,dt\\&=\frac{1}{\Gamma(-\alpha)\Gamma(1+\alpha)}\int_0^\infty T_tx\,dt\int_0^\infty\lambda^\alpha e^{-\lambda t}d\lambda\\&=\frac{1}{\Gamma(-\alpha)}\int_0^\infty t^{-\alpha-1}T_tx\,dt\,.\end{aligned}$$

Since all integrals converge absolutely, we may change the order of integration.

Now, the first part of Theorem is proved in the same way as Theorem 5.4. The only difference is that we use (1.3) instead of (1.4).

To prove the latter part, first we assume that A is bounded. Then $A_+^{-\alpha}$ is bounded and, as Theorem 5.4 of [3] shows, is given by

$$A_+^{-\alpha}x=\frac{1}{\Gamma(m+\alpha)}\int_0^\infty t^{m+\alpha-1}A^mT_tx\,dt\,,$$

where m is an integer greater than $-\operatorname{Re}\alpha$. Since $D(A_-^\alpha)$ coincides with $R(A_+^{-\alpha}|_{\overline{R(A)}})$, what we have to prove is that for $x\in\overline{R(A)}$

$$J_Nx=\int_0^N s^{-\alpha-1}T_sds\int_0^\infty t^{m+\alpha-1}A^mT_tx\,dt$$

converges strongly as N tends to infinity.

We have easily

$$\begin{aligned}J_Nx=&\int_0^N A^mT_rx\,dr\int_0^r s^{-\alpha-1}(r-s)^{m+\alpha-1}ds\\&+\int_N^\infty A^mT_rx\,dr\int_0^N s^{-\alpha-1}(r-s)^{m+\alpha-1}ds\end{aligned}$$

$$=\frac{\Gamma(-\alpha)\Gamma(m+\alpha)}{\Gamma(m)}\int_0^N r^{m-1}A^m T_r x\,dr$$

$$+\int_1^\infty N^m A^m T_{\rho N}x\,d\rho\int_0^1 \sigma^{-\alpha-1}(\rho-\sigma)^{m+\alpha-1}d\sigma\,.$$

The first term converges strongly to $\Gamma(-\alpha)\Gamma(m+\alpha)x$. In fact, if $m=1$, we have

$$\int_0^N AT_r x\,dr=(1-T_N)x\to x$$

from the ergodic Theorem 1.5. When $m\geqq 1$, integration by parts gives

$$\int_0^N r^m A^{m+1}T_r x\,dr=-N^m A^m T_N x+m\int_0^N r^{m-1}A^m T_r x\,dr\,.$$

That $N^m A^m T_N x$ converges to zero is proved in the same way as the ergodic theorem. Hence the induction on m proceeds.

The second term of $J_N x$ converges strongly to zero. For, the integral

$$\int_1^\infty \rho^{-m}d\rho\int_0^1 \sigma^{-\alpha-1}(\rho-\sigma)^{m+\alpha-1}d\sigma$$

converges absolutely and $(\rho N)^m A^m T_{\rho N}x$ converges strongly to zero as N tends to infinity uniformly on $(1, \infty)$.

Next assume that A is unbounded. For $\mu>0$ we denote by T_t^μ the bounded analytic semi-group generated by the bounded operator $-A_\mu=-\mu A(\mu+A)^{-1}$. According to Theorem 6.8 of [2] we have

$$D(A_-^\alpha)=D((A_\mu)_-^\alpha)=R((A_\mu)_+^{-\alpha}|_{\overline{R(A)}})\subset R_\infty^{-\mathrm{Re}\,\alpha}(A)\,.$$

Thus it suffices to show that for $x\in R_\infty^{-\mathrm{Re}\,\alpha}$

$$\int_0^\infty t^{-\alpha-1}(T_t-T_t^\mu)x\,dt$$

converges absolutely. For this purpose we will prove that

$$(T_t-T_t^\mu)x=O(t^{-\sigma-1})\quad\text{as } t\to\infty\quad\text{for } x\in R_\infty^\sigma \tag{6.7}$$

when $\sigma>0$ is not an integer. (Probably (6.7) holds true also for σ an integer.)

Suppose that A is a non-negative operator of type $(\omega, M(\theta))$ with $\omega<\pi/2$. Then it is easily shown that A_μ is also of type $(\omega, M(\theta))$. In this case the semi-group T_t (and hence T_t^μ) has the integral representation

$$T_t x=\frac{1}{2\pi i}\,\frac{m\,!}{t^m}\int_\Gamma e^{t\zeta}(\zeta+A)^{-m-1}x\,d\zeta\,, \tag{6.8}$$

where m is a non-negative integer and Γ is the path composed of the ray from $\infty e^{-i\theta}$ to $\varepsilon e^{-i\theta}$, the portion from $\varepsilon e^{-i\theta}$ to $\varepsilon e^{i\theta}$ of the circle with center at the origin, and the ray from $\varepsilon e^{i\theta}$ to $\infty e^{i\theta}$ with $\varepsilon>0$ and $\pi/2<\theta<\pi-\omega$.

In fact, if $m=0$, this is well known [1] and integration by parts gives the representation for $m>0$.

Choose the least integer m greater than σ. Then, we have

$$(T_t-T_t^\mu)x=\frac{1}{2\pi i}\frac{m!}{t^m}\int_\Gamma e^{t\zeta}(\mu+A)^{-1}A(\zeta+A)^{-1}A(\zeta+A_\mu)^{-1}$$
$$\times\{(\zeta+A)^{-m}+(\zeta+A)^{-m+1}(\zeta+A_\mu)^{-1}+\cdots+(\zeta+A_\mu)^{-m}\}x\,d\zeta\,.$$

Since $(\zeta+A_\mu)(\zeta+A)^{-1}=1-A(\mu+A)^{-1}A(\zeta+A)^{-1}$ is uniformly bounded on the sector $\Sigma=\{\zeta:|\arg\zeta|\leqq\theta\}$ and since $R^\sigma_\infty(A)=R^\sigma_\infty(A_\mu)$, we have

$$\{(\zeta+A)^{-m}+(\zeta+A)^{-m+1}(\zeta+A_\mu)^{-1}+\cdots+(\zeta+A_\mu)^{-m}\}x$$
$$=O(\|(\zeta+A_\mu)^{-m}x\|)=O(|\zeta|^{\sigma-m})\qquad\text{for } x\in R^\sigma_\infty$$

uniformly on Σ, similarly to the proof of Theorem 12.3 of [2].

Now, noting that $(\mu+A)^{-1}A(\zeta+A)^{-1}A(\zeta+A_\mu)^{-1}$ is uniformly bounded, we can let $\varepsilon\to0$ in the integral and obtain the desired estimate

$$\|(T_t-T_t^\mu)x\|\leqq Ct^{-m}\int_0^\infty e^{tr\cos\theta}r^{\sigma-m}dr$$
$$=C\Gamma(\sigma-m+1)(-\cos\theta)^{-\sigma+m-1}t^{-\sigma-1}\,.$$

University of Tokyo

References

[1] E. Hille and R. S. Phillips, Functional Analysis and Semigroups, Amer. Math. Soc. Colloq. Publ., 31, 1957.

[2] H. Komatsu, Fractional powers of operators, Pacific J. Math., **19** (1966), 285–346.

[3] H. Komatsu, Fractional powers of operators, II, Interpolation spaces, Pacific J. Math., **21** (1967), 89–111.

[4] H. Komatsu, An ergodic theorem, Proc. Japan Acad., **44** (1968), 46–48.

[5] T. Kōmura, Semi-groups of operators in locally convex spaces, J. Funct. Anal. **2** (1968), 258–296.

[6] J.-L. Lions et J. Peetre, Sur une classe d'espaces d'interpolation, Publ. Math. I. H. E. S., **19** (1964), 5–68.

[7] K. Masuda, an article to appear.

[8] J. Peetre, Sur le nombre de paramètres dans la définition de certains espaces d'interpolation, Ricerche Mat., **12** (1963), 248–261.

[9] P. L. Butzer and H. Berens, Semi-groups of Operators and Approximation, Grundlehren Bd. 145, Springer, 1967.