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An example for the theorem of W. Browder

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Introduction

W. Browder proved in his paper [2] that a simply connected finite CW-complex of dimension $4k$ $(k \neq 1)$ has the same homotopy type as a closed differentiable manifold[1] under the following conditions:

(1) Poincaré duality holds,

(2) there exists an oriented vector bundle ξ such that $T(\xi)$, the Thom space, has a spherical fundamental class,

(3) the Hirzebruch formula in the dual Pontrjagin classes of ξ gives the index.

In this paper we shall apply the above theorem to obtain the homotopy type classification of closed differentiable manifolds M which are simply connected and have homology groups $H^0(M)=H^4(M)=H^8(M)=Z$, $H^i(M)=0$ $i \neq 0, 4, 8$. This result is previously obtained by J. Eells and N. Kuiper in [3]. Their method makes use of the existence of certain non-degenerate functions so that it is quite different from our method. They also obtained some informations on Pontrjagin classes, for instance a counter example of homotopy type invariance of Pontrjagin numbers, and examples of closed differentiable manifolds which have the same homotopy type but are not diffeomorphic. These results can be proved more intuitively by our method. Moreover, we shall give a counter example to the problem (2) about combinatorial and differentiable structures on manifolds proposed by C. T. C. Wall in A. M. S. Summer Topology Institute, Seattle, 1963, [4].

Let X_f be a CW-complex $S^4 \cup_f e^8$. If $h: S^7 \to S^4$ is the Hopf fibering X_h is the quaternion projective plane. Now we fix the orientation of S^4 and determine the orientation of (E^8, S^7) such that the generator of $H^8(E^8, S^7)$ represented by (E^8, S^7) is equal to $\bar{h}^* j^{-1}(e_h^4 \cup e_h^4)$ where $\bar{h}: (E^8, S^7) \to (X_h, S^4)$ is the characteristic map of the cell e^8, j is the inclusion homomorphism $H^8(X_h, S^4) \to H^8(X_h)$ and e_h^4 is the generator of $H^4(X_h)$ represented by the oriented S^4. Let (f) denote the homotopy class of a map $f: S^7 \to S^4$.

Since $\pi_7(S^4)$ is the direct sum $\boldsymbol{Z}(h)+\boldsymbol{Z}_{12}(\tau)$ where $2(h)+(\tau)=[i_4, i_4]$ we have

1) "closed" means compact and unbounded.

$(f)=a(h)+b(\tau)$ by some integers a and b (mod 12). Let us determine the orientation of X_f such that the generator of $H^8(X_f)$ represented by e_f^8 is equal to $j\cdot\bar{f}^{*-1}(E^8, S^7)$ where (E^8, S^7) is the oriented generator of $H^8(E^8, S^7)$ as above. In this case we say that the oriented complex X_f has type (a, b). Now our purpose is to obtain necessary and sufficient conditions for a and b under which X_f satisfies (1), (2), and (3) and to obtain relations among a, b, a', b' such that X_f has the same homotopy type as $X_{f'}$. If X_f has type (a, b) it is clear that the cup product $e_f^4\cup e_f^4$ is ae_f^8 where e_f^l denotes the oriented generator of $H^l(X_f)$ determined as above.

Hence it is easy to see that Poincaré duality holds in X_f if and only if $a=\pm 1$.

In section 1 we consider the homotopy type of X_f. For our purpose it is sufficient to consider X_f of types $(-1, b)$ or $(1, b)$ and we obtain the well known result that the number of the different homotopy types of these complexes is six.

In section 2 we concern with the problem: which pair of classes of $H^4(X_f)$ and $H^8(X_f)$ are realizable as the pair of Pontrjagin classes of a vector bundle over X_f. It is known that a class of $H^4(X_f)$ is realizable as the first Pontrjagin class of a certain vector bundle over X_f if and only if it is divisible by 2. Therefore we are interested only in the second Pontrjagin class. In section 3 we shall obtain vector bundles over X_f of type $(1, b)$ which satisfy the condition (2) and it shall be shown that there exists a vector bundle over X which satisfies the conditions (2) and (3) if and only if b is congruent to 0 or 1 mod 4.

REMARK. The same argument holds in the case of a CW-complex which is like the Caley projective plane.

1. Homotopy type

Let X_f and X_g be complexes of type (a, b) and (c, d) respectively. Then we have

LEMMA 1.1. *There exists a map* $F: X_f\to X_g$ *such that* $F^*(e_g^4)=me_f^4$ *and* $F^*(e_g^8)=se_f^8$ *if and only if* $am=sc$ *and* $\dfrac{am(m-1)}{2}+mb=sd$ mod 12.

PROOF. Let $F_m: S^4\to S^4$ be a map with degree m and let $F_{m*}:\pi_7(S^4)\to\pi_7(S^4)$ be the induced homomorphism by F_m. Since we have

$$F_{m*}((f))=F_{m*}(a(h)+b(\tau))=aF_{m*}(h)+bF_{m*}(\tau)$$

$$=\frac{am(m-1)}{2}[i_4, i_4]+m(h)+bm(\tau)$$

$$=a(m(m-1)(h)-\frac{m(m-1)}{2}(\tau))+am(h)+bm(\tau)$$

$$=(am(m-1)+am)(h)+\left(\frac{am(m-1)}{2}+bm\right)(\tau)$$

$$=am^2(h)+\left(\frac{am(m-1)}{2}+bm\right)(\tau)$$

$$=sc(h)+sd(\tau)=s(c(h)+d(\tau))=s(g)$$

it is easy to see that F_m has an extention $F: X_f \to X_g$ such that $F^*(e_g^4)=me_f^4$ and $F^*(e_g^8)=se_f^8$.

Suppose that X_f has the same homotopy type as X_g. Then there exists a map $F: X_f \to X_g$ such that $F^*(e_g^4)=\pm e_f^4$ and $F^*(e_g^8)=\pm e_f^8$. Hence from lemma 1.1 we have

LEMMA 1.2. *X_f has the same homotopy type with X_g if and only if*

(1) $a=c,\ b=d$ (2) $a=c,\ b=c+d$

(3) $a=-c,\ b=-d$ (4) $a=-c,\ b=-c-d$.

Especially all complexes with type $(1, b)$, $(1, 1+b)$, $(-1, -b)$, $(-1, -b-1)$ have the same homotopy type, and therefore the number of different homotopy types of complexes for which Poincaré duality hold is six.

2. Pontrjagin classes

Let f be a map of S^7 to S^4 and let $\boldsymbol{Z}_6$ denote the module of integers mod 6. Consider a correspondence $P: f \to \boldsymbol{Z}_6$ defined as follows:

Choose a stable vector bundle ξ over X_f such that $p_1(\xi)$ is $2e_f^4$ where $p_i(\xi)$ denotes the i-th Pontrjagin class of ξ. Since $p_2(\xi)$ mod 6 is uniquely determined we put $P(f)=\langle p_2(\xi), e_8^f\rangle$ mod 6[2)].

LEMMA 2.1. *P depends only on the homotopy class of f and induces a homomorphism of $\pi_7(S^4)$ to $\boldsymbol{Z}_6$.*

PROOF. It is clear that P is determined by the homotopy class of f. Let $X_{f,g}$ be a complex which is obtained from X_f and X_g by identifying S^4.

It is easy to prove that there exists a map $G: X_{f+g} \to X_{f,g}$ which satisfies the conditions

$$(1)\quad G^*(e_{f,g}^4)=e_{f+g}^4 \qquad (2)\quad G^*(e_1^8)=e_{f+g}^8=G^*(e_2^8)\,.$$

where (e_1^8, e_2^8) denote the oriented generators of $H^8(X_{f,g})=\boldsymbol{Z}+\boldsymbol{Z}$.

Let ξ_f, ξ_g be stable vector bundles over X_f and X_g respectively such that

2) $\langle , \rangle$ denotes the Kronecker index and $e_8{}^f$ denotes the dual homology class of the oriented generator of $H^8(X_f)$.

$p_1(\xi_f)=2e_4^4$ and $p_1(\xi_g)=2e_g^4$ and let $\xi_f\,|\,S^4$, $\xi_g\,|\,S^4$ denote the restrictions of ξ_f, ξ_g on S^4. By identifying $\xi_f\,|\,S^4$ with $\xi_g\,|\,S^4$ we obtain a stable vector bundle ξ over $X_{f,g}$ whose $p_1(\xi)$ is $2e_{f,g}^4$. Let η be the induced bundle of ξ by G. Then from (1), (2) we have that $p_1(\eta)=2e_{f+g}^4$ and $\langle p_2(\eta), e_8^{f+g}\rangle=\langle p_2(\xi), e_1^{f,g}+e_2^{f,g}\rangle$ $=\langle p_2(\xi_f), e_8^f\rangle+\langle p_2(\xi_g), e_8^g\rangle$. These show that P is a homomorphism.

LEMMA 2.2. *$P(h)=1$ and $P(\tau)=2$.*

PROOF. First, since X_h is the quaternion projective plane there exists a stable vector bundle ξ_h over X_h such that $p_1(\xi_h)=2e_h^4$ and $p_2(\xi_h)=7e_h^8$. Hence we obtain $P(h)=1$. Secondly, by Lemma 2.1 $P(h+\tau)=P(h)+P(\tau)=1+P(\tau)$. On the other hand, if we put $a=c=b=1$, $d=0$ and $m=-1$ in Lemma 1.1 we have a map $F:X_{h+\tau}\to X_h$ such that $F^*(e_h^4)=-e_{h+\tau}^4$ and $F^*(e_h^8)=e_{h+\tau}^8$. Let η be the induced bundle of ξ_h by F. Then it is obvious that $p_1(\eta)=-2e_{h+\tau}^4$ and $p_2(\eta)=7e_{h+\tau}^8$. If we denote by $\tilde{\eta}$ the inverse bundle of η we have that $p_1(\tilde{\eta})=2e_{h+\tau}^4$ and $p_2(\tilde{\eta})=-3e_{h+\tau}^8$. Hence we obtain $P(h+\tau)=3$ and therefore $P(\tau)=2$.

By combining Lemma 2.1 and Lemma 2.2 we have

LEMMA 2.3[3]. *Let X_f be a complex of type (a, b) and let ξ be a stable vector bundle over X_f. Then $p_1(\xi)=2me_f^4$, $p_2(\xi)=(am(2m-1)+2bm+6n)e_f^8$ for some integers m and n. Conversely, a pair of cohomology classes $(2me_f^4, (am(2m-1)+2bm+6n)e_f^8)$ is realizable as $(p_1(\xi), p_2(\xi))$ of a certain vector bundle ξ over X_f.*

3. Reducibility of Thom complexes

Since it is sufficient for our purpose to consider only X_f of type $(1, b)$ we shall use the notation X_b instead of X_f in this section. Now the condition (2) in the introduction is equivalent to that $T(\xi)$ is reducible. It is known that the Thom complex of the stable normal bundle of a differentiable manifold is reducible. Then we have

LEMMA 3.1. *There exists a stable vector bundle ξ_0 over X_0 such that*

(1) $p_1(\xi_0)=-2e_0^4$ *and* $p_2(\xi_0)=-3e_0^8$

(2) $T(\xi_0)$ *is reducible.*

PROOF. X_0 may be concidered as the quaternion projective plane and it is sufficient to take ξ_0^0 as the stable normal bundle of the equaternion projective plane. Suppose $m(m+2b-1)=0 \bmod 24$. From Lemma 1.1 there exists a map $F:X_b\to X_0$ such that $F^*(e_0^4)=me_b^4$ and $F^*(e_0^8)=me_b^8$. Let ξ_m^b denote the induced bundle of ξ_0^0 by F. It is clear that $p_1(\xi_m^b)=-2me_b^4$ and $p_2(\xi_m^b)=-3m^2e_b^8$.

Let $\tilde{F}:T(\xi_m^b)\to T(\xi_0^0)$ be the map induced by F and let l be the dimension of ξ_0^0. By Thom isomorphism we know that $T(\xi_m^b)$ has a cell decomposition

3) A. Hattori has also obtained this result by another method.

$S^l \cup e^{l+4} \cup e^{l+8}$, and $F^*(e^l)=e^l$, $F^*(e^{l+4})=me^{l+4}$, and $F^*(e^{l+8})=m^2e^{l+8}$ hold. The subcomplex $S^l \cup e^{l+4}$ of $T(\xi_m^b)$ is $T(\xi_m^b \mid S^4)$ so that $T(\xi_m^b)$ is $T(\xi_m^b \mid S^4) \cup_{\alpha_m^b} e^{l+8}$ and reducibility of $T(\xi_m^b)$ is equivalent to $\alpha_m^b=0$ in $\pi_{l+7}(T(\xi_m^b \mid S^4))$. Since F is an extension of $F \mid T(\xi_m^b \mid S^4)$ and $\alpha_0^0=0$ we obtain $(F \mid T(\xi_m^b \mid S^4))_*(\alpha_m^b)=0$. Now consider the following commutative diagram of two exact sequences of the pairs $(T(\xi_0^0 \mid S^4), S^4)$ and $(T(\xi_m^b \mid S^4), S^4)$[4]:

$$\begin{array}{ccccccc}
0 \longrightarrow & \pi_{l+7}(S^l) & \longrightarrow & \pi_{l+7}(T(\xi_0^0 \mid S^4)) & \longrightarrow & \pi_{l+7}(S^{l+4}) & \longrightarrow \\
 & \uparrow id & & \uparrow F \mid T(\xi_m^b \mid S^4)_* & & \uparrow (mi)_* & \\
0 \longrightarrow & \pi_{l+7}(S^l) & \longrightarrow & \pi_{l+7}(T(\xi_m^b \mid S^4)) & \longrightarrow & \pi_{l+7}(S^{l+4}) & \longrightarrow
\end{array}$$

By $\pi_{l+7}(S^{l+4})=\boldsymbol{Z}_{24}$ and $(mi)_*(x)=mx$, we have

LEMMA 3.2. *If m is prime to 6, $F \mid T(\xi_m^b \mid S^4)_*$ is an isomorphism and we have $\alpha_m^b=0$. If m is odd, $\alpha_m^b=0$ holds only when $\mathcal{P}_3^1(e^{l+4})=0$ holds in $H^*(T(\xi_m^b))$*[5].

PROOF. The first part is clear from that $(mi)_*$ is an isomorphism. In the second part it suffices to show $j(\alpha_m^b)=0$ by the above diagram. If m is odd the kernel of $(mi)_*$ is contained in the 3-component. Hence $j(\alpha_m^b)$ is in the **3**-component. On the other hand, it is known that the 3-component of $\pi_{l+7}(S^{l+4})$ is determined by $\mathcal{P}_3^1$. Therefore $j(\alpha_m^b)=0$ is equivalent to $\mathcal{P}_3^1(e^{l+4})=0$.

LEMMA 3.3. *If $m=1-2b \bmod 24$ $T(\xi_m^b)$ is reducible.*

PROOF. If $b \not\equiv 2 \bmod 3$ m is prime to 6 so that Lemma follows from Lemma 3.2. If $b \equiv 2 \bmod 3$ m is odd. Then we must consider $\mathcal{P}_3^1(e^{l+4})$ in $H^*(T(\xi_m^b))$. First we compute $\mathcal{P}_3^1(e_b^4)$ in $H^*(X_6)$. We set $\mathcal{P}_3^1(e_b^4)=l_b e_b^8$. By the formula $\mathcal{P}_3^1(p_1(\xi))=-p_1(\xi)^2-p_2(\xi)$ for any vector bundle ξ over X_b we have $2l_b=-4-1-2b$, i.e. $\mathcal{P}_3^1(e_b^4)=(-1-b)e_b^8$, by considering as ξ the vector bundle over X such as $p_1(\xi)=2e_b^4$ and $p_2(\xi)=(1+2b)e_b^8$. Secondly, let E, p be the total space and the projection map of ξ_m^b and we denote by E_0 the set of non-zero elements of E. Since we may identify $H^*(E, E_0)$ with $H^*(T(\xi_m^b))$ we use the same notations for generators of $H^*(E, E_0)$ and $H^*(T(\xi_m^b))$. Then we have

$$\begin{aligned}
\mathcal{P}_3^1(e^{l+4}) &= \mathcal{P}_3^1(e^l \cup p^*(e_b^4)) = \mathcal{P}_3^1(e^l) \cup p^*(e_b^4) + e^l \cup p^*(\mathcal{P}_3^1(e_b^4)) \\
&= e^l \cup p^*(p_1(\xi_m^b)) \cup p^*(e_b^4) + e^l \cup (-1-b)p^*(e_b^8) \\
&= e^l \cup p^*(p_1(\xi_m^b) \cup e_b^4) + e^l \cup (-1-b)p^*(e_b^8) \\
&= (-1-2m-b)(e^l \cup p^*(e_b^8)) = (-1-2m-b)e^{l+8}.
\end{aligned}$$

Hence $\mathcal{P}_3^1(e^{l+4})=0$ is equivalent to $m \equiv 1+b \equiv 1-2b \bmod 3$.

Let λ_k^b be the stable vector bundle over X_b with $p_1(\lambda_k^b)=-2(1-2b+24k)e_b^4$, $p_2(\lambda_k^b)=-3(1-2b+24k)^2e_b^8$ and let η be the stable vector bundle obtained by

4) $\pi_{l+8}(S^{l+4})=0$ holds for sufficient large l.

5) $\mathcal{P}^1{}_3$ is the Steenrod operation.

Whitney sum of λ_k^b with γ_s which satisfies $p_1(\gamma_s)=0$ and $p_2(\gamma_s)=6se$. If $s=0$ mod 240 we have $J(\eta)=J(\lambda_k^b)+J(\gamma_s)=J(\lambda_k^b)$ where J denotes the stable fibre homotopy equivalence class of a fibre bundle. Therefore $T(\eta)$ is reducible. Let $\tilde{\eta}$ be the inverse stable vector bundle of η. From $p_1(\tilde{\eta})=2(1-2b+24k)e_b^4$ and $p_2(\tilde{\eta})=(7(1-2b+24k)^2-6s)e_b^8$ the Hirzebruch formula of the index of X for $\tilde{\eta}$ gives the following equality;

$$45=7\cdot 7(1-2b+24)^2-42s-4(1-2b+24k)^2=45(1-2b+24k)^2-42s\,.$$

LEMMA 3.4. *The Hirzebruch formula for $\tilde{\eta}$ holds if and only if $k\equiv 3b$ or $3b-3 \bmod 7$ and also $(12k-b)(1-2b+12k)=0 \bmod 8$.*

PROOF. By the above equality we have

$$45(24k-2b)(2-2b+24k)=0 \mod 42\cdot 240$$

$$4\cdot 9\cdot 5(12k-b)(1-b+12k)=0 \mod 2^5\cdot 3^2\cdot 7\cdot 5$$

$$(12k-b)(1-b+12k)=0 \mod 2^3\cdot 7\,.$$

Suppose that there exists a stable vector bundle μ over X_b which satisfies the conditions (2) and (3) in the introduction.

Since X_b has the same homotopy type as a closed differentiable manifold with the normal stable bundle μ we have $J(\mu)=J(\lambda_0^b)$ by the proposition 3.4 of [1].

Thus we obtain $p_1(\mu)=-2(1-2b+24k)e_b^4$ for some integer k by $J(\mu\,|\,S^4)=J(\lambda_0^b\,|\,S^4)$ so that there exists a stable vector bundle ν_s over X_b with $p_1(\nu_s)=0$, $p_2(\nu_s)=6se$ and $\mu=\lambda_k^b+\nu_s$. From $J(\mu)=J(\lambda_k^b)+J(\nu_s)$ and $J(\mu)=J(\lambda_k^b)$ we obtain $J(\nu_s)=0$ so that $s\equiv 0 \bmod 240$. Hence μ must be a stable vector bundle such as η in the above argument. It is easily obtained that the equation in Lemma 3.4 have solutions for $b\equiv 0$ or $1 \bmod 4$ and no solutions for $b\equiv 2$ or $3 \bmod 4$. Tnus we have the following

THEOREM. *X_b of type (a, b) has the same homotopy type as a closed differentiable manifold if and only if*

$$a=1 \quad \textit{and} \quad b=0, 1, 4, 5, 8, 9$$

or

$$a=-1 \;\textit{and}\; b=0, 11, 8, 7, 4, 3\,.$$

Moreover, we can choose (1, 0), (1, 4), (1, 8) *as representatives of the homotopy types.*

COROLLARY (*counter examples to Wall's problem*). *If $b\equiv 2, 3 \bmod 4$ there exist stable vector bundles over X_b whose Thom complexes are reducible but X_b has not the same homotopy type as a closed differentiable manifold.*

COROLLARY. *Let M be a closed differentiable manifold with $H^0(M)=H^4(M)=H^8(M)=\boldsymbol{Z}$, $H^i(M)=0$ $(i\neq 0, 4, 8)$ and let τ_M be the tangent vector bundle of*

M. If M is simply connected there exist integers b, s, k which satisfy

(1) $p_1(\tau_M)=2(1-2b+24k)e^4,\quad p_2(\tau_M)=(7(1-2b+24k)^2-6s)e^8$

(2) *if* $b\equiv 0 \bmod 4$ $k=7\frac{b}{4}-4b$ *or* $7\frac{b}{4}-4b+4 \bmod 14$

(3) *if* $b\equiv 1 \bmod 4$ $k=7\frac{b-1}{4}-4b$ *or* $7\frac{b-1}{4}-4b+4 \bmod 14$

(4) $$s=\frac{45}{42}((1-2b+24k)^2-1)\,.$$

Conversely, a stable vector bundle over X_b which satisfies the above conditions is the stable tangent vector bundle of a closed differentiable manifold of the same homotopy type as X_b.

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