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Sufficient conditions for p-valence of regular functions

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§ 1. Introduction.

An interesting sufficient condition for univalence due to Umezawa [**18**, p. 213], [**16**, p. 191] and Kaplan [**5**, p. 173] has been generalized by Ogawa in his paper [**7**] as 'Main criterion' or as 'Theorem 2', while the last result has also been extended by Sakaguchi [**13**] as follows.

THEOREM A. *Let* $f(z)=z^p+\cdots$, $\varphi(z)$ *be regular in* $|z|\leqq r$ *and* $|z|<+\infty$ *respectively, and let* $f'(z)\neq 0$ *for* $0<|z|\leqq r$. *If neither* $f(z)$ *nor* $\varphi'(\log f(z))$ *vanishes on* $|z|=r$ *and the inequality*

$$\int_C d\arg d\varphi(\log f(z))>-\pi$$

holds for any arc C *on* $|z|=r$, *then* $f(z)$ *is* p-*valent in* $|z|\leqq r$.

The purpose of this paper is to extend or improve the above results and some of other ones in [**6**], [**7**] and [**13**] by a systematic method. Some of our results may include, in a certain sense, a few new classes of uni- or multivalent functions.

§ 2. Fundamental propositions.

In this paper, we mainly consider the functions belonging to the class which is defined as follows.

DEFINITION 1. A function $f(z)$ is said to be a member of the class $\mathfrak{F}(p, D_z)$, where p is a positive integer and D_z is a simply connected closed domain whose boundary $\partial D_z\equiv C_z$ consists of a piecewise regular curve [**1**, p. 65] and whose interior contains the origin, if $f(z)$ is regular in D_z and has the expansion about the origin

$$f(z)=z^p+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\cdots,$$

and if $f(z)f'(z)\neq 0$ except at the origin in D_z.

Let C'_z denote any continuous, directed sub-arc of $C_z\equiv\partial D_z$, and let C'_w and C_w denote the images of C'_z and C_z by the mapping $w=f(z)$ respectively. The direction of C'_z is always generated, as usual, in the positive sense with respect to D_z, while the direction of C'_w is induced by that of C'_z. The opposite arc [1, p. 65] of an arc C is denoted by $-C$. Throughout this paper the above notations are used in the above sense unless otherwise stated. We note that an arc C'_w always corresponds to a continuous arc $C'_z \subset C_z$, and that in this paper we leave 'a point curve [1, p. 66]' out of consideration (cf. for example (4.15)).

DEFINITION 2. For any fixed D_z and $f(z) \in \mathfrak{F}(p, D_z)$, let $J[C'_w]$ be a functional with the following properties: (a) by a certain rule, a real number is associated with each directed arc C'_w, and (b) if C'_w (directed as before) is a simple closed curve whose interior does not contain the origin and whose direction is clockwise, then $J[C'_w] \geqq 0$. The family of such functionals is denoted by Ω, and such a simple closed curve C'_w as in (b) is denoted by γ.

A non-negative constant is the simplest element of Ω, but it is useless for our purpose if it is used separately. The quantity

$$J_0 \equiv J_0[C'_w] \equiv \int_{-C'_w} d \arg dw - \pi \tag{2.1}$$

has been used by Umezawa or Kaplan for their cases. While also for our case it is seen that (a) for any C'_w, J_0 exists, (b) if there exists a curve γ as in Def. 2 then $J_0[\gamma] \geqq 0$, and that $J_0 \in \Omega$.

Let us also put

$$J_\psi \equiv J_\psi[C'_w] \equiv \int_{-C'_w} d\psi(w)\,, \tag{2.2}$$

where $\psi(w)$ is a real-valued function of bounded variation for each C'_w and is subject to the relation

$$\int_{-\gamma} d\psi(w) \geqq 0\,,$$

when there exists γ as before. Then we see that $J_\psi \in \Omega$.

REMARK 1. The integrals as in (2.1) or (2.2) should be interpreted as Stieltjes integrals (cf. for example [4, 292-295]), and $\psi(w)$ is not necessarily single-valued or continuous and, when C'_z is represented by the equation $z = z(t)$, $t_1 \leqq t \leqq t_2$, $\psi(f(z(t)))$ is not necessarily differentiable for $t_1 \leqq t \leqq t_2$.

In the following section, some examples of such functionals are listed, while we can construct much more examples, by noting the following property which is easily deduced by Def. 2.

$$J_a, J_b \in \Omega \Rightarrow \begin{cases} J_a + J_b \in \Omega\,, \\ J_a \cdot J_b \in \Omega\,, \quad (qJ_a \in \Omega, \text{ where } q \geqq 0)\,, \\ J_a / J_b \in \Omega\,, \quad (J_b \neq 0 \text{ for any } C'_w)\,. \end{cases} \tag{2.3}$$

Now we establish the following:

Proposition 1. *Let* $f(z)\in\mathfrak{F}(p, D_z)$. *If a suitable functional* $J[C'_w]\in\Omega$ *can be found, such that*

$$J[C'_w]<0$$

for every C'_w *(induced by the above* $f(z)$ *and* D_z*), then* $f(z)$ *is* p*-valent in* D_z.

Proof. Suppose that $f(z)$ is at least $(p+1)$-valent in D_z. Then, taking a function $z=\phi(\zeta)$ which maps the unit circle $|\zeta|<1$ onto the interior of D_z one-to-one conformally with $\phi(0)=0$, and noting that the function $f(\phi(\zeta))$ extended to $|\zeta|\leqq 1$ with the boundary values is continuous for $|\zeta|\leqq 1$, we can prove, in a similar way as in [7, 432–434], that in the set of C'_w there exists a simple closed curve γ as in Def. 2. Consequently $J[\gamma]\geqq 0$ since $J[C'_w]\in\Omega$. This contradicts the hypothesis, and the proposition follows.

More concretely (and less generally), we have the following:

Proposition 2. *Let* $f(z)\in\mathfrak{F}(p, D_z)$. *If a suitable functional* $J_\psi\equiv J_\psi[C'_w]$ *as in* (2.2) *can be found, and if the relation*

$$q_0J_0+q_1J_\psi<0\,,$$

holds for every C'_w, *where* q_0, q_1 *are non-negative constants and* J_0 *is that of* (2.1), *then* $f(z)$ *is* p*-valent in* D_z.

Proof. This is clear from Prop. 1 and the relation (2.3).

Remark 2. Even if $p=1$, Prop. 2 is an extension of 'Main criterion' in [7] as is seen from Remark 1.

Thus our problem is reduced to seeking the J's which belong to Ω and which are anyhow effective for our purpose. Each of such functionals we shall call an 'element of criteria', for the present.

§ 3. Elements of criteria.

In this section, some elements of criteria are listed. Previous to this we prepare the following two definitions.

Definition 3. Let Γ be a closed curve and let A, B be complex constants or the point at infinity. Then $A\in U(B,\Gamma)$ means that it is possible to connect the point A with the point B by a continuous curve none of whose points including the end points is on Γ.

Definition 4. Let ($w=f(z)$, C_z, C'_z, C_w and) C'_w be as before. Let A be a complex constant. Then $A\in E(C_w)$ means that $A\notin C'_w$ for every C'_w, and

$$\int_{C'_w} d\arg(w-A)\neq -2\pi\,.$$

Here and in what follows '$A\notin C$' means that A does not lie on C.

Remark 3. $|A|>\max_{z\in C_z}|f(z)|\Rightarrow A\in U(\infty, C_w)\cap E(C_w)$.

(3.1) $$J_1 \equiv J_1[C'_w] \equiv q_0 \int_{-C'_w} d \arg dw - q_0 \pi \in \Omega ,$$

where q_0 is a non-negative constant.

This is clear since $J_1 = q_0 J_0$ with J_0 in (2.1).

(3.2) $$J_{2i} \equiv J_{2i}[C'_w] \equiv \int_{-C'_w} d \arg (w-a_i)^{\lambda_i} \in \Omega ,$$

where λ_i, a_i are complex constants and $a_i \in U(0, C_w) \cup U(\infty, C_w) \cup E(C_w)$.

In fact, if there is γ as in Def. 2, let w_1 and w_2 denote the initial and terminal points of γ respectively. Then, from the assumption on a_i, we see that

$$J_{2i}[\gamma] = \Im[\lambda_i \{\log (w_1 - a_i) - \log (w_2 - a_i)\}] = 0 .$$

(3.2)′ $$J'_{2i} \equiv J'_{2i}[C'_w] \equiv \int_{-C'_w} d \arg (w-a'_i)^{\lambda'_i} \in \Omega ,$$

where λ'_i, a'_i are complex constants and $\Re \lambda'_i \geqq 0$, $a'_i \notin C_w$.

In fact, if there is γ as before, it holds that

$$J'_{2i}[\gamma] = \begin{cases} 0 \text{ if } \gamma \text{ does not contain } a'_i \text{ within,} \\ 2\pi \Re \lambda'_i \geqq 0 \text{ if } \gamma \text{ contains } a'_i \text{ within.} \end{cases}$$

(3.3) $$J_{3i} \equiv J_{3i}[C'_w] \equiv k_i \int_{-C'_w} d \,|\, (w-b_i)^{\mu_i} \,| \in \Omega ,$$

where k_i is a real constant, μ_i, b_i are complex ones, and

$$b_i \in U(0, C_w) \cup U(\infty, C_w) \cup E(C_w) .$$

In fact, for $\gamma \equiv \widehat{w_1 w_2}$ as before,

$$J_{3i}[\gamma] = \exp \{\Re(\mu_i \log (w_1 - b_i))\} - \exp \{\Re(\mu_i \log (w_2 - b_i))\} = 0 .$$

(3.3)′ $$J'_{3i} \equiv J'_{3i}[C'_w] \equiv k'_i \int_{-C'_w} d \,|\, (w-b'_i)^{\mu'_i} \,| \in \Omega ,$$

where k'_i is a real constant, μ'_i, b'_i are complex ones and, $k'_i \Im \mu'_i \leqq 0$, $b'_i \notin C_w$.

In fact, for $\gamma = \widehat{w_1 w_2}$ as before,

$$\begin{aligned} J'_{3i}[\gamma] &= k'_i \exp \{\Re(\mu'_i \log (w_1 - b'_i))\} - k'_i \exp \{\Re(\mu'_i \log (w_2 - b'_i))\} \\ &= k'_i \exp \{\Re(\mu'_i \log (w_1 - b'_i))\}[1 - \exp \{\Re(\mu'_i \times (-2\pi i \text{ or } 0))\}] \end{aligned}$$

according as the point b'_i is inside or outside of γ. Since, $k'_i \Im \mu'_i \leqq 0$, the value of the above equality cannot be negative.

(3.4) $$J_{4i} \equiv J_{4i}[C'_w] \equiv q_i \int_{-C'_w} d \arg F_i(\log (w - A_i)) \in \Omega ,$$

where q_i is a non-negative constant, $F_i(\zeta)$ is an integral function, A_i is a complex constant, $A_i \in U(0, C_w) \cup U(\infty, C_w) \cup E(C_w)$ and $F_i(\log (w - A_i)) \neq 0$ on C_w.

In fact, let γ_ζ be the map of γ as before by $\zeta=\log(w-A_i)$, then γ_ζ is also a simple closed curve which has the negative direction with respect to its interior. Hence we have

$$J_{4i}[\gamma]=q_i\int_{-\gamma_\zeta} d\arg F_i(\zeta)=2nq_i\pi\geqq 0\,,$$

where n is the number of zeros of $F_i(\zeta)$ inside γ_ζ.

$$(3.5)\qquad J_{5i}\equiv J_{5i}[C'_w]\equiv r_i\int_{-C'_w} d\,|\,G_i(\log(w-B_i))\,|\in\Omega\,,$$

where r_i is a real constant, $G_i(\zeta)$ is an integral function, B_i is a complex constant and $B_i\in U(0,C_w)\cup U(\infty,C_w)\cup E(C_w)$.

In fact, as in the above case, the map γ_ζ of γ by $\zeta=\log(w-B_i)$ is a closed curve and the map of γ_ζ by $G_i(\zeta)$ is also a closed curve. Hence

$$J_{5i}[\gamma]=r_i\int_{-\gamma_\zeta} d\,|\,G_i(\zeta)\,|=0\,.$$

§ 4. Some criteria for p-valence.

Now we have the following main theorem.

THEOREM 1. *Let $f(z)\in\mathfrak{F}(p,D_z)$. If the following relation* (4.1) *holds for any arc $C'_z\subset C_z\equiv\partial D_z$, then $f(z)$ is p-valent in D_z:*

$$(4.1)\qquad \int_{-C'_z} d\Big[q_0\arg df(z)+\sum_{i=1}^{n_1}\arg(f(z)-a_i)^{\lambda_i}+\sum_{i=1}^{n_2}k_i\,|\,(f(z)-b_i)^{\mu_i}\,| + \sum_{i=1}^{n_3}q_i\arg F_i(\log(f(z)-A_i))+\sum_{i=1}^{n_4}r_i\,|\,G_i(\log(f(z)-B_i))\,|\Big]<q_0\pi\,,$$

where $F_i(z)$, $G_i(z)$ are integral functions, $F_i(\log(f(z)-A_i))\neq 0$ on C_z, and q_0, q_i are non-negative, k_i, r_i are real, λ_i, μ_i, a_i, b_i, A_i and B_i are all complex constants, and further

(a) $[a_i\in U(0,C_w)\cup U(\infty,C_w)\cup E(C_w)]$ *or* $[a_i\notin C_w$ *and* $\Re\lambda_i\geqq 0]$,

(b) $[b_i\in U(0,C_w)\cup U(\infty,C_w)\cup E(C_w)]$ *or* $[b_i\notin C_w$ *and* $k_i\Im\mu_i\leqq 0]$,

(A) $A_i\in U(0,C_w)\cup U(\infty,C_w)\cup E(C_w)$,

(B) $B_i\in U(0,C_w)\cup U(\infty,C_w)\cup E(C_w)$.

PROOF. Using the same notations as in the previous section, we can write the relation (4.1) in the form

$$(4.2)\qquad J_1+\sum_{i=1}^{n_1}(J_{2i}\text{ or }J'_{2i})+\sum_{i=1}^{n_2}(J_{3i}\text{ or }J'_{3i})+\sum_{i=1}^{n_3}J_{4i}+\sum_{i=1}^{n_4}J_{5i}<0\,.$$

Each term in the above sum belongs to Ω as is shown in § 3, and so, by the relation (2.3), the sum itself belongs to Ω. Consequently, by Prop. 1, $f(z)$ is

p-valent in D_z, and the theorem follows.

COROLLARY 1. *Let $f(z) \in \mathfrak{F}(p, D_z)$. Let $\varphi(z)$ be an integral function such that $\varphi'(\log(f(z)-A)) \neq 0$ on $\partial D_z \equiv C_z$, where A complex, $A \in U(0, C_w) \cup U(\infty, C_w) \cup E(C_w)$. If the inequality*

$$\int_{C'_z} d \arg d\varphi(\log(f(z)-A)) > -\pi \tag{4.3}$$

holds for any arc $C'_z \subset C_z$, then $f(z)$ is p-valent in D_z.

PROOF. In Th. 1, let us put $q_0 = 1$, $\lambda_1 = -1$, $q_1 = 1$, and the other λ_i, k_i, q_i and r_i are all equal to zero, and let us also put $a_1 = A_1 = A$ and $F_1(z) = \varphi'(z)$. Then, after a simple calculation, we have this corollary.

Cor. 1 is an extention of Th. A.

Henceforth, we denote the image of $|z| = r$ under $f(z)$ by C_r, and we abbreviate the part 'for any pair of t_1, t_2 such that $0 \leqq t_1 < 2\pi$, $0 < t_2 - t_1 < 2\pi$' by 'for any $t_1 < t_2$'.

COROLLARY 2. *Let $f(z) \in \mathfrak{F}(p, |z| \leqq r)$. If the inequality*

$$\int_{t_2}^{t_1} \Re\Big\{q\Big(1+\frac{zf''(z)}{f'(z)}\Big) + \sum_{i=1}^{m}\Big(\lambda_i \frac{zf'(z)}{f(z)-a_i}\Big) + i\sum_{i=1}^{n}\Big(k_i\mu_i \frac{|(f(z)-b_i)^{\mu_i}|}{f(z)-b_i} zf'(z)\Big)\Big\} dt < q\pi, \quad z = re^{it}, \tag{4.4}$$

holds for any $t_1 < t_2$, where q is non-negative, k_i are real, λ_i, μ_i, a_i and b_i are all complex, and the conditions (a) *and* (b) *in Th. 1 are satisfied with C_r instead of C_w, then $f(z)$ is p-valent in $|z| \leqq r$.*

PROOF. In Th. 1, let us set $D_z: |z| \leqq r$, $q_0 = q$, and q_i and r_i are all equal to zero. Then a simple calculation leads this corollary.

COROLLARY 3. *Let $f(z) \in \mathfrak{F}(p, |z| \leqq r)$. If there holds*

$$\int_0^{2\pi} \Big|\Re\Big\{1+\frac{zf''(z)}{f'(z)} + \sum_{i=1}^{m}\Big(\lambda_i \frac{zf'(z)}{f(z)-a_i}\Big) + i\sum_{i=1}^{n}\Big(k_i \frac{|f(z)-b_i|}{f(z)-b_i} zf'(z)\Big)\Big\}\Big| dt < 2\pi\Big\{1+p+\sum_{i=1}^{m}(n(a_i)\Re\lambda_i)\Big\}, \quad z = re^{it}, \tag{4.5}$$

where k_i are real, λ_i, a_i, b_i are complex, and

$$[a_i \in U(0, C_r) \cup U(\infty, C_r) \cup E(C_r)] \text{ or } [a_i \notin C_r \text{ and } \Re\lambda_i \geqq 0], \quad b_i \notin C_r,$$

and

$$2\sum_{i=1}^{m}(n(a_i)\Re\lambda_i) > -(1+2p),$$

here $n(a_i)$ denotes the number of a_i-points of $f(z)$ in $|z| < r$; then $f(z)$ is p-valent in $|z| \leqq r$.

PROOF. In Cor. 2, let us put $q = 1$ and μ_i are all equal to 1, then Cor. 3 follows in a similar way to the proof of Cor. 2 in [13].

Cor. 3 is an extension of Cor. 2 in [**13**].

COROLLARY 4. *Let* $f(z)\in\mathfrak{F}(p, D_z)$. *If there holds, for any arc* $C'_z\subset C_z\equiv\partial D_z$,

$$\int_{C'_z}[d\arg df(z)+d\arg(f(z)-A)^{\lambda}]>-\pi\,, \tag{4.6}$$

where λ, A *are complex constants and* $A\in U(0, C_w)\cup U(\infty, C_w)\cup E(C_w)$ *or* $[A\notin C_w$ *and* $\Re\lambda\geqq 0]$, *then* $f(z)$ *is p-valent in* D_z.

PROOF. In Th. 1, let us put $q_0=1$, $a_1=A$, $\lambda_1=\lambda$ and the other λ_i, k_i, q_i and r_i are all equal to zero. Then the corollary follows readily.

Cor. 4 is an extension of Cor. 1 in [**13**] and 'a fortiori' of Th. 2 in [**7**].

COROLLARY 5. *Let* $f(z)\in\mathfrak{F}(p, |z|\leqq r)$. *If there holds, for any* $t_1<t_2$,

$$\int_{t_2}^{t_1}\Re\Big(1+z\frac{f''(z)}{f'(z)}+ik\frac{|f(z)-A|}{f(z)-A}zf'(z)\Big)dt<\pi\,,\ z=re^{it}\,, \tag{4.7}$$

where k real and A complex such that $A\notin C_r$, *then* $f(z)$ *is p-valent in* $|z|\leqq r$.

PROOF. In Cor. 2, let us put $q=1$, $\mu_1=1$, $k_1=k$ and the other k_i, λ_i are all equal to zero. Then Cor. 5 follows readily.

Cor. 5 is an extension of Th. 2 in [**6**] (even if $p=1$). In fact, in Cor. 5 let us set $A=\rho e^{i(3\pi/2-\omega)}$, $\rho>0$, ω real, and $|A|>\max_{z\in D_z}|f(z)|$. Then by tending $\rho\to+\infty$ we have the following:

COROLLARY 6. *Let* $f(z)\in\mathfrak{F}(p, |z|\leqq r)$. *If there holds, for any* $t_1<t_2$,

$$\int_{t_2}^{t_1}\Re\Big(1+z\frac{f''(z)}{f'(z)}+ke^{i\omega}zf'(z)\Big)dt<\pi,\ z=re^{it}\,, \tag{4.8}$$

where k, ω *real, then* $f(z)$ *is p-valent in* $|z|\leqq r$.

COROLLARY 7. *Let* $f(z)\in\mathfrak{F}(p, D_z)$. *If there holds*

$$\int_C d\arg df(z)>-\pi\,, \tag{4.9}$$

for all arcs $C\subset C_z\equiv\partial D_z$, *then* $f(z)$ *is p-valent in* D_z, *and is 'at most* π*-concave'* [**15**] *on* C_z.

PROOF. This is obtained by Cor. 4 by setting $\lambda=0$.

The special case of Cor. 7 in which $p=1$ and C_z is a regular curve is essentially equivalent to Kaplan-Umezawa's theorem [**5**], [**18**].

COROLLARY 8. *Let* $f(z)\in\mathfrak{F}(p, |z|\leqq r)$. *If there holds*

$$\Re\Big\{\sum_{i=1}^{m}\lambda_i\frac{zf'(z)}{f(z)-a_i}+i\sum_{i=1}^{n}\Big(k_i\mu_i\frac{|(f(z)-b_i)^{\mu_i}|}{f(z)-b_i}zf'(z)\Big)\Big\}>0,\ |z|=r\,, \tag{4.10}$$

where λ_i, k_i, μ_i, a_i *and* b_i *are constants as in Cor.* 2, *then* $f(z)$ *is p-valent in* $|z|\leqq r$.

PROOF. In Cor. 2. let us put $q=0$. Then Cor. 8 follows easily.

COROLLARY 9. *Let* $f(z)\in\mathfrak{F}(p, |z|\leqq r)$. *If there holds*

$$\Re\left\{\lambda\frac{zf'(z)}{f(z)-A}+ik\frac{|f(z)-B|}{f(z)-B}zf'(z)\right\}>0,\ |z|=r, \tag{4.11}$$

where k is real, λ, A and B are complex, $A\in U(0, C_r)\cup U(\infty, C_r)\cup E(C_r)$ or $[A\notin C_r$ and $\Re\lambda\geqq 0]$ and $B\notin C_r$; then $f(z)$ is p-valent in $|z|\leqq r$.

PROOF. In Cor. 8, let us put $\mu_1=1$, $\lambda_1=\lambda$, $k_1=k$ and the other λ_i, k_i are all equal to zero. Then Cor. 9 follows readily.

COROLLARY 10. *Let $f(z)\in\mathfrak{F}(p, |z|\leqq r)$. If there holds*

$$\Re\sum_{i=1}^{n}\left(\lambda_i\frac{zf'(z)}{f(z)-a_i}\right)>0,\quad |z|=r, \tag{4.12}$$

for complex constants λ_i, a_i subject to (a) *in Th.* 1 *with C_r instead of C_w, then $f(z)$ is p-valent in $|z|\leqq r$.*

PROOF. In Cor. 8, let us put $k_i=0$, $i=1, 2, \cdots, n$. Then we have Cor. 10.

COROLLARY 11. *Let $f(z)=z^p+\cdots$ be regular in $|z|<r$. If for some real α, $|\alpha|<\pi/2$, the relation*

$$\Re\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)>0,\quad |z|<r, \tag{4.13}$$

holds, then $f(z)$ is p-valent and spiral-like in $|z|<r$, [7], [8].

PROOF. The assumption shows that neither $f(z)$ nor $f'(z)$ vanishes for $0<|z|\leqq\rho$, where ρ is an arbitrary number such that $0<\rho<r$. Hence we can appeal to Cor. 10 with $n=1$, $a_1=0$ and $\lambda_1=e^{i\alpha}$ to conclude that $f(z)$ is p-valent in $|z|\leqq\rho$. The spiral-likeness is due to the definition; cf. [3], [17] or Def. 5 which will later be stated. The inequality $|\alpha|<\pi/2$ is a necessary condition that (4.13) should hold. Thus the corollary follows.

COROLLARY 12. *Let $f(z)\in\mathfrak{F}(p, |z|\leqq r)$. If the relation*

$$\Re\frac{zf'(z)}{f(z)-A}>k\Im\frac{zf'(z)}{f(z)-A},\quad |z|=r, \tag{4.14}$$

holds for k real and A complex, then $f(z)$ is p-valent in $|z|\leqq r$.

PROOF. Our assumption shows that $f(z)\neq A$ on $|z|=r$. Hence we can appeal to Cor. 10 with $n=1$, $\lambda_1=1+ki$ and $a_1=A$ to conclude that $f(z)$ is p-valent in $|z|\leqq r$.

Now, setting $p=1$ for the sake of simplicity, we give a few examples for some of our results.

EXAMPLE 1. Let D_z be the rectangle $|\Re z|\leqq M$ $(M>0)$, $|\Im z|\leqq\pi-\varepsilon$ $(0<\varepsilon<\pi)$, and let $f(z)\equiv e^z-1=z+\cdots$. If we put $\varphi(z)\equiv z$ and $A=-1$, then we have the following relations.

$$f(z)f'(z)\neq 0 \text{ for } z\neq 0 \text{ in } D_z,\ \varphi'(\log(f(z)-A))\neq 0 \text{ on } C_z\equiv\partial D_z,$$

and for any arc $C\subset C_z$

$$\int_C d\arg(f(z)-A)=\int_C d\Im z \neq -2\pi \text{ i.e. } A=-1\in E(C_w),$$

and

$$\int_C d\arg d\varphi(\log(f(z)-A))=\int_C d\arg dz \geqq 0 > -\pi .$$

Hence by Cor. 1, $f(z)$ is univalent in D_z.

EXAMPLE 2. Let D_w be the closed domain whose boundary curve C_w consists of two curves

$$C_1: \quad \rho=1-3\theta/4,\ 0\geqq\theta\geqq-2\pi ,$$
$$C_2: \quad \rho=1+4\pi/3-2\theta/3,\ -2\pi\leqq\theta\leqq 2\pi ,$$

where ρ, θ are the polar coordinates of a point w. Let the direction of C_w, as usual, generate to be positive with respect to its interior. Then there holds

$$\int_{C'_w}(d\arg w+d|w|)>0 \tag{4.15}$$

for every arc (different from a point) $C'_w\subset C_w$. Let D^*_w be a domain (open) whose interior contains D_w and whose boundary consists of a bounded Jordan curve. Let $w=f(z)=z+\cdots$ be the function which maps the circle $|z|<r$ with a suitable r one-to-one conformally onto the domain D^*_w, and let C_z be the map of C_w by $z=f^{-1}(w)$, where f^{-1} is the inverse function of f, and further let D_z be the closed domain bounded by C_z. Then, with these $f(z)$, C_z and D_z, a special case of the assumption of Th. 1 which is similar to that of Cor. 9 is satisfied since we have (4.15) for $w=f(z)$.

Clearly $f(z)$ is neither starlike [**2**], [**12**] nor close-to-convex (i.e. at most π-concave [**15**]) on the directed curve C_z. Now, in order to compare with the spiral-like case, we prepare the following:

DEFINITION 5. Let Γ denote a directed rectifiable curve. Suppose that $f(z)$ is regular and $f(z)\neq A$ on Γ and that $\lambda\neq 0$ (A, λ complex). Then $f(z)$ is said to be spiral-like with λ and with respect to A on Γ if

$$\int_{\Gamma'} d\arg(f(z)-A)^\lambda \geqq 0 \tag{4.16}$$

for all arcs $\Gamma'\subset\Gamma$. If $A=0$, we shall omit reference to A and say, briefly, that $f(z)$ is spiral-like (with λ) on Γ [**3**], [**17**].

Now, let C be the part of C_1 such that $-\frac{1}{2}\pi\geqq\theta\geqq-\pi$. The direction of the curve is that generated by decreasing θ. Then we have

$$\int_C (d\arg w+d|w|)=-\frac{\pi}{2}+\frac{2}{3}\pi=\frac{1}{6}\pi .$$

On the other hand, since

$$d \log |w| = d|w|/|w| < d|w|/3, \; w \in C,$$

we have that

$$\int_C d \arg w^{1+i} = \int_C (d \arg w + d \log |w|) < -\frac{5}{18}\pi .$$

Accordingly, $f(z)$ is not spiral-like with $(1+i)$ on C_z.

EXAMPLE 3. Let D_w be the complement of the domain

$$\{|w|>1\} \cup \{|\arg(w-1/3)-\pi/2|<\varepsilon\} \cup \{|(\arg(w+1/2)-\pi|<\varepsilon\},$$

where $\varepsilon > 0$ is a sufficiently small constant. Let us denote the boundary of D_w by C_w. Then there holds the inequality

$$\int_{C'_w} \left(d \arg w + d \arg \left(w - \frac{2}{3}\right)\right) > 0 \tag{4.17}$$

for every arc (different from a point) $C'_w \subset C_w$. This may be proved by noting that either the boundary of the domain

$$|\arg(w-1/3)-\pi/2| < \varepsilon$$

or the two points $w=0$ and $w=2/3$ are symmetric with respect to the straight line $\Re w = 1/3$. Consider (one of) the function $f(z)$ and the curve C_z which are obtained from the closed domain D_w as in the above example. Then, for these $w=f(z)$ and C_z, we have (4.17) a special case of (4.1). Clearly, on C_z, $f(z)$ is starlike neither with respect to the point $w=0$ nor with respect to $w=2/3$, though it is starlike with respect to the point $w=1/3-\delta i$, where $\delta > 0$ is a sufficiently small constant.

§ 5. Some remarks for the above results.

In Th. 1, if only one element of criteria, for example J_1, is used, we have the following slightly more precise result.

THEOREM 2. *Let $f(z) \in \mathfrak{F}(p, D_z)$ (without the assumption $f(z) \neq 0$). If there holds the relation*

$$\int_{C'_z} d \arg df(z) \geqq -\pi , \tag{5.1}$$

for any arc $C'_z \subset C_z \equiv \partial D_z$, then $f(z)$ is p-valent in D_z.

PROOF. If $f(z)$ is at least $(p+1)$-valent in D_z, then as in the proof of Prop. 1, there exists such a simple closed piecewise regular curve γ which is the image of a curve $C_z^* \subset C_z$ by $w=f(z)$ and for which we have the inequality

$$\int_{-\gamma} d \arg dw \geqq \pi .$$

Here we note that, from the geometrical property of γ, there also exists a sub-curve C of γ for which we have

$$\int_{-C} d \arg dw > \pi . \tag{5.2}$$

From this fact the theorem follows easily.

Th. 2 is more general or precise than Umezawa-Kaplan's result to which we referred before or than the result due to Reade [**10**, p. 255].

Next we refer to Cor. 5 from which the following corollary follows easily.

COROLLARY 13. *Let* $f(z) \in \mathfrak{F}(1, |z| \leqq r)$. *If there holds*

$$\Re\left\{1+z\frac{f''(z)}{f'(z)}+ik\frac{|f(z)-A|}{f(z)-A}zf'(z)\right\}>0,\ |z|=r, \tag{5.3}$$

where k *real,* A *complex and* $A \notin C_r$, *then* $f(z)$ *is univalent in* $|z| \leqq r$.

In the above corollary, if (5.3) holds then we have

$$\int_C \{d \arg df(z)+kd|f(z)-A|\}>0 \tag{5.4}$$

for every arc C on $|z|=r$. Now, let w_1, w_2, if exist, be the intersections of C_r and the circle $K_\rho : |w-A|=\rho$. Then, since (5.4) holds, the argument of the tangent vector of C_r at w_2 is larger than the previous value at w_1. This must hold for any $\rho, 0<\rho<+\infty$. Now we put $A=-ae^{i(\pi/2-\omega)}$, $a>0$, ω real, and we consider the case in which (5.3) remains for $a \to +\infty$. In this case, if we make $a \to +\infty$, then, for example, the part of $K_a : |w-A|=a$ inside C_r tends to a part of the straight line $L : \Im(we^{i\omega})=0$, and from the fact stated above, it is seen that C_r has no intersecting points with any line parallel to L more than two. Moreover we have

$$i|f(z)-A|/(f(z)-A) \to e^{i\omega}, \quad \text{when} \quad a \to +\infty .$$

Thus, we have the following:

COROLLARY 14. *Let* $f(z) \in \mathfrak{F}(1, |z| \leqq r)$. *If there folds*

$$\Re\left\{1+z\frac{f''(z)}{f'(z)}+ke^{i\omega}zf'(z)\right\}>0,\ |z|=r, \tag{5.5}$$

where k, ω *real, then* $f(z)$ *is univalent, convex in one direction* (cf. [**11**]) *in* $|z| \leqq r$, *and this direction coincides with that of the vector representing* $e^{i(\pi-\omega)}$.

Cor. 14 is equivalent to Th. 3 in [**6**, p. 10] and which has been generalized as Cor. 13 in a certain sense. But it is unnatural. Indeed, under the assumption of Cor. 13, there is a case such that C_r is cut by some K_ρ as before in more than two points, as the case $f(z) \equiv z$ and $k=A=0$. So, we generalize the definition of the class of functions convex in one direction, which is denoted by (C), as follows.

DEFINITION 6. We shall say $f(z) \in C(A)$ if $f(z)$ is regular for $|z| \leqq r$, $f(0)=0$, and if C_r as before is cut by any one of circles of center $A \in U(\infty, C_r)$ in not more than two points. We interpret as $C(-\infty)=(C)$-with the direction of the vector i.

Using the above definition we have the following:

THEOREM 3. *Let* $f(z)\in\mathfrak{F}(1, |z|\leqq r)$. *If there holds*

$$\mathfrak{R}\left\{1+z\frac{f''(z)}{f'(z)}+(\kappa i-1)\frac{zf'(z)}{f(z)-A}\right\}>0,\ |z|=r, \tag{5.6}$$

for suitable constants κ *and* A *such that* κ *real and* $A\in U(\infty, C_r)$; *then* $f(z)$ *belongs to* $C(A)$ *and is univalent in* $|z|\leqq r$.

PROOF. Since we have (5.6), the relation (4.6) holds for $\lambda=\kappa i-1$ and all arcs C_z' on $|z|=r$. Hence by Cor. 4, $f(z)$ is univalent in $|z|\leqq r$. Now let us set

$$g(z)=-A\{\log(f(z)-A)-\log(-A)\}=z+\cdots, \tag{5.7}$$

then we see that $g(z)\in\mathfrak{F}(1, |z|\leqq r)$ and a simple calculation shows that (5.6) is reduced to

$$\mathfrak{R}\left\{1+z\frac{g''(z)}{g'(z)}-\frac{\kappa i}{A}zg'(z)\right\}>0,\ |z|=r. \tag{5.8}$$

Hence by Cor. 14, $g(z)$ is convex in the direction of the vector $e^{i(\pi-\omega)}$, where $\omega=\pi/2-\arg(-A)$. On the other hand, the part of the circles $|w-A|=\rho$ inside C_r is univalently mapped by the function $-A\{\log(w-A)-\log(-A)\}$ onto the corresponding part of the lines parallel to the above vector $e^{i(\pi-\omega)}$. Noting the above facts we can deduce $f(z)\in C(A)$. Thus, the theorem follows.

EXAMPLE 4. Let $f(z)\equiv e^z-1$, $r=\pi-\varepsilon$ $(0<\varepsilon<\pi)$ and $A=-1$. Then (5.6) is reduced to

$$\mathfrak{R}(1+\kappa iz)>0,\ |z|=r, \tag{5.9}$$

which holds for a sufficiently small $|\kappa|$, and so we see $f(z)\in C(-1)$.

REMARK 4. In Th. 3, set $A=-aie^{-i\omega}$ $(a>0)$ and $\kappa=ka$, then by making $a\to+\infty$ we again have Cor. 14. Th. 3 is more natural than Cor. 13 as an extension of Cor. 14.

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