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On Pontrjagin classes modulo q

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Introduction. In this paper we shall deal with some properties of Pontrjagin classes modulo q, where q denotes some prime number larger than 2. In [1] and [2] Massey obtained many results concerning the vanishment of Stiefel-Whitney classes. We shall apply the Massey's method for the case of Pontrjagin classes modulo q. For this purpose we shall make use of the Hirzebruch's relation ([3], [4]) which is an analog of the Wu's relation in the case of Stiefel-Whitney classes.

§ **1.** Let q be a prime number larger than 2 and let X_n be a compact orientable differentiable n-manifold. For any cohomology class $v \in H^{n-2r(q-1)}(X_n, Z_q)$ it holds that

$$\mathcal{P}_q^r v = s_q^r v \qquad ([3],\ [4]) \tag{1.1}$$

where $\mathcal{P}_q^r$ denotes the Steenrod power

$$\mathcal{P}_q^r: \quad H^i(X_n, Z_q) \rightarrow H^{i+2r(q-1)}(X_n, Z_q) \tag{1.2}$$

and s_q^r denotes a mod q polynomial of Pontrjagin classes:

$$s_q^r = q^r L_{\frac{1}{2}r(q-1)}(p_1, \cdots, p_t) \quad \mod q, \quad t = \frac{1}{2}r(q-1), \tag{1.3}$$

where

$$\sum_{j \geqq 0} L_j(p_1, \cdots, p_j) = \prod_i \frac{\sqrt{\gamma_i}}{\mathrm{tgh}\sqrt{\gamma_i}}, \tag{1.4}$$

$$p = \sum_{i \geqq 0} p_i = \prod_i (1+\gamma_i) \tag{1.5}$$

and

$$p_i \in H^{4i}(X_n, Z). \tag{1.6}$$

It is needless to say that

$$s_q^r \in H^{2r(q-1)}(X_n, Z_q). \tag{1.7}$$

We put

$$\sum_{j \geqq 0} b_{q,j} = \prod_i (1+\gamma_i^l), \qquad b_{q,j} \in H^{4j}(X_n, Z_q) \tag{1.8}$$

where

(1.9) $$l=\frac{1}{2}(q-1).$$

It is known that

(1.10) $$b_{q,j}=\sum \mathcal{P}_q^i s_q^r \qquad \bmod q$$

where the sum is extended over the set of all the pairs (i, r) such that

(1.11) $$2j=(i+r)(q-1) \qquad ([4]).$$

In the case $q=3$ (1.8) takes the form

(1.12) $$\sum_{j\geqq 0} b_{3,j}=\prod_i (1+\gamma_i)=\sum_{i\geqq 0} p_i$$

and we have from (1.10)

(1.13) $$p_j=\sum_{j=i+r} \mathcal{P}_3^i s_3^r \qquad \bmod 3.$$

We define $\bar{b}_{q,j}$ and $\bar{s}_q^r$ by

(1.14) $$(\sum_{r\geqq 0} s_q^r)(\sum_{r\geqq 0} \bar{s}_q^r)=1, \qquad \bar{s}_q^r \in H^{2r(q-1)}(X_n, Z_q)$$

and

(1.15) $$(\sum_{j\geqq 0} b_{q,j})(\sum_{j\geqq 0} \bar{b}_{q,j})=1, \qquad \bar{b}_{q,j} \in H^{4j}(X_n, Z_q).$$

We recall the following relations: ([5])

(1.16)
$$\begin{cases} \text{(i)} & \mathcal{P}_q^0 = \text{identity}, \\ \text{(ii)} & \mathcal{P}_q^r(uv)=\sum_{s=0}^{r} \mathcal{P}_q^s u \mathcal{P}_q^{r-s} v, \\ \text{(iii)} & \mathcal{P}_q^r \mathcal{P}_q^s=\sum_i (-1)^{r+i}\binom{(s-i)(q-1)-1}{r-qi} \mathcal{P}_q^{r+s-i} \mathcal{P}_q^i \quad (r<qs), \\ \text{(iv)} & \mathcal{P}_q^i u_k=0, \quad 2i>k, \quad u_k \in H^k(X_n, Z_q). \end{cases}$$

We have from (1.14), (1.15), (1.10) and (1.16)

(1.17) $$\bar{b}_{q,j}=\sum_{2j=(i+r)(q-1)} \mathcal{P}_q^i \bar{s}_q^r \qquad \bmod q$$

because

(1.18)
$$\begin{aligned} 1 &= (\sum_{i\geqq 0} \mathcal{P}^i)((\sum_{r\geqq 0} s_q^r)(\sum_{\bar{r}\geqq 0} \bar{s}_q^{\bar{r}})) \\ &= ((\sum \mathcal{P}^i)(\sum s_q^r))((\sum \mathcal{P}^i)(\sum \bar{s}_q^{\bar{r}}))=(\sum_{j\geqq 0} b_{q,j})(\sum_{i,\bar{r}\geqq 0} \mathcal{P}^i \bar{s}_q^{\bar{r}}). \end{aligned}$$

§ **2.** Let us prove

LEMMA 1. *For any* $x \in H^{n-4k}(X_n, Z_q)$ $(0<k<n/4)$ *it holds that*

(2.1) $$x\bar{b}_{q,k}=-\sum_{r=1}^{2k/(q-1)} \mathcal{P}_q^r x \bar{b}_{q,k-r(q-1)/2} \qquad \bmod q.$$

PROOF. We have from (1.17)

$$(2.2)\qquad \bar{b}_{q,j}=\sum_{2j=(i+r)(q-1)}\mathcal{P}_q^i\bar{s}_q^r=\bar{s}_q^{2j/(q-1)}+\sum_{i=1}^{2j/(q-1)}\mathcal{P}_q^i\bar{s}_q^{2j/(q-1)-i}.$$

On the other hand we have from (1.14)

$$(2.3)\qquad 0=\bar{s}_q^{2j/(q-1)}+\sum_{i=1}^{2j/(q-1)}s_q^i\bar{s}_q^{2j/(q-1)-i}.$$

We have from (2.2) and (2.3)

$$(2.4)\qquad \bar{b}_{q,j}=\sum_{i=1}^{2j/(q-1)}(\mathcal{P}_q^i\bar{s}_q^{2j/(q-1)-i}-s_q^i\bar{s}_q^{2j/(q-1)-i})$$

which leads to

$$(2.5)\qquad x\bar{b}_{q,j}=\sum_{i=1}^{2j/(q-1)}(x\mathcal{P}_q^i\bar{s}_q^{2j/(q-1)-i}-xs_q^i\bar{s}_q^{2j/(q-1)-i}).$$

Now we put $j=k$. Then we have from (1.1) and (1.16) (iii)

$$(2.6)\qquad xs_q^i\bar{s}_q^{2k/(q-1)-i}=s_q^ix\bar{s}_q^{2k/(q-1)-i}=\mathcal{P}_q^i(x\bar{s}_q^{2k/(q-1)-i})$$
$$=\sum_{r=0}^{i}\mathcal{P}_q^rx\cdot\mathcal{P}_q^{i-r}\bar{s}_q^{2k/(q-1)-i}.$$

We have from (2.5) and (2.6)

$$(2.7)\qquad x\bar{b}_{q,k}=\sum_{i=1}^{2k/(q-1)}(x\mathcal{P}_q^i\bar{s}_q^{2k/(q-1)-i}-\sum_{r=0}^{i}\mathcal{P}_q^rx\mathcal{P}_q^{i-r}\bar{s}_q^{2k/(q-1)-i})$$
$$=\sum_{i=1}^{2k/(q-1)}(-\sum_{r=1}^{i}\mathcal{P}_q^rx\mathcal{P}_q^{i-r}\bar{s}_q^{2k/(q-1)-i})$$
$$=-\sum_{r=1}^{2k/(q-1)}\mathcal{P}_q^rx\sum_{i=r}^{2k/(q-1)}\mathcal{P}_q^{i-r}\bar{s}_q^{2k/(q-1)-i}$$
$$=-\sum_{r=1}^{2k/(q-1)}\mathcal{P}_q^rx\bar{b}_{q,k-r(q-1)/2}.\qquad \text{Q. E.}$$

The repeated use of (2.1) implies

$$(2.8)\qquad x\bar{b}_{q,k}=\sum\mathcal{P}_q^Ix\qquad \mod q$$

where $\mathcal{P}_q^I$ runs over the set of all iterated powers.

If $4k=n$, *then* $\bar{b}_{q,k}=0$ mod q, because we have from (2.2) and (1.1)

$$(2.9)\qquad \bar{b}_{q,k}=\bar{s}_q^{2k/(q-1)}+\sum_{i=1}^{2k/(q-1)}\mathcal{P}_q^i\bar{s}_q^{2k/(q-1)-i}$$
$$=\bar{s}_q^{2k/(q-1)}+\sum_{i=1}^{2k/(q-1)}s_q^i\bar{s}_q^{2k/(q-1)-i}=0\qquad \mod q.$$

Moreover it holds that $\bar{b}_{q,k}=0$ mod q, if $n-4k=1$.

For, if $\bar{b}_{q,k}\neq 0$ mod q there exists some $x\in H^1(X_n, Z_q)$ and

$$(2.10)\qquad x\bar{b}_{q,k}\neq 0\qquad \mod q.$$

We have from (2.8)

$$(2.11)\qquad \mathcal{P}_q^Ix\neq 0\qquad \mod q$$

for some iterated Steenrod power $\mathcal{P}_q^I$. However it is impossible from (1.16) (iv).

§ **3.** By means of (1.16) (iii) any iterated Steenrod power can be expressed as a sum of admissible powers:

(3.1) $$\mathcal{P}_q^I=\mathcal{P}_q^{i_1}\cdots\mathcal{P}_q^{i_r}\qquad (i_1\geqq qi_2,\ i_2\geqq qi_3,\cdots,i_{r-1}\geqq qi_r)\,.$$

We put

(3.2) $$\begin{cases} n(I)=i_1+\cdots+i_r\,, \\ e(I)=\alpha_1+\cdots+\alpha_r \end{cases}$$

where

(3.3) $$i_1=qi_2+\alpha_1\,,\quad i_2=qi_3+\alpha_2,\cdots,i_{r-1}=qi_r+\alpha_{r-1}\,,\quad i_r=\alpha_r\,.$$

We have from (3.3)

(3.4) $$n(I)=e(I)+q(n(I)-i_1)$$

which leads to

(3.5) $$qi_1=e(I)+(q-1)n(I)\,.$$

LEMMA 2. *If* $s=$ *degree* $x<2e(I)$, *then* $\mathcal{P}_q^I x=0$.

PROOF. We have from (3.5)

(3.6) $$i_1-(q-1)i_2-\cdots-(q-1)i_r=e(I)>s/2\,,$$

from which we have

(3.7) $$2i_1-2(q-1)i_2-\cdots-2(q-1)i_r=2e(I)>s\,,$$

i. e.

(3.8) $$2i_1>2(q-1)i_2+\cdots+2(q-1)i_r+s\,.$$

Hence we have

(3.9) $$2i_1>\text{degree}\,(\mathcal{P}_q^{i_2}\cdots\mathcal{P}_q^{i_r}x)\,.$$

We have from (3.9) and (1.16) (iv)

(3.10) $$\mathcal{P}_q^I x=0\,.$$ Q. E. D.

Next we consider the case where

(3.11) $$\bar{b}_{q,k}\neq 0\quad \bmod q\qquad s=n-4k>1\,.$$

In this case we have

(3.12) $$x\bar{b}_{q,k}\neq 0\quad \bmod q\qquad\text{and}\quad \mathcal{P}_q^I x\neq 0$$

for some $x\in H^s(X_n, Z_q)$ and some admissible $\mathcal{P}_q^I$. Thus we have

THEOREM 1. *Let* X_n *be a compact orientable differentiable manifold. If* $\bar{b}_{q,k}\neq 0 \bmod q$ $(s=n-4k>1)$, *then* $\mathcal{P}_q^I x\neq 0$ *holds for some admissible iterated Steenrod power* $\mathcal{P}_q^I$ *and some* $x\in H^s(X_n, Z_q)$.

By means of Lemma 2 it suffices to deal with the case where

(3.14) $$e(I) \leqq s/2 .$$

We have from (3.5), (3.3) and (3.13)

(3.15) $$n = \text{degree } \mathcal{P}_q^I x = 2(q-1)n(I) + s = 2qi_1 - 2e(I) + s$$
$$= 2(q\alpha_1 + q^2\alpha_2 + \cdots + q^r\alpha_r) - 2e(I) + s .$$

First we consider the case where $2e(I) \leqq s-1$. We put

(3.16) $$\alpha_0 + 2e(I) = s-1 , \qquad 0 \leqq \alpha_0 \leqq s-3 .$$

Then (3.15) becomes

(3.17) $$n = \{2(q\alpha_1 + q^2\alpha_2 + \cdots + q^r\alpha_r) + q\alpha_0\} + 1 - (q-1)\alpha_0$$
$$= (q^{h_1} + q^{h_2} + \cdots + q^{h_{s-1}}) - (q-1)\alpha_0 + 1$$
$$(h_1 \geqq h_2 \geqq \cdots \geqq h_{s-1} \geqq 1)$$

because the number of q in the $\{ \quad \}$ of (3.17) is equal to

(3.18) $$2(\alpha_1 + \cdots + \alpha_r) + \alpha_0 = 2e(I) + \alpha_0 = s-1 .$$

Next we consider the case where $2e(I) = s$. In this case we have from (3.15)

(3.19) $$n = 2(q\alpha_1 + q^2\alpha_2 + \cdots + q^r\alpha_r) = 2(q^{h_1} + \cdots + q^{h_{s/2}})$$
$$(h_1 \geqq h_2 \geqq \cdots \geqq h_{s/2} \geqq 1) .$$

Thus we have

THEOREM 2. *Let X_n be a compact orientable differentiable n-manifold. If $\bar{b}_{q,k} \not\equiv 0 \bmod q$ $(s = n - 4k > 1)$, then we have either*

$$n = (q^{h_1} + \cdots + q^{h_{s-1}}) - (q-1)\alpha_0 + 1 \qquad (h_1 \geqq h_2 \geqq \cdots \geqq h_{s-1} \geqq 1) ,$$

$(s-3 \geqq \alpha_0 \geqq 0)$ *or*

$$n = 2(q^{h_1} + \cdots + q^{h_{s/2}}) \qquad (h_1 \geqq \cdots \geqq h_{s/2} \geqq 1) .$$

§ 4. Define the dual-Pontrjagin classes by

(4.1) $$1 = (\sum_{k \geqq 0} \bar{p}_k)(\sum_{k \geqq 0} p_k) , \qquad \bar{p}_k \in H^{4k}(X_n, Z) .$$

We have from (1.12) and (1.15)

(4.2) $$\bar{b}_{3,k} = \bar{p}_k .$$

We consider the case $q = 5$. In this case we have

(4.3) $$\begin{cases} \bar{b}_{5,1} = \bar{p}_1^2 - 2\bar{p}_2 , \\ \bar{b}_{5,2} = 2\bar{p}_4 - 2\bar{p}_3\bar{p}_1 + \bar{p}_2^2 . \end{cases}$$

COROLLARY. *If $n = 19$, then we have $\bar{b}_{5,2} = 0 \bmod 5$.*

PROOF. If $\bar{b}_{5,2} \not\equiv 0 \bmod 5$, we have from Theorem 1

(4.4) $$\mathcal{P}_5^I x_3 \not\equiv 0 \mod 5 , \qquad x_3 \in H^3(X_{19}, Z_5)$$

for some admissible $\mathcal{P}_5^I$. However the only admissible $\mathcal{P}_5^I$ is $\mathcal{P}_5^2$ and we have from (1.16) (iv)

$$\mathcal{P}_5^2 x_3 = 0 \quad \text{mod } 5 \tag{4.5}$$

which contradicts (4.4). Q. E. D.

We can prove this corollary by Theorem 2 too. The same thing holds for the case $n=18$.

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