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On a problem of Alexandroff concerning the dimension of product spaces I.

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§ 1. Introduction.

Let X and Y be finite dimensional compact metric spaces. It is well known that the equality

$$(\boldsymbol{A}) \qquad \dim(X\times Y)=\dim X+\dim Y$$

does not hold generally. The known cases for which the equality ($\boldsymbol{A}$) holds are as follows:

1) X is a polytope and Y is any space [1].
2) X is a 1-dimensional space and Y is any space [10].
3) X is a 2-dimensional ANR and Y is any space [12].
4) X is an n-dimensional ANR containing a point which is HL^{n-2} and $(n-1)$-HS, and Y is an m-dimensional ANR containing a point which is HL^{m-2} and $(m-1)$-HS [12].
5) X and Y are spaces which have the property Δ in the sense of K. Borsuk [6].

The following problem is proposed by P. Alexandroff [1], Problem XII, p. 236 (cf. Hurewicz and Wallman [11], p. 34).

(*) *To determine a finite dimensional compact metric space X such that, whenever Y is a compact metric space, the equality* ($\boldsymbol{A}$) *holds.*

In this paper, we shall give an answer to this problem by determining a necessary and sufficient condition which a compact metric space X should satisfy in order that the equality (A) holds for every compact metric space Y.

A sequence $\mathfrak{a}=\{q_1, q_2,\cdots\}$ of positive integers is called a *k-sequence* if q_i is a divisor of q_{i+1}, $i=1,2,\cdots$, and $q_i>1$ for some i. There exists a natural homomorphism $h(\mathfrak{a}, i)$ from $Z_{q_{i+1}}$ onto Z_{q_i}, $i=1,2,\cdots$, where Z_q means the factor group Z/qZ and Z means the additive group of all integers. Let us denote by $Z(\mathfrak{a})$ the inverse limit group of the inverse system $\{Z_{q_i}: h(\mathfrak{a}, i)\}$. Let (X, A) be a pair of compact metric spaces. We shall denote by $H_n(X, A: Z(\mathfrak{a}))$ the n-dimensional Čech homology group of (X, A) with $Z(\mathfrak{a})$ as a coefficient group. Consider the following property $\boldsymbol{P}$ for an n-dimensional compact metric space X.

P. *For every k-sequence* $\mathfrak{a}$ *there exists a closed subset* $A_{\mathfrak{a}}$ *of* X *such that* $H_n(X, A_{\mathfrak{a}}: Z(\mathfrak{a}))\neq 0$.

A 1-dimensional space has the property $\boldsymbol{P}$ (cf. § 5, Lemma 24). If X contains a closed subset A such that $H_n(X, A: Z)\neq 0, X$ has the property $\boldsymbol{P}$ (cf. § 5, Lemma 20). Accordingly, if X is a space in the above-mentioned cases 1)–5), X has the property $\boldsymbol{P}$ (cf. § 5, Lemmas 21–23). Our main theorem is stated as follows.

THEOREM. *Let X be a finite dimensional compact metric space. In order that the equality* ($\boldsymbol{A}$) *hold for every compact metric space Y it is necessary and sufficient that X have the property* $\boldsymbol{P}$.

In § 2 we shall prove several lemmas and introduce the notations that are used later. In § 3 we shall construct some examples that are used in the proof of the main theorem and may be of interest in itself. These examples are modifications of Pontrjagin's surfaces (cf. [18] and [7]). Our main theorem will be proved in § 4. In § 5 we shall give some consequences of the main theorem.

§ 2. Lemmas and notations.

Let X be a topological space. By a *covering* of X we mean a covering by a finite collection of open sets. By the *nerve* of a covering we mean the nerve realized as a space with the Euclidean metric as defined by S. Lefschetz [13], p. 5. Let $\mathfrak{U}$ be a covering of a space X and let K be the nerve of $\mathfrak{U}$. A mapping[1] ϕ of X into K is called a *cannonical mapping*[2] of X into K if the inverse image of the open star of each vertex is contained in the open set of U corresponding to this vertex. If X is a normal space and $\mathfrak{U}$ is a covering of X, it is well known that there exists a canonical mapping of X into the nerve of $\mathfrak{U}$ (cf., for example, [9], Chap. X, Theorem 11.8). A covering $\mathfrak{U}$ of X is a *refinement* of a covering $\mathfrak{V}$ (this relation we denote by $\mathfrak{V}<\mathfrak{U}$), if every open set of $\mathfrak{U}$ is contained in some open set of $\mathfrak{V}$. Let $\mathfrak{V}=\{V_\beta\}$ and $\mathfrak{U}=\{U_\alpha\}$ be two coverings of X such that $\mathfrak{V}<\mathfrak{U}$ and let L and K be the nerves of $\mathfrak{V}$ and $\mathfrak{U}$ respectively. Let us denote by $\{v_\beta\}$ and $\{u_\alpha\}$ the vertexes of L and K corresponding to the open sets V_β and U_α of $\mathfrak{V}$ and $\mathfrak{U}$ respectively. A simplicial mapping $\Pi^{\mathfrak{U}}_{\mathfrak{V}}$ of K into L is called a *projection*[3] if, in case $\Pi^{\mathfrak{U}}_{\mathfrak{V}}(u_\alpha)=v_\beta$, we have $U_\alpha\subset V_\beta$. A collection $\boldsymbol{U}=\{\mathfrak{U}_\alpha\}$ of coverings of X is called a *cofinal collection* of coverings of X if for any covering $\mathfrak{V}$ of X there exists a member $\mathfrak{U}_\alpha$ of $\boldsymbol{U}$ such that $\mathfrak{V}<\mathfrak{U}_\alpha$. If X is a compact metric

1) Throughout this paper we mean by a mapping a continuous transformation.
2) Cf. [8], p. 202.
3) Cf. for example [9], p. 234.

space, there exists a countable and cofinal collection $\{\mathfrak{U}_i\}$ of coverings of X such that $\mathfrak{U}_i < \mathfrak{U}_{i+1}$, $i=1, 2, \cdots$. We shall mean by a cofinal collection of coverings of a compact metric space a countable cofinal collection $\{\mathfrak{U}_i\}$ of coverings such that $\mathfrak{U}_i < \mathfrak{U}_{i+1}$, $i=1, 2, \cdots$. The *order* of a covering is the largest integer n such that there exists $n+1$ members of the covering which have a non-empty intersection. The nerve of a covering whose order is n is n-dimensional. By the *dimension*[4] of a normal space X, which we shall denote by $\dim X$, we mean the least integer n such that every covering of X has a refinement of order n. If X is a separable metric space, this dimension is equal to the usual Brouwer-Menger-Urysohn's dimension[5].

Let (X, A) be a pair[6] of topological spaces. We shall denote by $H_n(X, A: G)$ the n-dimensional Čech homology group with coefficients in G[7]. Let R_1 be the additive group of rational numbers mod 1. The following lemmas are proved in [3].

LEMMA 1[8]. (*Hopf's extension theorem*). *Let A be a closed subset of an $(n+1)$-dimensional compact metric space X. In order that a mapping f of A into the n-dimensional sphere S^n be extensible to a mapping of X into S^n, it is necessary and sufficient that the condition $f_*\partial H_{n+1}(X, A: R_1)=0$ holds, where f_* is the homomorphism*[9] *of $H_n(A: R_1)$ into $H_n(S: R_1)$ induced by the mapping f and ∂ is the boundary homomorphism*[9] *of $H_{n+1}(X, A: R_1)$ into $H_n(A: R_1)$.*

LEMMA 2[8]. *Let X be a compact metric space. In order that* $\dim X=n$ *it is necessary and sufficient that*

(1) *there exists a closed subset A of X such that $H_n(X, A: R_1)\neq 0$,*

(2) *for every closed subset A and every integer $j>n$ we have $H_j(X, A: R_1)=0$.*

The following lemma is a consequence of [11], Chap. III, § 4, Theorem III, 4.

LEMMA 3[8]. *Let X and Y be two compact metric spaces. Then* $\dim(X\times Y) \leqq \dim X + \dim Y$.

Let (X, A) and (Y, B) be pairs of compact metric spaces. By $(X, A)\times(Y, B)$ we mean a pair of spaces $(X\times Y, X\times B\cup A\times Y)$. Let $\boldsymbol{U}=\{\mathfrak{U}_i \mid i=1, 2, \cdots\}$ and $\boldsymbol{V}=\{\mathfrak{V}_i \mid i=1, 2, \cdots\}$ be cofinal collections of coverings of X and Y respectively. Let us denote the nerves of $\mathfrak{U}_i$ and $\mathfrak{V}_i$ corresponding to (X, A) and (Y, B) by

4) Cf. [8], p. 206 and [15], p. 7.

5) See, for instance, [11], Chap. V, Theorem V 7.

6) By a *pair* of topological spaces (X, A) we mean a pair of X and a closed subset A of X.

7) See, for instance, [9], Chap. IX.

8) It is known ([15], Theorems 5.2 and 5.3 and [17]) that Lemmas 1, 2 and 3 hold in case X is a more general space, but we do not need these generalizations in this paper.

9) Cf. [9], Chap. I and Chap. IX.

(K_i, L_i) and (M_i, N_i) respectively. There exist projections $\phi_i^{i+1}: (K_{i+1}, L_{i+1}) \to (K_i, L_i)$ and $\psi_i^{i+1}: (M_{i+1}, N_{i+1}) \to (M_i, N_i)$ for $i=1, 2, \cdots$. Let us denote by Π_i^{i+1} the product mapping[10] $\phi_i^{i+1} \times \psi_i^{i+1}$ of the pair $(K_{i+1}, L_{i+1}) \times (M_{i+1}, N_{i+1})$ of cell complexes into the pair $(K_i, L_i) \times (M_i, N_i)$ of cell complexes. Since Π_i^{i+1} is a cellular mapping[11], it induces a homomorphism $(\Pi_i^{i+1})_*: H_n((K_{i+1}, L_{i+1}) \times (M_{i+1}, N_{i+1}): G) \to H_n((K_i, L_i) \times (M_i, N_i): G)$. Let us denote by (S_i, T_i) the pair of the nerves of the product covering[12] $\mathfrak{U}_i \times \mathfrak{V}_i$ of $X \times Y$ corresponding to $(X, A) \times (Y, B)$. The following lemma is proved in the same way as [4], Theorem 12.42.

LEMMA 4. *For each i, there exist a homeomorphism into $\theta_i: (K_i, L_i) \times (M_i, N_i) \to (S_i, T_i)$ and a homotopy $F_t^i: (S_i, T_i) \to (S_i, T_i)$ such that F_0^i=identity, $F_t^i | \theta_i(K_i \times M_i)$=identity, $F_1^i(S_i) \subset \theta_i(K_i \times M_i)$ and $F_1^i(T_i) \subset \theta(K_i \times N_i \cup L_i \times M_i)$. Moreover, commutativity relation holds in each square of the following diagram:*

$$\begin{array}{ccccc} (K_{i+1}, L_{i+1}) \times (M_{i+1}, N_{i+1}) & \xrightarrow{\theta_{i+1}} & (S_{i+1}, T_{i+1}) & \xrightarrow{F_t^{i+1}} & (S_{i+1}, T_{i+1}) \\ \downarrow \Pi_i^{i+1} & & \downarrow h_i & & \downarrow h_i \\ (K_i, L_i) \times (M_i, N_i) & \xrightarrow{\theta_i} & (S_i, T_i) & \xrightarrow{F_t^i} & (S_i, T_i) \end{array} ,$$

where h_i is the simplicial mapping of (S_{i+1}, T_{i+1}) into (S_i, T_i) induced by Π_i^{i+1}.

The following lemma is proved, in view of Lemma 4, by a straightforward computation.

LEMMA 5[13]. *Let (X, A) and (Y, B) be pairs of compact metric spaces. Then we have the following isomorphism:*

$$H_n((X, A) \times (Y, B): G) \approx \varprojlim \{H_n((K_i, L_i) \times (M_i, N_i): G): (\Pi_i^{i+1})_*\}^{14)} .$$

For each positive integer p let us denote the factor group Z/pZ by Z_p, where Z is the additive group of all integers. A sequence $\mathfrak{a}=(q_1, q_2, \cdots)$ of positive integers is called a *k-sequence* if q_i is a divisor of q_{i+1} for each i and $q_i > 1$ for some i. If $\mathfrak{a}=(q_1, q_2, \cdots)$ is a k-sequence, there exists a sequence of natural homomorphisms $\{h(\mathfrak{a}, i) | i=1, 2, \cdots\}$, where $h(\mathfrak{a}, i)$ is a natural homomorphism from $Z_{q_{i+1}}$ onto Z_{q_i}. Let (X, A) be a pair of compact metric spaces.

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10) Let f and g be mappings of (X, A) and (Y, B) into (X', A') and (Y', B'). By the *product mapping* $f \times g$ of f and g we understand the mapping ϕ of $(X, A) \times (Y, B)$ into $(X', A') \times (Y', B')$ defined by $\phi(x, y) = (f(x), g(y))$ for $(x, y) \in X \times Y$.

11) A mapping f of a cell complex K into a cell complex M is called a *cellular mapping* if $f(K^i) \subset M^i$, where K^i means the i-section of K.

12) Let $\mathfrak{U}=\{U_\alpha\}$ and $\mathfrak{V}=\{V_\alpha\}$ be coverings of X and Y respectively. By the *product covering* $\mathfrak{U} \times \mathfrak{V}$ of $\mathfrak{U}$ and $\mathfrak{V}$ we mean the covering $\{U_\alpha \times V_\beta\}$ of $X \times Y$.

13) It is proved that this lemma holds in case X and Y are compact Hausdorff spaces.

14) By $\varprojlim \{X_i : \Pi_i^{i+1}\}$ we mean a) the inverse limit space if X_i is a space and Π_i^{i+1} is a mapping and b) the inverse limit group if X_i is a group and Π_i^{i+1} is a homomorphism.

For each k-sequence $\mathfrak{a}=(q_1, q_2,\cdots)$ let us define a group $H_n(X, A: \mathfrak{a})$ as follows: Let $\{\mathfrak{U}_i\}$ be a cofinal collection of coverings of X. There exists a projection $\Pi_i^{i+1}: (K_{i+1}, L_{i+1})\to(K_i, L_i)$, where (K_i, L_i) means the pair of the nerves of $\mathfrak{U}_i$ corresponding to (X, A). Define a homomorphism $\mathfrak{P}_i^{i+1}: H_n(K_{i+1}, L_{i+1}: Z_{q_{i+1}})\to H_n(K_i, L_i: Z_{q_i})$ by a composition of homomorphisms $(h(\mathfrak{a}, i))_*$ and $(\Pi_i^{i+1})_*$, where $(h(\mathfrak{a}, i))_*$ is the homomorphism of $H_n(K_{i+1}, L_{i+1}: Z_{q_{i+1}})$ into $H_n(K_{i+1}, L_{i+1}: Z_{q_i})$ induced by the homomorphism $h(\mathfrak{a}, i)$ and $(\Pi_i^{i+1})_*$ is the homomorphism of $H_n(K_{i+1}, L_{i+1}: Z_{q_i})$ into $H_n(K_i, L_i: Z_{q_i})$ induced by the mapping Π_i^{i+1}. The group $H_n(X, A: \mathfrak{a})$ is defined to be the inverse limit group of the inverse system $\{H_n(K_i, L_i: Z_{q_i}): \mathfrak{P}_i^{i+1}\}$.

LEMMA 6. *The group $H_n(X, A: \mathfrak{a})$ is independent of the choice of a cofinal collection $\{\mathfrak{U}_i\}$ of coverings of X.*

PROOF. Let $\{\mathfrak{U}_i\}$ and $\{\mathfrak{V}_j\}$ be two cofinal collections of coverings of X. By $H_n(X, A: \mathfrak{a}, \{\mathfrak{U}_i\})$ and $H_n(X, A: \mathfrak{a}, \{\mathfrak{V}_j\})$ we denote the groups defined by means of $\{\mathfrak{U}_i\}$ and $\{\mathfrak{V}_i\}$. Since $\{\mathfrak{U}_i\}$ and $\{\mathfrak{V}_j\}$ are cofinal, there exists a sequence of coverings $\{\mathfrak{U}_{i_1}, \mathfrak{V}_{j_1}, \mathfrak{U}_{i_2},\cdots, \mathfrak{U}_{i_k}, \mathfrak{V}_{j_k},\cdots\}$ such that $\mathfrak{U}_{i_1}<\mathfrak{V}_{j_1}<\mathfrak{U}_{i_2}<\cdots<\mathfrak{U}_{i_k}<\mathfrak{V}_{j_k}<\cdots$ and $i_1<j_1<i_2<\cdots<i_k<j_k<\cdots$. For the pairs (i_k, j_k) and (j_{k-1}, i_k) there exist natural homomorphisms $\nu_k: H_n(M_{j_k}, N_{j_k}: Z_{q_{j_k}})\to H_n(K_{i_k}, L_{i_k}: Z_{q_{i_k}})$ and $\varepsilon_k: H_n(K_{i_k}, L_{i_k}: Z_{q_{i_k}})\to H_n(M_{j_{k-1}}, N_{j_{k-1}}: Z_{q_{j_{k-1}}})$, where (K_{i_k}, L_{i_k}) and (M_{j_k}, N_{j_k}) are pairs of the nerves of coverings $\mathfrak{U}_{i_k}$ and $\mathfrak{V}_{j_k}$ respectively. It is obvious that $\mathfrak{P}_{i_k}^{i_{k+1}}=\nu_k\varepsilon_{k+1}: H_n(K_{i_{k+1}}, L_{i_{k+1}}: Z_{q_{i_{k+1}}})\to H_n(K_{i_k}, L_{i_k}: Z_{q_{i_k}})$ and $\mathfrak{P}_{j_{k-1}}^{j_k}=\varepsilon_k\nu_k: H_n(M_{j_k}, N_{j_k}: Z_{q_{j_k}})\to H_n(M_{j_{k-1}}: N_{j_{k-1}}: Z_{q_{j_{k-1}}})$, where $\mathfrak{P}_{i_k}^{i_{k+1}}$ and $\mathfrak{P}_{j_{k-1}}^{j_k}$ are homomorphisms used in the definition of the groups $H_n(X, A: \mathfrak{a}, \{\mathfrak{U}_i\})$ and $H_n(X, A: \mathfrak{a}, \{\mathfrak{V}_j\})$ respectively. Therefore we have $H_n(X, A: \mathfrak{a}, \{\mathfrak{U}_i\})=\varprojlim\{H_n(K_i, L_i: Z_{q_i}): \mathfrak{P}_i^{i+1}\}=\varprojlim\{H_n(K_{i_k}, L_{i_k}: Z_{q_{i_k}}): \mathfrak{P}_{i_k}^{i_{k+1}}\}=\varprojlim\{H_n(M_{j_k}, N_{j_k}: Z_{q_{j_k}}): \mathfrak{P}_{j_k}^{j_{k+1}}\}=\varprojlim\{H_n(M_{j_k}, N_{j_k}: Z_{q_{j_k}}): \mathfrak{P}_{j_k}^{j_{k+1}}\}=\varprojlim\{H_n(M_j, N_j: Z_{q_j}): \mathfrak{P}_j^{j+1}\}=\varprojlim H_n(X, A: \mathfrak{a}, \{\mathfrak{V}_j\})$. This completes the proof.

LEMMA 7. *Let X be an n-dimensional normal space. Let G be an abelian group. Suppose that $H_n(X, A: G)=0$ for every closed subset A of X. Then we have $H_n(B, C: G)=0$ for every pair (B, C) of closed subsets of X.*

PROOF. Let $\{\mathfrak{U}_\alpha\}$ be a cofinal collection of coverings of X. Assume that each covering $\mathfrak{U}_\alpha$ has the order n. Then we have $H_n(B, C: G)=\varprojlim\{H_n(M_\alpha, N_\alpha: G)\}=\varprojlim\{Z_n(M_\alpha, N_\alpha: G)\}$, where (M_α, N_α) is the pair of the nerve of $\mathfrak{U}_\alpha$ corresponding to (B, C) and $Z_n(M, N: G)$ is the group of n-dimensional cycles of (M, N) with coefficients in G. Similarly, $H_n(X, C: G)$ is considered as the inverse limit of $\{Z_n(K_\alpha, N_\alpha: G)\}$, where K_α is the nerve of $\mathfrak{U}_\alpha$. Therefore the homomorphism $i_*: H_n(B, C: G)\to H_n(X, C: G)$ induced by the inclusion mapping $i: (B, C)\to(X, C)$ is an isomorphism into. Since $H_n(X, C: G)=0$, we have $H_n(B,$

$C:G)=0$. This completes the proof.

Let $\mathfrak{a}=\{q_1, q_2, \cdots\}$ be a k-sequence. By $Z(\mathfrak{a})$ we denote the inverse limit group of the inverse system $\{Z_{q_i}: h(\mathfrak{a}, i)\}$, where $h(\mathfrak{a}, i)$ is the natural homomorphism from $Z_{q_{i+1}}$ onto Z_{q_i}. Consider two groups $H_n(X, A: Z(\mathfrak{a}))$ and $H_n(X, A: \mathfrak{a})$. We have the following lemma.

LEMMA 8. *There exists an isomorphism $H_n(X, A: Z(\mathfrak{a}))\approx H_n(X, A: \mathfrak{a})$.*

Before proving this lemma it is convenient to prove the following lemmas.

LEMMA 9. *Let (K, L) be a pair of n-dimensional simplicial complexes. There exists an isomorphism J_*: $H_n(K, L: Z(\mathfrak{a}))\approx H_n(K, L: \mathfrak{a})$. Moreover the isomorphism J_* is natural in the following sense: Let f be a simplicial mapping of (K, L) into another pair (M, N) of n-dimensional simplicial complexes. The following commutative diagram holds:*

$$\begin{array}{ccc} H_n(K, L: Z(\mathfrak{a})) & \xrightarrow{f_*} & H_n(M, N: Z(\mathfrak{a})) \\ \downarrow J_* & & \downarrow J_* \\ H_n(K, L: \mathfrak{a}) & \xrightarrow{f_*} & H_n(M, N: \mathfrak{a}) \end{array}$$

PROOF. Let us denote by $(K(j+1), L(j+1))$ the j-th barycentric subdivision of (K, L) and by Π_j^{j+1} a simplicial mapping from $(K(j+1), L(j+1))$ into $(K(j), L(j))$, $j=1, 2, \cdots$, with the usual property, where $(K(1), L(1))=(K, L)$. Let P_j^{j+1} be the homomorphism of $H_n(K(j+1), L(j+1): Z_{q_{j+1}})$ into $H_n(K(j), L(j): Z_{q_j})$ defined by $\mathfrak{P}_j^{j+1}=(\Pi_j^{j+1})_*(h(\mathfrak{a}, j))_*$, where $(h(\mathfrak{a}, j))_*$ is the homomorphism of $H_n(K(j+1), L(j+1): Z_{q_{j+1}})$ into $H_n(K(j+1), L(j+1): Z_{q_j})$ induced by the homomorphism $h(\mathfrak{a}, j)$ and $(\Pi_j^{j+1})_*$ is the homomorphism of $H_n(K(j+1), L(j+1): Z_{q_j})$ into $H_n(K(j), L(j): Z_{q_j})$ induced by the mapping Π_j^{j+1}. Since the homology group is invariant by a subdivision [2], the homomorphism $(\Pi_j^{j+1})_*$ is an isomorphism. Therefore we have an isomorphism $\Theta_1: H_n(K, L: \mathfrak{a})\approx \varprojlim\{H_n(K, L: Z_{q_i}): (h(\mathfrak{a}, i))_*\}$, which is natural in the sense of the lemma. Put $G=\varprojlim\{H_n(K, L: Z_{q_i}): (h(\mathfrak{a}, i))_*\}$. Take an element $g=\{g_i\}$ of G. Since $\dim K=n$, we have $g_i=\sum_{j=1}^k t_j(i)\sigma_j\in Z_n(K, L: Z_{q_i})$, where $t_j(i)\in Z_{q_i}$ and σ_j is an n-dimensional simplex of $K-L$, $j=1, 2, \cdots, k$. Since $(h(\mathfrak{a}, i))_* g_{i+1}=g_i$, we have $h(\mathfrak{a}, i)t_j(i+1)=t_j(i)$ for $j=1, 2, \cdots, k$ and $i=1, 2, \cdots$. Accordingly the sequence $\{t_j(i)\,|\,i=1, 2, \cdots\}$ determines an element $\boldsymbol{t}_j$ of $Z(\mathfrak{a})$. Put $\mathfrak{g}=\sum_{j=1}^k \boldsymbol{t}_j\sigma_j$. Obviously we have $\mathfrak{g}\in Z_n(K, L: Z(\mathfrak{a}))$. Define a transformation Θ_2 of G into $H_n(K, L: Z(\mathfrak{a}))$ by $\Theta_2 g=\mathfrak{g}$. It is obvious that Θ_2 is a homomorphism. Let $\Theta_2 g=0$. We have $\boldsymbol{t}_j\equiv 0$ in $Z(\mathfrak{a})$ for $j=1, 2, \cdots, k$. Therefore we have $t_j(i)\equiv 0$ in Z_{q_i} for $j=1, 2, \cdots, k$ and $i=1, 2, \cdots$. Thus Θ_2 is an isomorphism. Take an element $\mathfrak{g}\in Z_n(K, L: Z(\mathfrak{a}))$. Let $\mathfrak{g}=\sum_{j=1}^k \boldsymbol{t}_j\sigma_j$, where $\boldsymbol{t}_j\in Z(\mathfrak{a})$ and σ_j is an n-dimensional simplex of $K-L$ for $j=1, 2, \cdots, k$. Let $\boldsymbol{t}_j=\{t_j(i)\,|\,i=1, 2, \cdots\}$, where $t_j(i)\in Z_{q_i}$. Put $g_i=\sum_{j=1}^k t_j(i)\sigma_j$. It is obvious that $g_i\in Z_n(K, L: Z_{q_i})$. Since $h(\mathfrak{a}, i)t_j(i+1)=t_j(i)$, we have $(h(\mathfrak{a}, (i))_* g_{i+1}$

$=g_i, i=1, 2, \cdots$. Therefore $\{g_i | i=1, 2, \cdots\}$ determines an element g of G. By the definition of Θ_2 we have $\Theta_2 g=\mathfrak{g}$. It is obvious that Θ_2 is natural. Put $J_*=\Theta_2\Theta_1$. Then J_* is an isomorphism required in the lemma..

LEMMA 10. *Let (X, A) be a pair of n-dimensional compact metric spaces and let $\{\mathfrak{U}_i\}$ be a cofinal collection of coverings of X each member of which has the order n. Let us denote by (K_i, L_i) the pair of the nerves of $\mathfrak{U}_i$ corresponding to (X, A). Then there exists an isomorphism I_*:*

$$\varprojlim\{H_n(K_i, L_i : \mathfrak{a})\} \approx H_n(X, A : \mathfrak{a}).$$

PROOF. By Lemma 9 there exists a natural isomorphism $\Theta_1 : H_n(K_i, L_i : \mathfrak{a}) \approx \varprojlim\{H_n(K_i, L_i : Z_{q_j}) : (h(\mathfrak{a}, j))_*\}$, where $\mathfrak{a}=(q_1, q_2, \cdots)$. Take an element $g=\{g_i\} \in \varprojlim\{H_n(K_i, L_i : \mathfrak{a})\}$. Then we can assume that $g_i \in \varprojlim\{H_n(K_i, L_i : Z_{q_j})\}$. Let $g_i=\{g_j(i)\}$, where $g_j(i) \in H_n(K_i, L_i : Z_{q_j})\}$ and $(h(\mathfrak{a}, j))_* g_{j+1}(i)=g_j(i)$, $j=1, 2, \cdots$ and $i=1, 2, \cdots$. Then the element $g_i(i)$ belongs to $H_n(K_i, L_i : Z_{q_i})$. Moreover we have $\mathfrak{P}_i^{i+1} g_{i+1}(i+1)=(\Pi_i^{i+1})_*(h(\mathfrak{a}, i))_* g_{i+1}(i+1)=(\Pi_i^{i+1})_* g_i(i+1)=g_i(i)$. Accordingly $\{g_i(i)\}$ determines an element $\mathfrak{g}$ of $H_n(X, A : \mathfrak{a})$. Define a transformation I_* of $\varprojlim\{H_n(K_i, L_i : \mathfrak{a})\}$ into $H_n(X, A : \mathfrak{a})$ by $I_* g=\mathfrak{g}$. It is obvious that I_* is a homomorphism. Let $I_* g=0$. Then $g_i(i)=0, i=1, 2, \cdots$. Since $(\Pi_i^{j})_* g_j(j)=g_j(i)$ for $j>i$, where $\Pi_i^{j}=\Pi_i^{i+1} \cdots \Pi_{j-1}^{j}$, we have $g_i=0$ for $i=1, 2, \cdots$. This shows that I_* is an isomorphism into. Take an element $\mathfrak{a}=\{g_i\}$ of $H_n(X, A : \mathfrak{a})$, where $g_i \in H_n(K_i, L_i : Z_{q_i}), i=1, 2, \cdots$. For each $j>i$, consider the element $g_j(i)=(\Pi_i^{j})_* g_j$ of $H_n(K_i, L_i : Z_{q_j})$. We have $(h(\mathfrak{a}, j))_* g_{j+1}(i)=(h(\mathfrak{a}, j))_*(\Pi_i^{j+1})_* g_{j+1}=(\Pi_i^{j})_*(h(\mathfrak{a}, j))_* (\Pi_j^{j+1})_* g_{j+1}=(\Pi_i^{j})_* g_j=g_j(i)$ for each $j>i$. Therefore $\{g_j(i)\}$ determines an element $\tilde{g}_i$ of $H_n(K_i, L_i : \mathfrak{a})$. Since Θ_1 is natural, we have $(h(\mathfrak{a}, j))_* \tilde{g}_{i+1}=\tilde{g}_i$. Accordingly $\{\tilde{g}_i\}$ determines an element g of $\varprojlim\{H_n(K_i, L_i : \mathfrak{a})\}$. It is obvious that $I_* g=\mathfrak{g}$ and I_* is the required isomorphism.

We have the following isomorphisms:

$$\begin{aligned} H_n(X, A : Z(\mathfrak{a})) &\approx \varprojlim\{H_n(K_i, L_i : \mathfrak{a})\} && \text{by Lemma 9,} \\ &\approx H_n(X, A : \mathfrak{a}) && \text{by Lemma 10.} \end{aligned}$$

This completes the proof of Lemma 8.

Let X be a metric space and let ε be a positive number. By an *ε-mapping* of X into a topological space Y we mean a mapping f of X into Y such that for each point y of Y we have the diameter of $f^{-1}(y)<\varepsilon$. The following lemma is well known.[15)]

LEMMA 11. *Let X be a compact metric space. In order that* $\dim X \leqq n$ *it is necessary and sufficient that, for each positive number ε there exists an ε-mapping of X into an n-dimensional polytope.*

15) See, for instance, [11], Chap. V.

The following lemma is a consequence of [9], Chap. X, Lemma 3.7.

LEMMA 12. *Let* $\{X_i : \Pi_i^{i+1}\}$ *be an inverse system of n-dimensional compact metric spaces and let X be the limit space of* $\{X_i : \Pi_i^{i+1}\}$. *Then we have* $\dim X \leqq n$.

§ 3. Examples.

In [18] L. Pontrjagin has constructed two dimensional compact metric spaces P_1 and P_2 which we call *Pontrjagin's surfaces*, such that $\dim(P_1 \times P_2)=3$. In this article we shall construct 2-dimensional compact metric spaces which are considered as generalizations of Pontrjagin's surfaces.

1) Möbius band mod (p, q)--$M(p, q)$.

Let (p, q) be a pair of positive integers. Let S be the 1-dimensional sphere and let I be the interval [0,1]. By a *Möbius band mod* (p, q) we understand the continuum $M(p, q)$ obtained from the product space $S\times I$ by identifying on the circumference $S_0=S\times(0)$ points corresponding to each other under the rotation of angle $2\pi/p$ and by identifying on the circumference $S_1=S\times(1)$ points corresponding to each other under the rotation of angle $2\pi/q$. Let f be an identification mapping. Put $T_0=f(S_0)$ and $T_1=f(S_1)$. We shall call T_0 and T_1 the outer- and inner-boundaries of $M(p, q)$ respectively. The outer- and inner-boundaries are homeomorphic to a circumference. In general, Möbius band $M(p, q)$ mod (p, q) is a homogeneously 2-dimensional curvilinear polytope[16]. For each pair (p, q) of positive integers we shall consider $M(p, q)$ as a simplicial polytope with a fixed triangulation.

LEMMA 13. *Let* $M(p, q)$ *be a Möbius band mod* (p, q) *with the outer-boundary* T_0. *Let us give an orientation to each 2-dimensional simplex* σ_j, $j=1, 2,\cdots, k$, *of* $M(p, q)$ *such that the integral chain which has the value* 1 *on each 2-dimensional simplex—we call the fundamental chain of* $M(p,q)$*—is a cycle mod g relative to* T_0. *Then an element* $C=\sum_{j=1}^k t_j\sigma_j$ *of* $C_2(M(p, q), T_0 : R_1)$ *belongs to* $Z_2(M(p, q), T_0 : R_1)$ *if and only if* $t_j=t$ *for* $j=1, 2,\cdots, k$ *and* $qt\equiv 0 \bmod 1$, *where* $C_n(K, L : G)$ *means the group of n-dimensional chains of* (K, L) *with coefficients in G. Moreover* $H_2(M(p, q), T_0 : R_1)$ *is generated by the chain* $\frac{1}{q}\delta$[17], *where* δ *is a funda-*

16) Cf. [7], p. 56.

17) Let G_1 and G_2 be two abelian groups *paired* to a third group G, that is, there is given a function ψ of $G_1\times G_2$ into G which is distributive in both variables and whose values are in G (cf. [14], p. 59). Let $c=\Sigma_i t_i\sigma_i$ be an element of $C_n(K, L : G)$, where $t_i\in G_1$ and σ_i's are n-dimensional simplexes of a complex K and let q be an element of G_2. By qc we understand the chain $\Sigma_i\psi(t_i, q)\sigma_i$ of (K, L) with coefficients in G. Let $d=\Sigma_j s_j\tau_j$ be an element of $C_m(M, N : G_2)$, where $s_j\in G_2$ and τ_j's are m-dimensional simplexes of a complex M. By $c\times d$ we understand the $(n+m)$-dimensional chain $\Sigma_i \Sigma_j \psi(t_i, s_j)(\sigma_i\times s_j)$ of the pair $(K, L)\times(K, N)$ of cell complexes with coefficients in G.

mental chain of $M(p, q)$.

This lemma is a consequence of [2], Kap. IV, 5, Satz VII.

LEMMA 14. *Let* τ *be a* 2-*dimensional element*[18] *with the boundary* $\dot{\tau}$. *Then there exists a mapping* f *of* $(M(p, q), T_0)$ *into* $(\tau, \dot{\tau})$ *such that the restricted mapping* $f|T_0$ *is topological and* $f(T_1)$ *is a point of* τ, *where* T_1 *is the inner-boundary of* $M(p, q)$. *The fundamental chain* δ *of* $M(p, q)$ *is mapped by* f *onto the integral chain* pz, *where* z *is a generator of the group* $H_2(\tau, \dot{\tau}: Z)$.

PROOF. Since τ is contractible in itself[19], it is obvious that there exists a mapping f such that $f|T_0$ is topological and $f(T_1)$ is a point x of τ. Consider the following commutative diagram:

$$\begin{array}{ccc} H_2(M(p,q), T_0\cup T_1: Z) & \xrightarrow{f_*} & H_2(\tau, \tau\cup x: Z) \\ \downarrow \partial & & \downarrow \partial_1 \\ H_1(T_0\cup T_1: Z) & \xrightarrow{(f|T_0\cup T_1)_*} & H_1(\tau\cup x: Z), \end{array}$$

where ∂ and ∂_1 are the boundary homomorphisms[20]. Since ∂_1 is an isomorphism onto and $\partial(\delta)=p\nu_0+q\nu_1$, where ν_0 and ν_1 are generators of the group $H_1(T_0: Z)$ and $H_1(T_1: Z)$ respectively, we have $f_*(\delta)=(\partial_1)^{-1}(f|T_0\cup T_1)_*\partial(\delta)=pz$. This completes the proof.

LEMMA 15. *Let* p *and* q *be two integers such that* $1<p<q$ *and* p *is a divisor of* q. *Put* $P=M(1, p)$ *and* $Q=M(p, q)$. *Let* τ *and* μ *be* 2-*dimensional elements. Let* f *and* g *be mappings of* P *and* Q *into* τ *and* μ *which are topological on the outer-boundaries of* P *and* Q, *constructed in Lemma* 14, *respectively. Then the product mapping*[21] $\phi=f\times g$ *of* $P\times Q$ *into the* 4-*dimensional element* $\tau\times\mu$ *is inessential*[22].

PROOF. Let S_0 and S_1 be the outer- and inner-boundaries of Q and let T_0 be the outer-boundary of P. Consider the group $H_4((P, T_0)\times(Q, S_0): R_1)$. Take an element a of $Z_4((P, T_0)\times(Q, S_0): R_1)$. Let $a=\sum_{i=1}^{k}\sum_{j=1}^{l} t_{ij}\tau(i)\times\mu(j)$, where $t_{ij}\in R_1$, $\tau(i)$ and $\mu(j)$ are 2-dimensional simplexes of P and Q, $i=1, 2,\cdots, k$ and $j=1, 2,\cdots, l$, respectively. Since ∂a[23] $=\sum_{i=1}^{k}\sum_{j=1}^{l} t_{ij}(\partial\tau(i)\times\mu(j))+\sum_{i=1}^{k}\sum_{j=1}^{l} t_{ij}(\tau(i)$

18) By an *n-dimensional element* we understand a set homeomorphic to an n-dimensional closed simplex.

19) Let A and B be subsets of a topological space such that $A\subset B$. It is called that A is *contractible in* B if there exists a mapping F of $A\times I$ into B such that $F(a, 0)=a$ and $F(a, 1)=$ a point a_0 of B for each point a of A.

20) See footenote 9).

21) See footenote 10).

22) A mapping f of a topological space X into an n-dimensional element σ is called *inessential* if there exists a mapping F of $X\times I$ into σ such that $F(x, 0)=f(x)$ and $F(x, 1)\in\sigma$ for $x\in X$, $F(x', t)=f(x')$ for $x'\in f^{-1}(\dot{\sigma})$ and $t\in I$, where $\dot{\sigma}$ is the boundary of the element σ.

23) Let a be a chain or an oriented simplex. By ∂a we mean the boundary chain of a.

$\times \partial \mu(j)))$[24] belongs to $C_3(P\times S_0 \cup T_0\times Q : R_1)$, the chains $\sum_{i=1}^{k} t_{ij}\tau(i)$ and $\sum_{j=1}^{l} t_{ij}\mu(j)$ are elements of $Z_2(P, T_0 : R_1)$ and $Z_2(Q, S_0 : R_1)$ respectively. By Lemma 13 we have $t_{ij}=\frac{m_j}{p}$ for $j=1, 2,\cdots, l$, and $t_{ij}=\frac{n_i}{q}$ for $i=1, 2,\cdots, k$, where m_j and n_i are integers. Therefore we have $t_{ij}=\frac{s}{p}$ for $i=1, 2,\cdots, k$ and $j=1, 2,\cdots, l$, where s is an integer, and we can write $a=\frac{s}{p}(\delta_1\times\delta_2)$,[25] where δ_1 and δ_2 are the fundamental chain of P and Q respectively. Since $\partial\delta_2=p\nu_0+q\nu_1$, we have $\partial a=\frac{s}{p}(\nu'\times\delta_2)$, where ν_0, ν_1 and ν' are generators of $H_1(S_0 : Z), H_1(S_1 : Z)$ and $H_1(T_0 : Z)$ respectively. Accordingly the chain $f_\#(\partial a)$[26] has a carrier[27] $\dot{\tau}\times\mu$ and it is homologous to zero in $\tau\times\dot{\mu}\cup\dot{\tau}\times\mu$. That f is inessential is a consequence of Hopf's extension theorem for polytopes[28].

2) Polytope $P(p_1,\cdots, p_n)$ and continuum $P(p_1, p_2,\cdots)$.

Let $(p_1, p_2\cdots)$ be a sequence of positive integers. Let $P(p_1)$ be a Möbius band mod $(1, p_1)$. Let us denote by $\tau_{h_1}(1), h_1=1, 2,\cdots, l_1$, all 2-dimensional simplexes of $P(p_1)$. We replace every triangle $\tau_{h_1}(1)$ of $P(p_1)$ by a Möbius band $M_{h_1}(1, p_2)$ mod $(1, p_2)$ such that $M_{h_1}(1, p_2)\cap M_{h_1'}(1, p_2)=S_{h_1}\cap S_{h_1'}=\tau_{h_1}(1)\cap\tau_{h_1'}(1)$ for $h_1\neq h_1'$, where S_{h_1} and $S_{h_1'}$ are the outer-boundaries of $M_{h_1}(1, p_2)$ and $M_{h_1'}(1, p_2)$. Put $P(p_1, p_2)=\sum_{h_1=1}^{l_1} M_{h_1}(1, p_2)$. The polytope $P(p_1, p_2)$ contains the 1-section $\varDelta_1$ of $P(p_1)$. By Lemma 14 there exists a mapping $\varPi_1^2 : P(p_1, p_2)\rightarrow P(p_1)$ such that $\varPi_1^2 | \varDelta_1$ is a homeomorphism. Let us suppose that a 2-dimensional simplicial polytope $P(p_1,\cdots, p_i)$ consisting of Möbius bands $M_{h_1\cdots h_{i-1}}(1, p_i)$ mod $(1, p_i)$, $h_1=1, 2,\cdots, l_1, h_2=1, 2,\cdots, l_2,\cdots. h_{i-1}=1, 2,\cdots, l_{i-1}$, and a mapping $\varPi_{i-1}^i : P(p_1,\cdots, p_i)\rightarrow P(p_1,\cdots, p_{i-1})$ such that $\varPi_{i-1}^i | \varDelta_{i-1}$ is a homeomorphism, where $\varDelta_{i-1}$ is the 1-section of $P(p_1,\cdots, p_{i-1})$, are constructed for some i. Let us denote by $\tau_{h_1\cdots h_i}(i)$, $h_i=1, 2,\cdots, l_i$, all 2-dimensional simplexes of $M_{h_1\cdots h_{i-1}}(1, p_i)$ for $h_1=1, 2,\cdots, l_1$, $h_2=1, 2,\cdots, l_2, \cdots, h_{i-1}=1, 2,\cdots, l_{i-1}$. We replace every triangle $\tau_{h_1\cdots h_i}(i)$ by a Möbius band $M_{h_1\cdots h_i}(1, p_{i+1})$ mod $(1, p_{i+1})$ such that $M_{h_1\cdots h_i}(1, p_{i+1})\cap M_{h_1'\cdots h_i'}(1, p_{i+1})=S_{h_1\cdots h_i}\cap S_{h_1'\cdots h_i'}=\tau_{h_1\cdots h_i}(i)\cap\tau_{h_1'\cdots h_i'}(i)$ for $(h_1,\cdots, h_i)\neq(h_1',\cdots, h_i')$. Put $P(p_1, p_2,\cdots, p_{i+1})=\sum_{h_1=1}^{l_1}\sum_{h_2=1}^{l_2}\cdots\sum_{h_i=1}^{l_i} M_{h_1\cdots h_i}(1, p_{i+1})$. Then $P(p_1,\cdots, p_{i+1})$ contains the 1-section $\varDelta_i$

24) Cf. [2], Kap. VII, § 3.

24a) Strictly speaking, $t_{ij}\equiv n_i/q$ (mod 1), but we shall write simply $t_{ij}=n_i/q$ when there is no danger of confusion.

25) See footnote [17].

26) Let (K, L) and (M, N) be pairs of simplicial complexes. Let f be a simplicial mapping of (K, L) into (M, N). For any abelian group G we denote by $f_\#$ a homomorphism of $C_n(K, L : G)$ into $C_n(M, N : G)$ induced by f. We shall use the same notation for groups with different coefficients too.

27) Let $c=\sum_i t_i\sigma_i$ be a chain of (K, L) with coefficients in G. By a *carrier* of the chain c we mean a subcomplex M of K such that $\sigma_i\in M$ if $t_i\neq 0$.

28) See, for instance, [2], Kap. XIII.

of $P(p_1,\cdots,p_i)$. By Lemma 14 there exists a mapping $\Pi_i^{i+1}: P(p_1,\cdots,p_{i+1})\rightarrow P(p_1,\cdots,p_i)$ such that $\Pi_i^{i+1}|\Delta_i$ is a homeomorphism. Put $P(p_1,p_2,\cdots)=\varprojlim\{P(p_1,\cdots,p_i):\Pi_i^{i+1}\}$. Δ_i defines a subset of $P(p_1,p_2,\cdots)$ which is homeomorphic to Δ_i; this set will be denoted by the same letter Δ_i. By Π_i we denote the projection from $P(p_1,p_2,\cdots)$ onto $P(p_1,p_i)$. Then the restricted mapping $\Pi_i|\Delta_i$ is topological. If the integers p_j, $j=i,i+1,\cdots$. have no common divisor for each $i=1,2,\cdots$, then we have $\dim P(p_1,p_2,\cdots)=1$ and each projection Π_i, $i=1,2,\cdots$, is a monotone dimension-raising mapping from $P(p_1,p_2,\cdots)$ onto $P(p_1,p_2,\cdots,p_i)$. If the integers p_i, $i=1,2,\cdots$, have a common divisor, then we have $\dim P(p_1,p_2,\cdots)=2$. We omit the proof since these facts will not be used in this paper. Especially if $p_i=p$ for all i, $P(p_1,p_2,\cdots)$ is Pontrjagin's surface mod p (cf. [18] and [7]).

3) Polytope $Q(q_1,\cdots,q_n)$ and continuum $Q(\mathfrak{a})$.

Let $\mathfrak{a}=(q_1,q_2,\cdots)$ be a k-sequence. Put $p_i=q_{i+1}/q_i$, $i=1,2,\cdots$. Let $Q(q_1)$ be a Möbius band mod $(1,q_1)$. Let $\delta(1)$ be a fundamental chain of $Q(q_1)$ (Lemma 13). Let us denote by $\mu_{h_1}(1)$, $h_1=1,2,\cdots,l_1$, all 2-dimensional simplexes of $Q(q_1)$. We replace every triangle $\mu_{h_1}(1)$ of $Q(q_1)$ by a Möbius band $M_{h_1}(p_1,q_2)$ mod (p_1,q_2) such that $M_{h_1}(p_1,q_2)\cap M_{h_1'}(p_1,q_2)=S_{h_1}\cap S_{h_1'}=\mu_{h_1}(1)\cap\mu_{h_1'}(1)$ for $h_1\neq h_1'$, where S_{h_1} and $S_{h_1'}$ are the outer-boundaries of $M_{h_1}(p_1,q_2)$ and $M_{h_1'}(p_1,q_2)$ respectively. Put $Q(q_1,q_2)=\sum_{h_1=1}^{l_1}M_{h_1}(p_1,q_2)$. Let δ_{h_1} be the fundamental chain of $M_{h_1}(p_1,q_2)$ (Lemma 13). Define an integral chain $\delta(2)$ of $Q(q_1,q_2)$ by $\delta(2)=\sum_{h_1=1}^{l_1}\delta_{h_1}$. Then the chain $\frac{1}{q_2}\delta(2)$ is a cycle mod 1 relative to S, where S is the outer-boundary of $Q(q_1)$. By Lemma 14 there exists a mapping $\theta_1^2: Q(q_1\,q_2)\rightarrow Q(q_1)$ such that $(\theta_1^2)_\#\delta(2)=p_1\delta(1)$. To proceed by induction, let us suppose that for some i we have constructed a 2-dimensional simplicial polytope $Q(q_1,\cdots,q_i)$ consisting of Möbius bands $M_{h_1\cdots h_{i-1}}(p_{i-1},q_i)$ mod (p_{i-1},q_i) $h_1=1,2,\cdots,l_1$, $h_2=1,2,\cdots,l_2,\cdots,h_{i-1}=1,2,\cdots,l_{i-1}$, the integral chain $\delta(i)=\sum_{h_1=1}^{l_1}\sum_{h_2=2}^{l_2}\cdots\sum_{h_{i-1}=1}^{l_{i-1}}\delta_{h_1\cdots h_{i-1}}$ and a mapping $\theta_{i-1}^i: Q(q_1,q_i)\rightarrow Q(q_1,\cdots,q_{i-1})$ such that $(\theta_{i-1}^i)_\#\delta(i)=p_{i-1}\delta(i-1)$, where $\delta_{h_1\cdots h_{i-1}}$ is the fundamental chain of $M_{h_1\cdots h_{i-1}}(p_{i-1},q_i)$, $h_1=1,2,\cdots,l_1,\cdots,h_{i-1}=1,2,\cdots,l_{i-1}$. Let us denote by $\mu_{h_1\cdots h_i}(i)$, $h_i=1,2,\cdots,l_i$, all 2-dimensional simplexes of $M_{h_1\cdots h_{i-1}}(p_{i-1},q_i)$, $h_1=1,2,\cdots,l_1$, $h_2=1,2,\cdots,l_2,\cdots,h_{i-1}=1,2,\cdots,l_{i-1}$. We replace every triangle $\mu_{h_1\cdots h_i}(i)$ by a Möbius band $M_{h_1\cdots h_i}(p_i,q_{i+1})$ mod (p_i,q_{i+1}) such that $M_{h_1\cdots h_i}(p_i,q_{i+1})\cap M_{h_1'\cdots h_i'}(p_i,q_{i+1})=S_{h_1\cdots h_i}\cap S_{h_1'\cdots h_i'}=\mu_{h_1\cdots h_i}(i)\cap\mu_{h_1'\cdots h_i'}(i)$ for $(h_1,\cdots,h_i)\neq(h_1',\cdots,h_i')$, where $S_{h_1\cdots h_i}$ and $S_{h_1'\cdots h_i'}$ are the outer-boundaries of $M_{h_1\cdots h_i}(p_i,q_{i+1})$ and $M_{h_1'\cdots h_i'}(p_i,q_{i+1})$ respectively, $h_1=1,2,\cdots,l_1$, $h_2=1,2,\cdots,l_2,\cdots,h_i=1,2,\cdots,l_i$. Put $Q(q_1,\cdots,q_{i+1})=\sum_{h_1=1}^{l_1}\cdots\sum_{h_i=1}^{l_i}M_{h_1\cdots h_i}(p_i,q_{i+1})$. Let $\delta_{h_1\cdots h_i}$ be the fundamental chain of $M_{h_1\cdots h_i}(p_i,q_{i+1})$ (Lemma 13). Put $\delta(i+1)=\sum_{h_1=1}^{l_1}\cdots\sum_{h_i=1}^{l_i}\delta_{h_1\cdots h_i}$. Then $\delta(i+1)$ is an integral chain and $\frac{1}{q_{i+1}}\delta(i+1)$ is a cycle mod 1 relative to S, where S is the outer-boundary of $Q(q_1)$. By Lemma 14 there exists a mapping $\theta_i^{i+1}: Q(q_1,\cdots,q_{i+1})\rightarrow Q(q_1,\cdots,q_i)$ such that $(\theta_i^{i+1})_\sharp\delta(i+1)=p_i\delta(i)$, $i=1,2,\cdots$. Put $Q(\mathfrak{a})=\varprojlim\{Q(q_1,\cdots,q_i):\theta_i^{i+1}\}$. Let us denote by $\varDelta_i$ the 1-section of $Q(q_1,\cdots,q_i)$ and by θ_i the projection from $Q(\mathfrak{a})$ onto $Q(q_1,\cdots,q_i)$. For each $i=1,2,\cdots$, the restricted mapping $\theta_i|\varDelta_i$ is a homeomorphism, where we assume that $\varDelta_i\subset Q(\mathfrak{a})$.

Lemma 16. *The continuum $Q(\mathfrak{a})$ is 2-dimensional.*

Proof. Let us denote by S the outer-boundary of $Q(q_1)$. By the continuity theorem of Čech homology groups [9], Chap. X we have an isomorphism $H_n(Q(\mathfrak{a}),S:R_1)\approx\varprojlim\{H_2(Q(q_1,\cdots,q_i),S:R_1):(\theta_i^{i+1})_*\}$. Consider the chain $c_i=\dfrac{1}{q_i}\delta(i)$ of $C_2(Q(q_1,\cdots,q_i),S:R_1)$ for $i=1,2,\cdots$. It follows from our construction of the polytope $Q(q_1,\cdots,q_i)$ that c_i belongs to $Z_2(Q(q_1,\cdots,q_i),S:R_1)$. Moreover, we have $(\theta_i^{i+1})_\sharp c_{i+1}=c_i$. Therefore, the sequence $\{c_i|i=1,2,\cdots\}$ determines an element a of $H_2(Q(\mathfrak{a}),S:R_1)$ which is not zero. By Lemma 2, this shows that $\dim Q(\mathfrak{a})\geqq 2$. Since $\dim Q(\mathfrak{a})\leqq 2$ by Lemma 12, we have $\dim Q(\mathfrak{a})=2$.

Let $\{p_i|i=1,2,\cdots\}$ be a sequence of positive integers such that $p_i>1$ for some i. Put $q_i=p_1p_2\cdots p_i$, $i=1,2,\cdots$. Then $\mathfrak{a}=\{q_1,q_2,\cdots\}$ is a k-sequence. We have the following lemma.

Lemma 17. *The product space $P(p_1,p_2,\cdots)\times Q(\mathfrak{a})$ is a 3-dimensional continuum.*

Proof. In the above notations $P(p_1,p_2,\cdots)=\varprojlim\{P(p_1,p_2,\cdots,p_i):\Pi_i^{i+1}\}$ and $Q(\mathfrak{a})=\varprojlim\{Q(q_1,q_2,\cdots,q_i):\theta_i^{i+1}\}$. Let τ and μ be 2-dimensional simplexes of $P(p_1,p_2,\cdots,p_j)$ and $Q(q_1,q_2,\cdots,q_i)$. Put $A=(\Pi_i^{i+1})^{-1}\tau$ and $B=(\theta_i^{i+1})^{-1}\mu$. Then A and B are subcomplexes of $P(p_1,p_2,\cdots,p_{i+1})$ and $Q(q_1,q_2,\cdots,q_{i+1})$ which are homeomorphic to Möbius bands mod $(1,p_{i+1})$ and (p_{i+1},q_{i+1}) respectively. Consider the restricted mapping $g=\Pi_i^{i+1}\times\theta_i^{i+1}|A\times B$ of $A\times B$ into $\tau\times\mu$. By Lemma 15 the mapping g is inessential. Accordingly there exists a mapping f of $P(p_1,\cdots,p_{i+1})\times Q(q_1,\cdots,q_{i+1})$ into $P(p_1,p_2,\cdots,p_i)\times\varDelta_i\cup\varDelta_i\times Q(q_1,\cdots,q_i)$ such that $f(x,y)\in\tau\times\dot{\mu}\cup\dot{\tau}\times\mu$ for $(x,y)\in(\Pi_i^{i+1})^{-1}\tau\times(\theta_i^{i+1})^{-1}\mu$. This shows that for each positive number ε there exists an ε-mapping of $P(p_1,p_2,\cdots)\times Q(\mathfrak{a})$ into a 3-dimensional simplicial complex. Since $\dim(P(p_1,p_2,\cdots)\times Q(\mathfrak{a}))\geqq 3$ by [10], we have $\dim(P(p_1,p_2,\cdots)\times Q(\mathfrak{a}))=3$ by Lemma 11. This completes the proof.

§ 4. Main theorem.

Theorem. *Let X be a finite dimensional compact metric space. In order that the equality* (**A**) *holds for every compact metric space Y it is necessary and sufficient that X has the property* **P**.

Proof of the sufficiency.

Let Y be an m-dimensional compact metric space. By Lemma 2, there exists a closed subset B such that $H_m(Y, B: R_1)\neq 0$. Let $\{\mathfrak{V}_i\}$ be a cofinal collection of coverings of Y each member of which has the order m. Let us denote by (M_i, N_i) the pair of the nerves of $\mathfrak{V}_i$ corresponding to (Y, B) and by ψ_i^{i+1} a projection of (M_{i+1}, N_{i+1}) into (M_i, N_i). Then $H_m(Y, B: R_1)=\varprojlim\{Z_m(M_i, N_i: R_1): (\psi_i^{i+1})_*\}$. Since $H_m(Y, B: R_1)\neq 0$, we can find a non-zero element $d=\{d_i\}$ of $H_m(Y, B: R_1)$ such that $d_i\in Z_m(M_i, N_i: R_1)$ and $(\psi_i^{i+1})_\# d_{i+1}=d_i$ for $i=1, 2,\cdots$. Let $d_i=\sum_{j=1}^{k_i}\frac{r_j^i}{t_j^i}\sigma_j(i)$, where r_j^i and t_j^i are coprime integers and $\sigma_j(i)$ is an m-dimensional oriented simplex of M_i-N_i for $j=1, 2,\cdots, k_i$ and $i=1, 2,\cdots$. Since $\dim M_i=m$ and d_i is an m-dimensional chain, the order of the element d_i, which we denote by q_i, is the least common multiple of a finite number of integers $\{t_j^i\,|\,j=1, 2,\cdots, k_i\}$. Then q_i is a divisor of q_{i+1} for $i=1, 2,\cdots$. Therefore, the sequence $\mathfrak{a}=(q_1, q_2,\cdots)$ is a k-sequence. Since X has the property $\boldsymbol{P}$, by Lemma 8 there exists a closed subset A of X such that $H_n(X, A: \mathfrak{a})\neq 0$. Let $\{\mathfrak{U}_i\}$ be a cofinal collection of coverings of X such that the order of $\mathfrak{U}_i$ is n for $i=1, 2,\cdots$. Let us denote by (K_i, L_i) the pair of the nerves of $\mathfrak{U}_i$ corresponding to (X, A) and by ϕ_i^{i+1} a projection of (K_{i+1}, L_{i+1}) into (K_i, L_i) for $i=1, 2,\cdots$. Let $c=\{c_i\}$ be a non-zero element of $H_n(X, A: \mathfrak{a})$. Since $\dim K_i=n$, we can suppose that c_i belongs to the group $Z_n(K_i, L_i: Z_{q_i})$. Let $c_i=\sum_{l=1}^{h_i}s_l^i\tau_l(i)$, where $s_l^i\in Z_{q_i}$ and $\tau_l(i)$ is an oriented n-dimensional simplex of K_i-L_i for $l=1, 2,\cdots, h_i$ and $i=1, 2,\cdots$. Take an integer $\tilde{s}_l^i$ such that $h_{q_i}(\tilde{s}_l^i)=s_l^i$ for $l=1, 2,\cdots, h_i$ and $i=1, 2,\cdots$, where h_{q_i} means the natural homomorphism from Z onto Z_{q_i}. Put $a_i=\tilde{c}_i\times d_i$[29], where $\tilde{c}_i$ is the integral chain $\sum_{l=1}^{h_i}\tilde{s}_l^i\tau_l(i)$ of (K_i, L_i) for $i=1, 2,\cdots$. Then the chain a_i is an element of the group $C_{m+n}((K_i, L_i)\times(M_i, N_i): R_1)$. Moreover, a_i belongs to the group $Z_{m+n}((K_i, L_i)\times(M_i, N_i): R_1)$. For, we have $\partial a_i=\partial\tilde{c}_i\times d_i\pm\tilde{c}_i\times\partial d_i$[30]. Suppose that an $(n-1)$-dimensional simplex of K_i-L_i has a coefficient l in $\partial\tilde{c}_i$. Since c_i belongs to $Z_n(K_i, L_i: Z_{q_i})$, we have $l\equiv 0 \bmod q_i$. Accordingly the chain $\partial\tilde{c}_i\times d_i$ is an integral chain, since the integer q_i is the least common multiple of t_j^i for $j=1, 2,\cdots, k_i$. Moreover any $(m-1)$-dimensional simplex of M_i-N_i has an integral coefficient in ∂d_i since d_i belongs to $Z_m(M_i, N_i: R_1)$. Therefore any $(m+n-1)$-dimensional cell of $K_i\times M_i-(K_i\times N_i\cup L_i\times M_i)$ has an integral coefficient in ∂a_i. This shows that $a_i\in Z_{m+n}((K_i, L_i)\times(M_i, N_i): R_1)$. The cycle a_i is independent of the choice of an integer s_l^i such that $h_{q_i}(s_l^i)=s_l^i$ for $l=1, 2,\cdots, h_i$. For, take another integer $'s_l^i$ such that $h_{q_i}('s_l^i)=s_l^i$ for $l=1, 2,\cdots, h_i$. Put $'\tilde{c}_i=\sum_{i=1}^{h_i}{'s_l^i}\tau_l(i)$ and $'a_i={'\tilde{c}_i}\times d_i$, $i=1, 2,\cdots$. Then we have $a_i-{'a_i}=\tilde{c}_i\times d_i-{'\tilde{c}_i}\times d_i=(\tilde{c}_i-{'\tilde{c}_i})\times d_i$. Since $s_l^i-{'s_l^i}\equiv 0 \bmod q_i$ and q_id_i is an integral chain, we have $a_i\equiv{'a_i} \bmod 1$. Let Π_i^{i+1} be the

29) See footnote [17].
30) See footnote [21].

product mapping[31] $\phi_i^{i+1}\times\psi_i^{i+1}$ of $(K_{i+1}, L_{i+1})\times(M_{i+1}, N_{i+1})$ into $(K_i, L_i)\times(M_i, N_i)$. Consider the chain $(\Pi_i^{i+1})_\# a_{i+1}-a_i$ for $i=1, 2,\cdots$. Since $a_{i+1}=\tilde{c}_{i+1}\times d_{i+1}$, we have $(\Pi_i^{i+1})_\# a_{i+1}=(\phi_i^{i+1})_\#\tilde{c}_{i+1}\times(\psi_i^{i+1})_\# d_{i+1}$. Since $\mathfrak{P}_i^{i+1}(c_{i+1})=c_i$, we have $(\phi_i^{i+1})_\#\tilde{c}_{i+1}-c_i\equiv 0$ mod q_i, where $\mathfrak{P}_i^{i+1}$ is the homomorphism of $H_n(K_{i+1}, L_{i+1}: Z_{q_{i+1}})$ into $H_n(K_i, L_i: Z_{q_i})$ used in the definition of the group $H_n(X, A:\mathfrak{a})$ (cf. § 2). Since $(\psi_i^{i+1})_\# d_{i+1}-d_i\equiv 0$ mod 1 and $q_i d_i$ is an integral chain, we have

$$
\begin{aligned}
&(\Pi_i^{i+1})_\# a_{i+1}-a_i\\
&=(\phi_i^{i+1})_\#\tilde{c}_{i+1}\times(\psi_i^{i+1})_\# d_{i+1}-\tilde{c}_i\times d_i\\
&=(\phi_i^{i+1})_\#\tilde{c}_{i+1}\times(\psi_i^{i+1})_\# d_{i+1}-(\phi_i^{i+1})_\#\tilde{c}_{i+1}\times d_i\\
&\quad+(\phi_i^{i+1})_\#\tilde{c}_{i+1}\times d_i-\tilde{c}_i\times d_i\\
&=(\phi_i^{i+1})_\#\tilde{c}_{i+1}\times\{(\psi_i^{i+1})_\# d_{i+1}-d_i\}\\
&\quad+\{(\phi_i^{i+1})_\#\tilde{c}_{i+1}-\tilde{c}_i\}\times d_i\equiv 0 \qquad \text{mod } 1.
\end{aligned}
$$

This shows that $(\Pi_i^{i+1})_* a_{i+1}=a_i$ for $i=1, 2,\cdots$. Therefore $\{a_i\}$ determines an element a of $H_{m+n}((X, A)\times(Y, B): R_1)$ by Lemma 5. Suppose that $a=0$. Then we have $a_i=0$ for $i=1, 2,\cdots$. Therefore we have $\frac{r_j^i}{t_j^i}\times s_j^i\equiv 0$ mod 1 for $j=1, 2, \cdots, k_i, l=1, 2,\cdots, h_i$ and $i=1, 2,\cdots$. Accordingly we have $s_l^i\equiv 0$ mod q_i for $l=1, 2, \cdots, h_i$ and $i=1, 2,\cdots$. This shows that $c_i=0$ for $i=1, 2,\cdots$. This contradicts our assumption $c\neq 0$. Thus we have proved that $H_{m+n}((X, A)\times(Y, B): R_1)\neq 0$. Therefore we have $\dim(X\times Y)\geqq m+n$ by Lemma 2. Since $\dim(X\times Y)\leqq m+n$ by Lemma 3, we have $\dim(X\times Y)=\dim X+\dim Y$. This completes the proof of the sufficiency part of the theorem.

To prove the necessity part of the theorem, it is sufficient to prove the following lemma.

Lemma 18. *If X has not the property* $\boldsymbol{P}$, *there exists a 2-dimensional compact metric space Y such that* $\dim(X\times Y)=\dim X+1$.

Proof. Let $\dim X=n$. Since X has not the property $\boldsymbol{P}$, by Lemma 7 there exists a k-sequence $\mathfrak{a}=(q_1, q_2,\cdots)$ such that for every pair of (A, B) of closed subsets of X we have $H_n(A, B: Z(\mathfrak{a}))=0$. By Lemma 8, we see that $H_n(A, B:\mathfrak{a})=0$ for each pair (A, B) of closed sets of X. Let us construct the continuum $Y=Q(\mathfrak{a})$ described in 3) of § 3. We shall prove that $\dim(X\times Y)=\dim X+1$. Let $\{\mathfrak{U}_i\}$ be a cofinal collection of coverings of X each member of which has the order n. Let K_i be the nerve of $\mathfrak{U}_i$ and let ϕ_i be a canonical mapping of X into $K_i, i=1, 2,\cdots$. Let us denote by ϕ_i^{i+1} a projection of K_{i+1} into K_i. Let f_i be the product mapping $\phi_i\times\theta_i$ of $X\times Y$ into the cell complex $K_i\times Q(q_1,\cdots, q_i), i=1, 2,\cdots$, where θ_i is the projection from $Y=Q(\mathfrak{a})$ onto $Q(q_1,\cdots, q_i)$ described in 3) of § 3. Take a positive number ε. There exists some integer i such that for each cell e of $K_i\times Q(q_1,\cdots, q_i)$ we have the diameter of

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31) See footnote [10].

$f_i^{-1}(e)<\varepsilon$. Let σ be an n-dimensional simplex of K_i and let μ be a 2-dimensional simplex of $Q(q_1,\cdots,q_i)$. Put $A=\phi_i^{-1}(\sigma)$, $B=\phi_i^{-1}(\dot\sigma)$, $C=\theta_i^{-1}(\mu)$ and $D=\theta_i^{-1}(\dot\mu)$. Let us denote by (A_j, B_j) the pair of the subcomplexes of K_j corresponding to (A, B). By $\sigma_k(j)$, $k=1,2,\cdots,k_j$, we mean all n-dimensional and oriented simplexes of A_j-B_j. Let us denote by (C_j, D_j) the pair of the subcomplexes of $Q(q_1,\cdots,q_j)$ which is the image of (C, D) under θ_j for each $j>i$. Then we have $(C_{i+1}, D_{i+1})=(M(p_i, q_{i+1}), T)$, where $M(p_i, q_{i+1})$ is a Möbius band mod (p_i, q_{i+1}) and T is its outer-boundary, and we have $D_j=T$, $j=i+1, i+2,\cdots$. We shall use similar notations as in 3) of § 3. Let us denote by $\mu_{h_1}(i+1)$, $h_1=1,2,\cdots,l_1$, all 2-dimensional and oriented simplexes of $M(p_i, q_{i+1})$. Then we have $\delta(i+1)=\sum_{h_1=1}^{l_1}\mu_{h_1}(i+1)$, where $\delta(i+1)$ is the fundamental chain[32] of $M(p_i, q_{i+1})$. In general, for $j>1$, we have $C_{i+j}=\sum_{h_1=1}^{l_1}\cdots\sum_{h_{j-1}}^{l_{j-1}}M_{h_1\cdots h_{j-1}}(p_{i+j-1}, q_{i+j})$, where $M_{h_1\cdots h_{j-1}}(p_{i+j-1}, q_{i+j})$ is a Möbius band mod (p_{i+j-1}, q_{i+j}), $h_1=1,2,\cdots,l_1$, $h_2=1,2,\cdots,l_2,\cdots,h_{j-1}=1,2,\cdots,l_{j-1}$. Let us denote by $\mu_{h_1\cdots h_j}(i+j)$ all 2-dimensional and oriented simplexes of $M_{h_1\cdots h_{j-1}}(p_{i+j-1}, q_{i+1})$, $h_j=1,2,\cdots,l_j$, $h_1=1,2,\cdots,l_2\cdots,h_{j-1}=1,2,\cdots,l_{j-1}$. Then we have $\delta_{h_1\cdots h_{j-1}}=\sum_{h_j=1}^{l_j}\mu_{h_1\cdots h_j}(i+j)$, where $\delta_{h_1\cdots h_j}$ is the fundamental chain[32] of $M_{h_1\cdots h_{j-1}}(p_{i+j-1}, q_{i+j})$, $h_1=1,2,\cdots,l_1,\cdots,h_{j-1}=1,2,\cdots,l_{j-1}$. Consider the restricted mapping $f(\sigma,\mu)=f_i\,|\,A\times D\cup B\times C: A\times D\cup B\times C\to(\sigma\times\dot\mu)\cup(\dot\sigma\times\mu)$. We shall prove that there exists an extension of $f(\sigma,\mu)$ over $A\times C$. By Lemma 5 we have $H_{n+2}((A, B)\times(C, D): R_1)=\underleftarrow{\lim}\{H_{n+2}((A_j, B_j)\times(C_j, D_j):(\Pi_j^{j+1})_*=(\phi_j^{j+1}\times\theta_j^{j+1})_*\}$, where ϕ_j^{j+1} and θ_j^{j+1} are the restricted projections $\phi_j^{j+1}|A_{j+1}$ and $\theta_j^{j+1}|C_{j+1}$ of A_{i+1} and C_{j+1} into A_j and C_j respectively. Take an element $a=\{a_j\}$ of $H_{n+2}((A, B)\times(C, D): R_1)$. Then a_j is an element of $H_{n+2}((A_j, B_j)\times(C_j, D_j): R_1)$. Since $\dim(A_j\times C_j)\leqq n+2$, we can consider that $a_j\in Z_{n+2}((A_j, B_j)\times(C_j, D_j): R_1)$. Let $a_{i+1}=\sum_{k=1}^{k_{i+1}}\sum_{h_1=1}^{l_1}t_{kh_1}^{i+1}(\sigma_k(i+1)\times\mu_{h_1}(i+1))$, where $t_{kh_1}^{i+1}\in R_1$. We have[33] $\partial a_{i+1}=\sum_{k=1}^{k_{i+1}}\sum_{h_1=1}^{l_1}t_{kh_1}^{i+1}(\partial\sigma_k(i+1)\times\mu_{h_1}(i+1))\pm\sum_{k=1}^{k_{i+1}}\sum_{h_1=1}^{l_1}t_{kh_1}^{i+1}(\sigma_k(i+1)\times\partial\mu_{h_1}(i+1))$. Put $b(i+1)=\sum_{k=1}^{k_{i+1}}\sum_{h_1=1}^{l_1}t_{kh_1}^{i+1}(\partial\sigma_k(i+1)\times\mu_{h_1}(i+1))$ and $c(i+1)=\sum_{k=1}^{k_{i+1}}\sum_{h_1=1}^{l_1}t_{kh_1}^{i+1}(\sigma_k(i+1)\times\partial\mu_{h_1}(i+1))$. Since ∂a_{i+1} is a chain of $C_{n+1}(A_{i+1}\times D_{i+1}\cup B_{i+1}\times C_{i+1}: R_1)$, if an $(n+1)$-cell e in $c(i+1)$ has a non-zero coefficient then e must belong to $A_{i+1}\times D_{i+1}$. Since $c(i+1)=\sum_{k=1}^{k_{i+1}}\sigma_k(i+1)\times\partial(\sum_{h_1=1}^{l_1}t_{kh_1}^{i+1}\mu_{h_1}(i+1))$, the chain $\sum_{h_1=1}^{l_1}t_{kh_1}^{i+1}\mu_{h_1}(i+1)$ is an element of the group $Z_2(C_{i+1}, D_{i+1}: R_1)$. By Lemma 13 we have $t_{kh_1}^{i+1}=t_k^{i+1}$, $h_1=1,2,\cdots,l_1$, and $q_{i+1}t_k^{i+1}\equiv 0 \bmod 1$. Put $d(i+1)=\sum_{k=1}^{k_{i+1}}t_k^{i+1}\sigma_k(i+1)$. Since $b(i+1)=\sum_{h_1=1}^{l_1}\partial d(i+1)\times\mu_{h_1}(i+1)$[34], the chain $d(i+1)$ belongs to $Z_n(A_{i+1}, B_{i+1}: R_1)$. Put $u(i+1)=q_{i+1}d(i+1)$. Then we have $a_{i+1}=u(i+1)\times\dfrac{1}{q_{i+1}}\delta(i+1)$. The chain $u(i+1)$ is an integral chain and, since $\partial u(i+1)=q_{i+1}(\partial d(i+1))$ and $\partial d(i+1)$ is an integral chain, the chain $u(i+1)$ determines an element

32) Cf. Lemma 13, § 3.

33) See footnote 21).

34) See footnote 17).

$\tilde{u}(i+1)$ of the group $Z_n(A_{i+1}, B_{i+1}: Z_{q_{i+1}})$. Consider the cycle a_{i+2}. Let $a_{i+2}=\sum_{k=1}^{k_{i+2}}\sum_{h_1=1}^{l_1}\sum_{h_2=1}^{l_2} t_{kh_1h_2}^{i+2}(\sigma_k(i+2)\times\mu_{h_1h_2}(i+2))$, where $\sigma_k(i+2)$ is an n-simplex of $A_{i+1}-B_{i+2}, k=1,\cdots,k_{i+2}$ and $\mu_{h_1h_2}(i+2)$ is a 2-simplex of $M_{h_1}(p_{i+1}, q_{i+2}), h_1=1,\cdots, l_1$ and $h_2=1,\cdots,l_2$. We have $\partial a_{i+2}=\sum_{k=1}^{k_{i+2}}\sum_{h_1=1}^{l_1}\sum_{h_2=1}^{l_2} t_{kh_1h_2}^{i+2}(\partial\sigma_k(i+2)\times\mu_{h_1h_2}(i+2)) \pm\sum_{k=1}^{k_{i+2}}\sum_{h_1=1}^{l_1}\sum_{h_2=1}^{l_2} t_{kh_1h_2}^{i+2}(\sigma_k(i+2)\times\partial\mu_{h_1h_2}(i+2))$. Put $b_{h_1}(i+2)=\sum_{k=1}^{k_{i+2}}\sum_{h_2=1}^{l_2} t_{kh_1h_2}^{i+2} (\partial\sigma_k(i+2)\times\mu_{h_1h_2}(i+2))$ and $c_{h_1}(i+2)=\sum_{k=1}^{k_{i+1}}\sum_{h_2=1}^{l_2} t_{kh_1h_2}^{i+2}(\sigma_k(i+2)\times\partial\mu_{h_1h_2}(i+2)), h_1=1, \cdots, l_1$. Then we have $c_{h_1}(i+2)=\sum_{k=1}^{k_{i+2}}\sigma_k(i+2)\times\partial(\sum_{h_2=1}^{l_2} t_{kh_1h_2}^{i+2}\mu_{h_1h_2}(i+2))$. Since ∂a_{i+2} is a chain of $C_{n+1}(A_{i+2}\times D_{i+2}\cup B_{i+2}\times C_{i+2}: R_1), c_{h_1}(i+2)$ is a chain of $C_{n+1}(A_{i+2}\times T_{h_1}: R_1)$, where T_{h_1} is the outer-boundary of $M_{h_1}(p_{i+1}, q_{i+2})$. Therefore the chain $\sum_{h_2=1}^{l_2} t_{kh_1h_2}^{i+2}\mu_{h_1h_2}(i+2)$ belongs to $Z_2(M_{h_1}(p_{i+1}, q_{i+2}), T_{h_1}: R_1)$. By Lemma 13 we have $t_{kh_1h_2}^{i+2}=t_{kh_1}^{i+2}$ and $q_{i+1}t_{kh_1}^{i+2}\equiv 0 \mod 1$, $h_1=1,2,\cdots,l_1$ and $h_2=1,2, \cdots,l_2$. Put $d_{h_1}(i+2)=\sum_{k=1}^{k_{i+2}} t_{kh_1}^{i+2}\sigma_k(i+2), h_1=1,2,\cdots,l_1$. Since $b_{h_1}(i+2)=\sum_{h_2=1}^{l_2}\partial d_{h_1}(i+2)\times\mu_{h_1h_2}(i+2)$ and $b_{h_1}(i+2)$ is a chain of $C_{n+1}(B_{i+2}\times M_{h_1}(p_{i+2}, q_{i+2}): R_1)$, the chain $d_{h_1}(i+2)$ belongs to $Z_n(A_{i+2}, B_{i+2}: R_1), h_1=1,2,\cdots,l_1$. Put $u_{h_1}(i+2)=q_{i+2}d_{h_1}(i+2)$, $h_1=1,2,\cdots,l_1$. The chain $u_{h_1}(i+2)$ is an integral chain and determines an element $\tilde{u}_{h_1}(i+2)$ of $Z_u(A_{i+2}, B_{i+2}: Z_{q_{i+2}})$. Since $a'_{i+2}=\sum_{h_1=1}^{l_1}(u_{h_1}(i+2)\times\frac{1}{q_{i+2}}\delta_{h_1}(i+2))$, where $\delta_{h_1}(i+2)$ is the fundamental chain of $M_{h_1}(p_{i+1}, q_{i+2})$, we have

$$
\begin{aligned}
(\Pi_{i+1}^{i+2})_* a_{i+2}-a_{i+1}\equiv 0&\\
&\equiv\sum_{h_1=1}^{l_1}(\Pi_{i+1}^{i+2})_\#(u_{h_1}(i+2)\times\frac{1}{q_{i+2}}\delta_{h_1}(i+2))\\
&\quad -u(i+1)\times\frac{1}{q_{i+2}}\delta(i+1)\\
&\equiv\sum_{h_1=1}^{l_1}(\phi_{i+1}^{i+2})_\# u_{h_1}(i+2)\times(\theta_{i+1}^{i+2})_\#\frac{1}{q_{i+2}}\delta_{h_1}(i+2)\\
&\quad -\sum_{h_1=1}^{l_1}u(i+1)\times\frac{1}{q_{i+1}}\mu_{h_1}(i+1)\\
&\equiv\sum_{h_1=1}^{l_1}(\phi_{i+2}^{i+2})_\#(u_{h_1}(i+2)\times\frac{1}{q_{i+1}}\mu_{h_1}(i+1)\\
&\quad -\sum_{h_1=1}^{l_1}u(i+1)\times\frac{1}{q_{i+1}}\mu_{h_1}(i+1)\\
&\equiv\sum_{h_1=1}^{l_1}\{(\phi_{i+1}^{i+2})_\# u_{h_1}(i+2)-u(i+1)\}\times\frac{1}{q_{i+1}}\mu_{h_1}(i+1) \qquad \mod 1.
\end{aligned}
$$

Therefore we have $(\phi_{i+1}^{i+2})_\# u_{h_1}(i+2)-u(i+1)\equiv 0 \mod q_{i+1}, h_1=1,2,\cdots,l_1$. This shows that $\mathfrak{P}_{i+1}^{i+2}\tilde{u}_{h_1}(i+2)=\tilde{u}(i+1), h_1=1,2,\cdots,l_1$, where $\mathfrak{P}_{i+1}^{i+2}$ is the homomorphism of $Z_n(A_{i+2}, B_{i+2}: Z_{q_{i+2}})$ into $Z_n(A_{i+1}, B_{i+1}: Z_{q_{i+1}})$, (cf. § 2). To proceed by induction, suppose that we can find integral chains $u_{h_1\cdots h_{j-1}}(i+j)$ of (A_{i+j}, B_{i+j}), which are considered as cycles mod q_{i+j}, such that $a_{i+j}=\sum_{h_1=1}^{l_1}\cdots\sum_{h_{j-1}}^{l_{j-1}} u_{h_1\cdots h_{j-1}}(i+j) \times\frac{1}{q_{i+j}}\delta_{h_1\cdots h_{j-1}}(i+j)$, where $\delta_{h_1\cdots h_{j-1}}(i+j)$ is the fundamental chain of $M_{h_1\cdots h_{j-1}} (p_{i+j-1}, q_{i+j})$ and $h_1=1,2,\cdots,l_1,\cdots,h_{j-1}=1,2,\cdots,l_{j-1}$. The chain $u_{h_1\cdots h_{j-1}}(i+j)$ determines an element $\tilde{u}_{h_1\cdots h_{j-1}}(i+j)$ of $Z_n(A_{i+j}, B_{i+j}: Z_{q_{i+j}})$. Moreover let us assume that $\mathfrak{P}^{i+j}_{i+j-1}\tilde{u}_{h_1\cdots h_{j-1}}(i+j)=\tilde{u}_{h_1\cdots h_{j-2}}(i+j-1)$. Let $a_{i+j+1}=\sum_{k=1}^{k_{i+j+1}}\sum_{h_1=1}^{l_1}\cdots\sum_{h_{j+1}=1}^{l_{j+1}} t^{i+j+1}_{kh_1\cdots h_{j-1}}(\sigma_k(i+j+1)\times\mu_{h_1\cdots h_{j+1}}(i+j+1))$, where $t^{i+j+1}_{kh_1\cdots h_{j+1}}\in R_1$, $k=1,2,\cdots,k_{i+j+1}$, $h_1=1,2,\cdots,l_1,\cdots, h_{j+1}=1,2,\cdots,l_{j+1}$. Put $b_{h_1\cdots h_j}(i+j+1)=\sum_{k=1}^{k_{i+j+1}}\sum_{h_{j+1}=1}^{l_{j+1}} t^{i+j+1}_{kh_1\cdots h_{j+1}}(\partial\sigma_k(i+j+1)\times\mu_{h_1\cdots h_{j+1}}(i+j+1))$ and $c_{h_1\cdots h_j}(i+j+1)=\sum_{k=1}^{k_{i+j+1}}\sum_{h_{j+1}=1}^{l_{j+1}} t^{i+j+1}_{kh_1\cdots h_j}(\sigma_k(i+j+1)\times\partial\mu_{h_1\cdots h_{j+1}}(i+j+1))$, $h_1=1,\cdots,l_1,\cdots,h_j=1,\cdots,l_j$. Then we have $\partial a_{i+j+1}=\sum_{h_1=1}^{l_1}\cdots\sum_{h_j=1}^{l_j}(b_{h_1\cdots h_j}(i+j+1)\pm c_{h_1\cdots h_j}(i+j+1))$. Since ∂a_{i+j+1} is a chain of $C_{n+1}(A_{i+j+1}\times D_{i+j+1}\cup B_{i+j+1}\times C_{i+j+1}: R_1)$, $c_{h_1\cdots h_j}(i+j+1)$ is a chain of $C_{n+1}(A_{i+j+1}\times T_{h_1\cdots h_j}: R_1)$, where $T_{h_1\cdots h_j}$ is the outer-boundary of $M_{h_1\cdots h_j}(p_{i+j}, q_{i+j+1})$, $h_1=1,\cdots,l_1,\cdots,h_j=1,\cdots,l_j$, Since $c_{h_1\cdots h_j}(i+j+1)=\sum_{k=1}^{k_{i+j+1}}\sigma_k(i+j+1)\times\partial(\sum_{h_{j+1}=1}^{l_{j+1}} t^{i+j+1}_{kh_1\cdots h_{j+1}}\mu_{h_1\cdots h_{j+1}}(i+j+1))$, the chain $\sum_{h_{j+1}=1}^{l_{j+1}} t^{i+j+1}_{kh_1\cdots h_{j+1}}\mu_{h_1\cdots h_{j+1}}(i+j+1)$ belongs to $Z_2(M_{h_1\cdots h_j}(p_{i+j}, q_{i+j+1}), T_{h_1\cdots h_j}: R_1)$. By Lemma 13 we have $t^{i+j+1}_{kh_1\cdots h_j h_{j+1}}=t^{i+j+1}_{kh_1\cdots h_j}$ and $q_{i+j+1}t^{i+j+1}_{kh_1\cdots h_j}\equiv 0 \bmod 1$, $k=1,\cdots,k_{i+j+1}$, $h_1=1,\cdots,l_1,\cdots,h_{j+1}=1,2,\cdots,l_{j+1}$. Put $d_{h_1\cdots h_j}(i+j+1)=\sum_{k=1}^{k_{i+j+1}} t^{i+j+1}_{kh_1\cdots h_j}\sigma_k(i+j+1)$, $h_1=1,\cdots,l_1,\cdots, h_j=1,\cdots,l_j$. Since $b_{h_1\cdots h_j}(i+j+1)\in C_{n+1}(B_{i+j+1}\times M_{h_1\cdots h_j}(i+j+1): R_1)$ and $b_{h_1\cdots h_j}(i+j+1)=\sum_{h_{j+1}=1}^{l_{j+1}}\partial d_{h_1\cdots h_j}(i+j+1)\times\mu_{h_1\cdots h_{j+1}}(i+j+1)$, the chain $d_{h_1\cdots h_j}(i+j+1)$ belongs to $Z_n(A_{i+j+1}, B_{i+j+1}: R_1)$, $h_1=1,\cdots,l_1,\cdots, h_j=1,\cdots,l_j$. Put $u_{h_1\cdots h_j}(i+j+1)=q_{i+j+1}d_{h_1\cdots h_j}(i+j+1)$, $h_1=1,\cdots,l_1,\cdots, h_j=1,\cdots,l_j$. Then chain $u_{h_1\cdots h_j}(i+j+1)$ is an integral chain and determines an element $\tilde{u}_{h_1\cdots h_j}(i+j+1)$ of $Z_n(A_{i+j+1}, B_{i+j+1}: Z_{q_{i+j+1}})$. Moreover we have $a_{i+j+1}=\sum_{h_1=1}^{l_1}\cdots\sum_{h_j=1}^{l_j} u_{h_1\cdots h_j}(i+j+1)\times\frac{1}{q_{i+j+1}}\delta_{h_1\cdots h_j}(i+j+1)$, where $\delta_{h_1\cdots h_j}(i+j+1)$ is the fundamental chain of $M_{h_1\cdots h_j}(p_{i+j}, q_{i+j+1})$. Consider the chain $(\Pi^{i+j+1}_{i+j})_\#a_{i+j+1}-a_{i+j}$, we have

$$
\begin{aligned}
&(\Pi^{i+j+1}_{i+j})_\#a_{i+j+1}-a_{i+j}\equiv 0\\
&\equiv(\Pi^{i+j+1}_{i+j})_\#\Big(\sum_{h_1=1}^{l_1}\cdots\sum_{h_j=1}^{l_j} u_{h_1\cdots h_j}(i+j+1)\times\frac{1}{q_{i+j+1}}\delta_{h_1\cdots h_j}(i+j+1)\Big)\\
&\quad-\sum_{h_1=1}^{l_1}\cdots\sum_{h_{j-1}=1}^{l_{j-1}} u_{h_1\cdots h_{j-1}}(i+j)\times\frac{1}{q_{i+j}}\delta_{h_1\cdots h_{j-1}}(i+j))\\
&\equiv\sum_{h_1=1}^{l_1}\cdots\sum_{h_j=1}^{l_j}(\phi^{i+j+1}_{i+j})_\#u_{h_1\cdots h_j}(i+j+1)\times(\theta^{i+j+1}_{i+j})_\#\frac{1}{q_{i+j+1}}\delta_{h_1\cdots h_j}(i+j+1)\\
&\quad-\sum_{h_1=1}^{l_1}\cdots\sum_{h_{j-1}=1}^{l_{j-1}}\sum_{h_j=1}^{l_j} u_{h_1\cdots h_{j-1}}(i+j)\times\frac{1}{q_{i+j}}\mu_{h_1\cdots h_j}(i+j)\\
&\equiv\sum_{h_1=1}^{l_1}\cdots\sum_{h_j=1}^{l_j}(\phi^{i+j+1}_{i+1})_\#u_{h_1\cdots h_j}(i+j+1)\times\frac{1}{q_{i+j}}\mu_{h_1\cdots h_j}(i+j)\\
&\quad-\sum_{h_1=1}^{l_1}\cdots\sum_{h_j=1}^{l_j} u_{h_1\cdots h_{j-1}}(i+j)\times\frac{1}{q_{i+j}}\mu_{h_1\cdots h_j}(i+j)\\
&\equiv\sum_{h_1=1}^{l_1}\cdots\sum_{h_j=1}^{l_j}\{(\phi^{i+j+1}_{i+j})_\#u_{h_1\cdots h_j}(i+j+1)-u_{h_1\cdots h_{j-1}}(i+j)\}\\
&\qquad\times\frac{1}{q_{i+j}}\mu_{h_1\cdots h_j}(i+j) \qquad\qquad \bmod 1\,.
\end{aligned}
$$

This shows that $(\phi^{i+j+1}_{i+j})_\#u_{h_1\cdots h_j}(i+j+1)\equiv u_{h_1\cdots h_{j-1}}(i+j) \bmod q_{i+j}$. Therefore, we have $\mathfrak{P}^{i+j+1}_{i+j}(\tilde{u}_{h_1\cdots h_j}(i+j+1))=\tilde{u}_{h_1\cdots h_{j-1}}(i+j)$, $h_1=1,2,\cdots,l_1,\cdots, h_j=1,2,\cdots,l_j$. Thus there exist sequences $\{\tilde{u}(i+1), \tilde{u}_{h_1}(i+2), \tilde{u}_{h_1h_2}(i+3),\cdots, \tilde{u}_{h_1\cdots h_j}(i+j+1),\cdots\}$ such that $\tilde{u}_{h_1\cdots h_j}(i+j+1)\in Z_n(A_{i+j+1}, B_{i+j+1}: Z_{q_{i+j+1}})$ and $\mathfrak{P}_{i+j}^{i+j+1}(\tilde{u}_{h_1\cdots h_j}(i+j+1))=\tilde{u}_{h_1\cdots h_{j-1}}(i+j)$, $h_1=1, 2,\cdots, l_1$, $h_2=1, 2,\cdots, l_2, \cdots$, $h_j=1, 2,\cdots, l_j, \cdots$ and $j=1, 2,\cdots$. Each sequence $\{\tilde{u}_{h_1\cdots h_j}(i+j+1)\}$ determines an element $u(h_1, h_2,\cdots, h_j,\cdots)$ of the group $H_n(A, B: \mathfrak{a})$. Since $H_n(A, B: \mathfrak{a})=0$, we have $u(h_1, h_2,\cdots, h_j,\cdots)=0, h_1=1,\cdots, l_1, h_2=1,\cdots, l_2, \cdots$. Especially we have $\tilde{u}(i+1)=0$. This shows that $u(i+1)\equiv 0 \bmod q_{i+1}$. Since $a_{i+1}=u(i+1)\times\frac{1}{q_{i+1}}\delta(i+1)$, we have $a_{i+1}\equiv 0 \bmod 1$. Let us denote by ∂ the boundary homomorphism of $H_{n+2}((A, B)\times(C, D): R_1)$ into $H_{n+1}(A\times D\cup B\times C: R_1)$. Consider the element $(f(\sigma, \mu))_*\partial a$ of $H_{n+1}(\sigma\times\dot{\mu}\cup\dot{\sigma}\times\mu: R_1)$. Let us denote by g the restricted mapping $f_{i+1}|A\times C: A\times C\rightarrow A_{i+1}\times C_{i+1}$. Since the mapping $\Pi_i^{i+1}g|A\times D\cup B\times C: A\times D\cup B\times C\rightarrow\sigma\times\dot{\mu}\cup\dot{\sigma}\times\mu$ is homotopic to the mapping $f(\sigma, \mu)$, we have

$$\begin{aligned}(f(\sigma, \mu))_*\partial a&=(\Pi_i^{i+1}g|A\times D\cup B\times C)_*\partial a\\&=(\Pi_i^{i+1}|A_{i+1}\times D_{i+1}\cup B_{i+1}\times C_{i+1})_*(g|A\times D\cup B\times C)_*\partial a\\&=(\Pi_i^{i+1}|A_{i+1}\times D_{i+1}\cup B_{i+1}\times C_{i+1})_*\partial g_* a\\&=(\Pi_i^{i+1}|A_{i+1}\times D_{i+1}\cup B_{i+1}\times C_{i+1})_*\partial a_{i+1}\\&=\partial(\Pi_i^{i+1})^* a_{i+1}=0^{35)}\ .\end{aligned}$$

Since a is any element of the group $H_{n+2}((A, B)\times(C, D): R_1)$, we have $(f(\sigma, \mu))_*\partial H_{n+2}((A, B)\times(C, D): R_1)=0$. By Hopf's extension theorem (Lemma 1) there exists an extension $g(\sigma, \mu)$ of $f(\sigma, \mu)$ over $A\times C$ such that $g(\sigma, \mu)(A\times C)\subset\sigma\times\dot{\mu}\cup\dot{\sigma}\times\mu$. Thus, for each $(n+2)$-dimensional cell $\sigma\times\mu$ of $K_i\times Q(q_1,\cdots, q_i)$, we have a mapping $g(\sigma, \mu)$ of $\phi_i^{-1}(\sigma)\times\theta_i^{-1}(\mu)$ into $\sigma\times\dot{\mu}\cup\dot{\sigma}\times\mu$ which is an extension of the mapping $f(\sigma, \mu)$. Define a mapping g of $X\times Y$ into the $(n+1)$-section of $K_i\times Q(q_1,\cdots, q_i)$ by $g(x, y)=g(\sigma, \mu)(x, y)$ for $(x, y)\in\phi_i^{-1}(\sigma)\times\theta_i^{-1}(\mu)$. For each point (x, y) of $X\times Y$, it is obvious that the diameter of $g^{-1}(x, y)<2\varepsilon$. Thus, for each positive number ε, we have constructed a 2ε-mapping of $X\times Y$ into an $(n+1)$-dimensional polytope. By Lemma 11 we have $\dim(X\times Y)\leqq n+1$. Since it is obvious that $\dim(X\times Y)\geqq n+1$[36], we have $\dim(X\times Y)=n+1$. This completes the proof of Lemma 18.

§ 5. Some consequences of the main theorem.

Let (X, A) be a pair of normal spaces. Let $c=\{c_\alpha\}$ be an element of the n-dimensional Čech homology group $H_n(X, A: Z)=\varprojlim\{H_n(K_\alpha, L_\alpha: Z)\}$ with coefficients in Z. By pc, where p is an integer, we mean the element of $H_n(X, A: Z)$ whose α-coordinate is pc_α. An integer p is called a *divisor* of an element b of $H_n(X, A: Z)$ if there exists an element a of $H_n(X, A: Z)$ such

35) Cf. [9], Chap. IX, Theorems 4.4, 5.1 and 7.4.
36) See, for instance, [10].

that $b=pa$. An element a of $H_n(X, A: Z)$ is called *irreducible* if there exists no divisor of a except ± 1.

LEMMA 19. *Let (X, A) be a pair of n-dimensional normal spaces. If $H_n(X, A: Z)\neq 0$, there exists an irreducible element of $H_n(X, A: Z)$.*

PROOF. Let $H_n(X, A: Z)=\varprojlim\{H_n(K_\alpha, L_\alpha: Z)\}$. We can assume that each K_α is an n-dimensional simplicial complex. Take a non-zero element $a=\{a_\alpha\}$ of $H_n(X, A: Z)$. Let $0\neq a_\alpha=\sum_{j=1}^{k} p_j\sigma_j$, where p_j is an integer and σ_j is an n-dimensional simplex of $K_\alpha-L_\alpha, j=1, 2,\cdots, k$. If an integer p is a divisor of a, p is a common divisor of integers $\{p_j \mid j=1,\cdots, k\}$ since K_α is n-dimensional. Therefore there exist only a finite number of divisors of a. Accordingly we can find an irreducible element b of $H_n(X, A: Z)$ and a divisor p of a such that $a=pb$. This completes the proof.

LEMMA 20. *Let X be an n-dimensional compact metric space. If $H_n(X, A: Z)\neq 0$ for some closed subset A of X, then the space X has the property* $\boldsymbol{P}$.

PROOF. Let $\mathfrak{a}=\{q_1, q_2,\cdots\}$ be any k-sequence. Let us denote the natural homomorphism from Z onto $Z_{q_i}=Z/q_iZ$ by h_{q_i}. Let $\{\mathfrak{U}_i\}$ be a cofinal collection of coverings of X each member of which has the order n. By our assumption we have $H_n(X, A: Z)=\varprojlim\{H_n(K_i, L_i: Z)\}=\varprojlim\{Z_n(K_i, L_i: Z)\}\neq 0$. Let $\{c_i\}$ be a non-zero element of $H_n(X, A: Z)$. We can suppose that $\{c_i\}$ is irreducible by Lemma 19 and $c_1\neq 0$. Let $c_i=\sum_{j=1}^{l_i} p_j(i)\sigma_j(i)$, where $p_j(i)$ is an integer and $\sigma_j(i)$ is an n-dimensional simplex of $K_i-L_i, j=1, 2,\cdots, l_i$ and $i=1, 2,\cdots$. Put $t_i{}^k{}_j=h_{q_k}(p_j(i)), j=1, 2,\cdots, l_i, i=1, 2,\cdots$ and $k=1, 2,\cdots$. Since the homomorphisms h_{q_k} and $h(\mathfrak{a}, k)$ are natural, we have $h(\mathfrak{a}, k)h_{q_{k+1}}=h_{q_k}, k=1, 2,\cdots$, where $h(\mathfrak{a}, k)$ is a natural homomorphism of $Z_{q_{k+1}}$ onto Z_{q_k}. Accordingly the sequence $\{t_i{}^k{}_j \mid k=1, 2,\cdots\}$ determines an element $t_j(i)$ of $Z(\mathfrak{a})$. Define an element c_i of $C_n(K_i, L_i: Z(\mathfrak{a}))$ by $c_i=\sum_{j=1}^{l_i} t_j(i)\sigma_j(i)$ for $i=1, 2,\cdots$. Obviously each c_i belongs to $Z_n(K_i, L_i: Z(\mathfrak{a}))$. Moreover, since $(\Pi_i^{i+1})_\# c_{i+1}$[37] $=c_i, i=1, 2,\cdots$, we have $(\Pi_i^{i+1})_\# c_{i+1}$[37] $=(\Pi_i^{i+1})_\# (\sum_{j=1}^{l_{i+1}} t_j(i+1)\sigma_j(i+1))=\sum_{j=1}^{l_{i+1}} t_j(i+1)(\Pi_i^{i+1}\sigma_j(i+1))=\sum_{j=1}^{l_i} t_j(i)\sigma_j(i)=c_i$. Therefore $\{c_i\}$ determines an element c of $H_n(X, A: Z(\mathfrak{a}))$. If $c=0$, we have $t_j(i)\equiv 0$ in $Z(\mathfrak{a})$ and $h_{q_i}(p_j(i))\equiv 0$ in Z_{q_k}. Therefore we have $p_j(i)\equiv 0 \bmod q_k, j=1, 2,\cdots, l_i, i=1, 2,\cdots$ and $k=1, 2,\cdots$. Since q_1 is a divisor of q_k for $k>1$ and $q_1\neq 1, q_1$ is a divisor of the element $\{c_i\}$. This contradicts the assumption that $\{c_i\}$ is irreducible. This completes the proof.

The following corollary is a consequence of the main theorem and Lemma 20.

COROLLARY 1[38]. *Let X be an n-dimensional compact metric space. If there*

37) See footnote [26].

38) It is proved that Corollary 1 is generalized as follows:

Corollary 1′. *Let X be an n-dimensional compact space. If there exists a closed subset A of X such that $H_n(X, A: Z)\neq 0$, then the equality* $(\boldsymbol{A})$ *holds for every locally compact fully normal space Y.*

exists a closed subset A of X such that $H_n(X, A: Z)\neq 0$, then the equality (***A***) *holds for every compact metric space Y.*

A metric space X is called an ANR[39] if, whenever X is a closed subset of a metric space Y, there exists a mapping from some neighborhood of X in Y into X which keeps X point-wise fixed. A point x_0 of a topological space X is n-HL[40] in X when for every neighborhood U of x_0 there exists a neighborhood V of x_0 which is contained in U and satisfies the following condition: Let E^{n+1} be an $(n+1)$-dimensional element whose boundary is an n-sphere S^n. Then every mapping $f: S^n\rightarrow V-x_0$ is extended to a mapping $f': E^{n+1}\rightarrow U-x_0$. A point x_0 of X is called n-HS[40] in X if it is not n-HL in X. If a point x_0 is k-HL for $k=0, 1,\cdots, n$, the point x_0 is called HL^n in X. The following lemma is proved easily in a similar way as [12], Theorem 6.

LEMMA 21. *Let X be a locally compact and m-dimensional* ANR *containing a point x_0 which is* HL^{m-2} *and* $(m-1)$-HS *in X. Then there exists a pair (A, B) of compact subsets of X such that $x_0\in A$ and $H_m(A, B: Z)\neq 0$.*

By K. Borsuk [6], a topological space X is said to have the property Δ if for each point x of X and each neighborhood U of x there exists a neighborhood V of x such that 1) $V\subset U$ and 2) every compact subset A of V is contractible[41] in a subset of U of the dimension $\leqq \dim A+1$. If a finite dimensional and locally compact metric space X has the property Δ, then X is an ANR[42]. The following lemma has been proved in [6], p. 92.

LEMMA 22. *Let X be a locally compact n-dimensional metric space which has the property Δ. Then there exists a pair (A, B) of compact subsets of X such that $H_n(A, B: Z)\neq 0$.*

The following lemma has been proved in [12], Theorem 8.

LEMMA 23. *If X is a 2-dimensional locally compact* ANR, *there exists a pair (A, B) of compact svbsets of X such that $H_n(A, B: Z)\neq 0$.*

Finally, we shall prove the following lemma.

LEMMA 24. *A 1-dimensional compact metric space X has the property* ***P***.

To prove this lemma, we need the following lemmas.

LEMMA 25. *Let $\{G_i: \Pi_i^{i+1}\}$ be an inverse system of finite abelian groups G_i and let G be its limit group. Let us denote by Π_i the projection of G into G_i for $i=1, 2,\cdots$. For each $i=1, 2,\cdots$, there exists an integer $k_i>i$ such that $\Pi_i(G)$ $=\Pi_i^{k_i}(G_{k_i})$, where $\Pi_i{}^j=\Pi_i^{i+1}\cdots\Pi_{j-1}^j$, $j>i$.*

The proof is obvious.

LEMMA 26. *Let K and L be 1-dimensional connected simplicial complexes.*

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39) Cf. [5].
40) Cf. [12], p. 172.
41) See footnote 19).
42) See Y. Kodama, On LC^n metric spaces, Proc. Japan Acad., **33** (1957), 79-83.

Let u_0 and u_1 be different vertexes of K and let v_0 and v_1 be different vertexes of L. Let f be a simplicial mapping of K into L such that $f^{-1}(v_i)=u_i$ for $i=1, 2$. Let G_1 and G_2 be non-trivial abelian groups and let h be a homomorphism from G_1 onto G_2. Then the homomorphism Π of $H_1(K, u_0 \cup u_1 : G_1)$ into $H_1(L, v_0 \cup v : G_2)$ induced by the mapping f and the homomorphism h is non-trivial.

PROOF. Since K is connected, there exists a 1-dimensional integral chain $c=\sum_{i=1}^{k}\sigma_k$ such that $\partial c=u_1-u_0$, $u_0\in\sigma_1$, $u_1\in\sigma_k$ and $\sigma_i\cap\sigma_j=\phi$ for $|i-j|\geqq 1$, where σ_i's are 1-dimensional oriented and closed simplexes of K. There exists an element g of G_1 such that $h(g)\neq 0$, since h is a homomorphism onto. Then gc is a non-zero element of $Z_1(K, u_0\cup u_1 : G_1)$. Since $\dim K=1$, we can consider $gc\in H_1(K, u_0\cup u_1 : G_1)$. Let $\Pi(gc)=\sum_{l=1}^{h} g_l\tau_l$, where $0\neq g_l\in G_2$ and τ_l is a 1-dimensional oriented simplex of L, $l=1, 2,\cdots, h$. Since $f^{-1}(v_0)=u_0$ and $f^{-1}(v_1)=u_1$, the simplexes $f(\sigma_1)$ and $f(\sigma_k)$ are non-degenerate. Therefore we can assume that $f(\sigma_1)=\tau_1$ and $f(\sigma_k)=\tau_h$. If $i\neq 1$ and $i\neq k$, we have $f(\sigma_i)\neq\tau_1$ and $f(\sigma_i)\neq\tau_h$. Accordingly we have $g_j=\pm h(g)$ for $j=1, h$. Since $\dim L=1$, this shows that $\Pi(gc)\neq 0$. This completes the proof.

PROOF OF LEMMA 24. Since $\dim X=1$ by [11], Chap. II, § 2, D) X is not totally disconnected[43]. Accordingly there exists a connected closed subset X_0 of X such that $\dim X_0=1$. Let x_1 and x_2 be different points of X_0. Let $\{\mathfrak{U}_i\}$ be a cofinal collection of coverings of X_0 such that the order of $\mathfrak{U}_i$ is $i=1, 2, \cdots$. We can assume that there exists two open sets U_{1i} and U_{2i} of $\mathfrak{U}_i$ such that $x_j\in U_{ji}$ and $x_j\notin U$ for every open set $U\neq U_{ji}$ of $\mathfrak{U}_i$, $j=1, 2$ and $i=1, 2,\cdots$. Let K_i be the nerve of $\mathfrak{U}_i$, $i=1, 2,\cdots$, and let u_{ji} be the vertex of K_i corresponding to the open set U_{ji} of $\mathfrak{U}_i$, $j=1, 2$ and $i=1, 2,\cdots$. Each K_i is a 1-dimensional connected complex. Let Π_i^l be a projection of K_l into K_i for $l>i$. We can assume that $(\Pi_i^l)^{-1}u_{ji}=u_{jl}$, $j=1, 2$ and $l>i$. Assume that X has not the property $\boldsymbol{P}$. By Lemmas 7 and 8, there exists a k-sequence $\mathfrak{a}=\{q_1, q_2,\cdots\}$ such that $H_1(A, B : \mathfrak{a})=0$ for every pair (A, B) of X. Since $\mathfrak{a}=\{q_1, q_2,\cdots\}$ is a k-sequence, there exists a positive integer i such that $q_i>1$. Since K_l is a 1-dimensional connected complex, $(\Pi_i^l)^{-1}u_{ji}=u_{jl}$ and $Z_{q_l}\neq 0$, $j=1, 2$ and $l=i+1, i+2,\cdots$, by Lemma 26 we have $0\neq\mathfrak{P}_i^l H_1(K_l, u_{1l}\cup u_{2l} : Z_{q_l})\subset H_1(K_i, u_{1i}\cup u_{2i} : Z_{q_i})$ for $l=i+1, i+2,\cdots$, where $\mathfrak{P}_i^l$ is the homomorphism of $H_1(K_l, u_{1l}\cup u_{2l} : Z_{q_l})$ into $H_1(K_i, u_{1i}\cup u_{2i} : Z_{q_i})$ defined in § 2 for $l>i$. On the other hand, by our assumption, we have $H_1(X_0, x_1\cup x_2 : \mathfrak{a})=\lim\{H_1(K_l, u_{1l}\cup u_{2l} : Z_{q_l})\}=0$. Since each $H_1(K_l, u_{1l}\cup u_{2l} : Z_{q_l})$ is a finite group, by Lemma 25 there exists a positive integer $l_0>i$ such that $\mathfrak{P}_i^{l_0}H_1(K_{l_0}, u_{1l_0}\cup u_{2l_0} : Z_{q_{l_0}})=0$. Thus we have obtained the contradictory relations. This completes the proof of Lemma 24.

Since every polytope has the property $\boldsymbol{P}$, by Corollary 1, Lemmas 21–24

43) A topological space is called *totally disconnected* if no connected subset contains more than one point.

and the main theorem we have the following corollary which is a generalization of [12], Theorem 9 and [6], Corollaire 16, p. 93.

COROLLARY 2. *In the following five cases the equality* (**A**) *holds for every compact metric space* Y.

1) X *is a polytope.*
2) X *is a* 1-*dimensional compact metric space.*
3) X *is a* 2-*dimensional locally compact* ANR.
4) X *is a locally compact* m-*dimensional* ANR *containing a point* x_0 *which is* HL^{m-2} *and* $(m-1)$-HS.
5) X *is a locally compact and finite dimensional* ANR *which has the property* $\varDelta$.

REMARK. The following lemma is a consequence of [16], Theorem 3.2.

LEMMA 27. *Let* X *be a fully normal*[44] *and locally compact space. In order that* $\dim X \leqq n$ *it is necessary and sufficient that for every compact subset* A *of* X *we have* $\dim A \leqq n$.

By this lemma, our main theorem is generalized in the following form.

THEOREM. *Let* X *be a locally compact* n-*dimensional metric space. In order that the equality* (**A**) *hold for every locally compact metric space* Y *it is necessary and sufficient that* X *have the following property* $\boldsymbol{P}'$:

$\boldsymbol{P}'$. *For every* k-*sequence* $\mathfrak{a}$ *there exists a pair* $(A_{\mathfrak{a}}, B_{\mathfrak{a}})$ *of compact subsets of* X *such that* $H_n(A_{\mathfrak{a}}, B_{\mathfrak{a}} : \mathfrak{a}) \neq 0$.

The following lemma is proved in a similar way as the main theorem.

LEMMA 28. *Let* X *be an* n-*dimensional fully normal and locally compact space. If the equality* (**A**) *holds for every fully normal and locally compact space* Y, *then for each prime number* p *there exists a pair* $(A(p), B(p))$ *of compact subsets of* X *such that* $H_n(A(p), B(p) : Z_p) \neq 0$.

But the condition of Lemma 28 is not a sufficient condition, that is, there exists a 2-dimensional compact metric space X satisfying the following conditions:

1) For each prime number p there exists a closed subset $A(p)$ of X such that $H_2(X, A(p) : Z_p) \neq 0$.
2) There exists a 2-dimensional compact metric space Y such that $\dim(X \times Y) = 3$.

To prove this, let $\{p_1, p_2, \cdots\}$ be a sequence of all prime numbers such that

44) A topological space is called *fully normal* ([19] and [20]) if for every (finite or infinite) open covering $\mathfrak{U}$ of X there exists an open covering $\mathfrak{V}$ of X satisfying the following conditions:

1) $\mathfrak{V}$ is a refinement of $\mathfrak{U}$.
2) Each point x of X has a neighborhood $V(x)$ intersecting only a finite number of open sets of $\mathfrak{V}$.

$p_i \neq p_j$, $i \neq j$. Put $q_i = p_1 p_2 \cdots p_i$ for $i=1, 2, \cdots$, and $X = P(p_1, p_2, \cdots)$, where $P(p_1, p_2, \cdots)$ is the 2-dimensional continuum described in 2) of § 3. It is obvious that the space X satisfies the above-stated condition 1). Put $\tilde{q}_i = q_1 q_2 \cdots q_i$ for $i=1, 2, \cdots$. The sequence $\mathfrak{a} = \{\tilde{q}_1, \tilde{q}_2, \cdots\}$ is a k-sequence. Put $Y = Q(\mathfrak{a})$, where $Q(\mathfrak{a})$ is the 2-dimensional continuum described in 3) of § 3. By Lemma 17 we have $\dim(X \times Y) = 3$.

Addendum. After this paper had been submitted for publication, I have learned by a letter from Prof. Bauer that the problem XII of Alexandroff was already solved by Boltyanskii: (1) On the dimensional fullvaluedness of compacta, Doklady Akad. Nauk SSSR (N. S.) **67**, 773–776 (1949), (Russian); (2) On the theorem of addition of dimensions, Uspehi Mat. Naut (N. S.) **6**, no. 3 (43), 99–128 (1951), (Russian). ((2) seems to be a detailed exposition of (1)). To our great regret these papers of Boltyanskii are not accessible to us in our country. There seems to be a little difference between Boltyanskii's solution and ours, but these two solutions are equivalent as will be proved below.

BOLTYANSKII'S THEOREM. *Let X be a finite dimensional compact metric space. In order that the equality (**A**) hold for every compact metric space Y it is necessary and sufficient that for each prime number p there exists a pair (A_p, B_p) of closed subsets of X such that $H^n(A_p, B_p : Q_p) \neq 0$, where Q_p means the additive group of all rational numbers of the form m/p^k reduced modulo 1 and $H^n(A_p, B_p : G)$ means the n-dimensional Čech cohomology group of (A_p, B_p) with G as a coefficient group.*

Consider the following two properties:

$\boldsymbol{P}_1$. *For every prime number p and every k-sequence $\mathfrak{a}$ each member of which is a power of p there exists a closed subset $A_{\mathfrak{a}}$ of X such that $H_n(X, A_{\mathfrak{a}} : Z(\mathfrak{a})) \neq 0$.*

$\boldsymbol{P}_2$. *For every prime p there exists a closed subset A_p of X such that $H_n(X, A_p : Z(\mathfrak{a}_p)) \neq 0$, where $\mathfrak{a}_p$ is the k-sequence $(p, p^2, \cdots)$.*

LEMMA 1. *The character group of the group Q_p is the group $Z(\mathfrak{a}_p)$ for each prime number p.*

PROOF. For each integer i, let us denote by G_i the subgroup of Q_p consisting of all rational numbers of the form m/p^i. If $j > i$, we have $G_j \supset G_i$. The group Q_p is considered as the direct limit group of the derected system $\{G_i ; i=1, 2, \cdots\}$. Since $G_i \approx \text{Char } G_i \approx Z_{p^i}$, we have $\text{Char } Q_p \approx \text{Char } \varinjlim \{G_i\} \approx \varprojlim \{Z_{p^i}\} = Z(\mathfrak{a}_p)$.

LEMMA 2. *An n-dimensional compact metric space X has the property $\boldsymbol{P}_1$ if and only if X has the property $\boldsymbol{P}_2$.*

PROOF. It is sufficient to prove "if" part. Let $\mathfrak{a} = (p^{\alpha_1}, \cdots, p^{\alpha_i}, \cdots)$ be a

k-sequence. If $\lim_i \alpha_i=\infty$, the proof is easy. Let $\lim_i \alpha_i=m$. We can assume that $\mathfrak{a}=(p^m, p^m, \cdots)$. Since X has the property $\boldsymbol{P}_2$, there exists a closed subset A of X such that $H_n(X, A: Z(\mathfrak{a}_p))\neq 0$. Since $H_n(X, A: Z(\mathfrak{a}_p))\approx H_n(X, A: \mathfrak{a}_p)=\varprojlim\{H_n(K_i, L_i: Z_{p^i})\}$ by Lemma 8 in § 2, we can find a non-zero element $\{z_i\}$ of $\varprojlim\{H_n(K_i, L_i: Z_{p^i})\}$, where (K_i, L_i) is the nerves of the i-th member $\mathfrak{V}_i$ from a countable cofinal system $\{\mathfrak{V}_i\}$ of coverings of X. For $i\geqq m$, let us denote by $h(i)$ the natural homomorphism from Z_{p^i} onto Z_{p^m}. The homomorphism $h(i)$ induces a homomorphism $h(i)$ from $H_n(K_i, L_i: Z_{p^i})$ onto $H_n(K_i, L_i: Z_{p^m})$. The sequence $\{h(i)z_i\}$ determines an element $z(1)$ of the group $H_n(X, A: \mathfrak{a})=H_n(X, A: Z_{p^m})$. If $z(1)\neq 0$, the proof is completed. Let $z(1)=0$. Then $z_i\equiv 0$ mod p^m for $i\geqq m$. Put $z_i^{(1)}=\frac{1}{p^m} z_{i+m}, i=1, 2, \cdots$. Since $z_i^{(1)}$ is a cycle mod p^i, $i=1, 2, \cdots, \{z_i^{(1)}\}$ determines an element of $H_n(X, A: \mathfrak{a}_p)$. Therefore the sequence $\{h(i)z_i^{(1)}\}$ determines an element $z(2)$ of $H_n(X, A: \mathfrak{a})$. If $z(2)\neq 0$, the proof is completed. If $z(2)=0$, by using the same process as above, we can find an element $z(3)$ of $H_n(X, A: \mathfrak{a})$. If we can repeat this process infinitely, we have $z_i=0$ for each i. This contradicts $\{z_i\}\neq 0$. This completes the proof.

Lemma 3. *An n-dimensional compact metric space X has the property $\boldsymbol{P}$ if and only if X has the property $\boldsymbol{P}_1$.*

Proof. It is sufficient to prove "if" part. Let $\mathfrak{a}=(q_1, q_2, \cdots)$ be a k-sequence such that $q_1\neq 1$. Let $q_1=p^{\alpha_i}r_i, i=1, 2, \cdots$, where p is a prime number, p and r_i are coprime numbers. Since q_i is a divisor of $q_{i+1}, \{p^{\alpha_i}\}$ is a k-sequence, which we denote by $\mathfrak{b}$. Since X has the property $\boldsymbol{P}_1$, there exists a closed subset A of X such that $H_n(X, A: Z(\mathfrak{b}))\approx\varprojlim\{H_n(K_i, L_i: Z_{p^{\alpha_i}})\}\neq 0$. Let $\{z_i\}$ be a non-zero element of $\varprojlim\{H_n(K_i, L_i: Z_{p^{\alpha_i}})\}$. We can assume $z_1\neq 0$. Since z_i is a cycle mod p^{α_i}, r_iz_i is a cycle mod q_i and determines an element $\bar{z}_i$ of the group $H_n(K_i, L_i: Z_{q_i}), i=1, 2, \cdots$. Since p and r_i are coprime numbers and $z_1\neq 0$, we have $\mathfrak{P}_1^2\cdots\mathfrak{P}_{i-2}^{i-1}\mathfrak{P}_{i-1}^i(\bar{z}_i)\neq 0$, where $\mathfrak{P}_{i-1}^i$ means the homomorphism from $H_n(K_i, L_i: Z_{q_i})$ into $H_n(K_{i-1}, L_{i-1}: Z_{q_{i-1}}), i=2, 3, \cdots$, defined in § 2. Since the group $H_n(K_1, L_1: Z_{q_1})$ is finite, there exist a non-zero element ζ_1 of $H_n(K_1, L_1: Z_{q_1})$ and a sequence $\{i_1<i_2<\cdots\}$ of integers such that $\mathfrak{P}_1^{i_j}\bar{z}_{i_j}=\zeta_1, j=1, 2, \cdots$, where $\mathfrak{P}_1^{i_j}=\mathfrak{P}_1^2\cdots\mathfrak{P}_{i_j-1}^{i_j}$. Since the group $H_n(K_2, L_2: Z_{q_2})$ is finite, we can find a non-zero element ζ_2 of $H_n(K_2, L_2: Z_{q_2})$ and a subsequence $\{i_1'<i_2'<\cdots\}$ of $\{i_1<i_2<\cdots\}$ such that $\mathfrak{P}_2^{i_j'}\bar{z}_{i_j'}=\zeta_2, j=1, 2, \cdots$. Obviously $\mathfrak{P}_1^2\zeta_2=\zeta_1$. By using this process repeatedly, we have a non-zero element $\{\zeta_i\}$ of the group $\varprojlim\{H_n(K_i, L_i: Z_{q_i})\}$. This completes the proof.

By Lemma 1 the cohomological dimension of X relative to the group Q_p is equal to the homological dimension of X relative to the group $Z(\mathfrak{a}_p)$. Lemmas 2 and 3 shows that Boltyanskii's solution and ours are equivalent.

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