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Additive prime number theory in an algebraic number field.

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Thanks to the remarkable work of Vinogradov [7], we know that every sufficiently large odd integer can be expressed as a sum of three primes. Less attention has been paid to the problem of representing numbers in an algebraic number field as a sum of primes. Rademacher [4] carried over the Hardy-Littlewood formula in the rational case to a real quadratic number field on a certain hypothesis concerning the distribution of the zeros of Hecke's $\zeta(s, \lambda)$ funcfions.

Let K be an algebraic number field of degree n with r_1 real conjugates $K^{(l)}$ $(l=1, 2, \cdots, r_1)$ and r_2 pairs of conjugate complex conjugates $K^{(m)}, K^{(m+r_2)}$ $(m=r_1+1,\ r_1+2, \cdots, r_1+r_2)$ so that $r_1+2r_2=n$. Let a, b be positive and μ, ν be in K. For convenience, we use the symbol

$$a\|\mu\| \leqq b\|\nu\|$$

in the sense that

$$a|\mu^{(i)}| \leqq b|\nu^{(i)}| \qquad (i=1, 2, \cdots, n).$$

For example, $\|\mu\| \leqq b$ means $|\mu^{(i)}| \leqq b$. Let $\mathfrak{a}$ be any principal ideal in K. By the theory of units, there exist a positive constant c_0 depending only on K and at least one ν in K such that

$$\mathfrak{a}=(\nu) \quad \text{and} \quad \|\nu\| \leqq c_0 \sqrt[n]{N(\nu)}. \tag{1}$$

In what follows we fix this constant c_0. We use a letter c to denote a positive constant depending only on K, not necessarily the same each time it occurs. The symbol

$$Y=O(X)$$

for positive X means that there exists c satisfying

$$|Y| \leqq cX$$

in the full domain under consideration. For example, the number of ν in (1) generating the same principal ideal $\mathfrak{a}$ is $O(1)$.

Let Γ and I be the integral domains consisting of all rational integers and all algebraic integers in K respectively. An algebraic integer in the field is called a prime when the principal ideal generated by it is a prime ideal. In [6] Siegel considered the generalized Waring problem in an algebraic number field. He constructed the ring J_m generated by m-th powers of all integers in the field. Using the generalized circle method, he proved that all totally positive integers in J_m with sufficiently large norms are sums of $(2^{m-1}+n)mn+1$ integral m-th powers of totally positive numbers. Moreover he noticed that J_m is an order in I, but not always equal to I, showing some theorems and some examples. Modeled on his idea, we construct the Γ-module J generated by all primes in K. By means of Brun's sieve method, Hecke's prime ideal theorem, and Schnirelmann's density theorem, we will prove without any hypothesis that J is of finite index in the additive group I and every element in J can be expressed as at most c sums of primes.

§ 1. On the number of representing integers in K as a sum of two primes.

Let $\mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_h$ be representatives from ideal classes of K. Then any ideal $\mathfrak{a}$ in K can be expressed in the form

$$\mathfrak{a} = (\nu)\mathfrak{b}_l$$

for some $\nu \in K$ with (1) and $\mathfrak{b}_l$. Let $\beta_{l1}, \beta_{l2}, \cdots, \beta_{ln}$ be an ideal basis of $\mathfrak{b}_l$ $(l=1, 2, \cdots, h)$. If we put $\alpha_j = \nu\beta_{lj}$ $(j=1, 2, \cdots, n)$, then

$$\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$$

and

$$\|\alpha_j\| = \|\nu\beta_{lj}\| \leqq c\sqrt[n]{N(\nu)}\ c\sqrt[n]{N\mathfrak{b}_l} \leqq c\sqrt[n]{N\mathfrak{a}}. \tag{2}$$

Let ζ be in I with

$$\|\zeta\| \leqq c_1 \sqrt[n]{N(\zeta)} \tag{3}$$

where

$$c_1 = 6c_0^n . \tag{4}$$

Let $\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be an integral ideal in K and $\beta \in I$. If we denote by $P(\mathfrak{a}, \zeta)$ the number of integers ξ in K subject to the conditions

$$\xi \equiv \beta \ (\text{mod. } \mathfrak{a}), \qquad \|\xi\| \leqq c_1 \|\zeta\| , \quad \|\zeta - \xi\| \leqq c_1 \|\zeta\| ,$$

then $P(\mathfrak{a}, \zeta)$ is the number of lattice points $(x_1, x_2, \cdots, x_n)$ in the n dimensional euclidean space S which lie in the domain

$$\begin{aligned} &|x_1 \alpha_1^{(i)} + x_2 \alpha_2^{(i)} + \cdots + x_n \alpha_n^{(i)} + \beta^{(i)}| \leqq c_1 |\zeta^{(i)}| , \\ &|x_1 \alpha_1^{(i)} + x_2 \alpha_2^{(i)} + \cdots + x_n \alpha_n^{(i)} + \beta^{(i)} - \zeta^{(i)}| \leqq c_1 |\zeta^{(i)}| . \end{aligned} \tag{5}$$

We change the variables as follows

$$u_l = x_1 \alpha_1^{(l)} + x_2 \alpha_2^{(l)} + \cdots + x_n \alpha_n^{(l)} + \beta^{(l)} ,$$

$$u_m = x_1 \Re(\alpha_1^{(m)}) + x_2 \Re(\alpha_2^{(m)}) + \cdots + x_n \Re(\alpha_n^{(m)}) + \Re(\beta^{(m)}) ,$$

$$u_{m+r_2} = x_1 \Im(\alpha_1^{(m+r_2)}) + x_2 \Im(\alpha_2^{(m+r_2)}) + \cdots + x_n \Im(\alpha_n^{(m+r_2)}) + \Im(\beta^{(m+r_2)}) .$$

The domain (5) is now described by

$$|u_l| \leqq c_1 |\zeta^{(l)}| , \qquad |u_l - \zeta^{(l)}| \leqq c_1 |\zeta^{(l)}|$$

$$u_m^2 + u_{m+r_2}^2 \leqq c_1^2 |\zeta^{(m)}|^2 , \qquad (u_m - \Re(\zeta^{(m)}))^2 + (u_{m+r_2} - \Im(\zeta^{(m+r_2)}))^2 \leqq c_1^2 |\zeta^{(m)}|^2 .$$

Since

$$\left| \frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} \right| = \frac{\sqrt{d}\, N\mathfrak{a}}{2^{r_2}}$$

where d is the discriminant of K, taking a basis of $\mathfrak{a}$ as in (2), we obtain

$$\iint\cdots\int_{u_l\in L_l,\ (u_m, u_{m+r_2})\in C_m} \frac{2^{r_2}}{\sqrt{d}\,N\mathfrak{a}}\, du_1\, du_2 \cdots du_n < P(\mathfrak{a}, \zeta)$$

$$< \iint\cdots\int_{u_l\in L_l',\ (u_m, u_{m+r_2})\in C_m'} \frac{2^{r_2}}{\sqrt{d}\,N\mathfrak{a}}\, du_1\, du_2 \cdots du_n ,$$

where L_l, L_l' are line segments with lengths

$$(2c_1-1)\,|\zeta^{(l)}|+O(\sqrt[n]{N\mathfrak{a}})$$

and C_m, C_m' are common domains surrounded by two circles having radii $c_1|\zeta^{(m)}|+O(\sqrt[n]{N\mathfrak{a}})$ and central distances $|\zeta^{(m)}|$ so that with areas

$$\left(2c_1^2 \sin^{-1}\sqrt{1-\frac{1}{4c_1^2}} - \sqrt{c_1^2-\frac{1}{4}}\right)|\zeta^{(m)}|^2+O(|\zeta^{(m)}|\sqrt[n]{N\mathfrak{a}}+(\sqrt[n]{N\mathfrak{a}})^2)\,.$$

From (3), we can easily deduce

$$P(\mathfrak{a}, \zeta)=c\,\frac{N(\zeta)}{N\mathfrak{a}}+O\left(\left(\frac{N(\zeta)}{N\mathfrak{a}}\right)^{1-\frac{1}{n}}+1\right). \tag{6}$$

This is a slight extention of Rademacher's work [5].

Let $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k$ be prime ideals in K. We denote by $P(\mathfrak{a}, \zeta\,;\,\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k)$ the number of integers in K satisfying

$$\xi\equiv\beta \ (\mathrm{mod.}\ \mathfrak{a}), \qquad \|\xi\|\leqq c_1\|\zeta\|, \qquad \|\zeta-\xi\|\leqq c_1\|\zeta\|$$

$$\xi\notin\mathfrak{p}_s, \qquad \zeta-\xi\notin\mathfrak{p}_s \qquad (s=1, 2, \cdots, k)\,.$$

If we define $v_s=2$ if $\zeta\notin\mathfrak{p}_s$ and $v_s=1$ if $\zeta\in\mathfrak{p}_s$, then

$$P(\mathfrak{a}, \zeta\,;\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k)=P(\mathfrak{a}, \zeta\,;\,\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_{k-1})-v_k\,P(\mathfrak{a}\mathfrak{p}_k, \zeta\,;\,\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_{k-1})\,.$$

By iteration,

$$P(\mathfrak{a}, \zeta\,;\,\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k)=P(\mathfrak{a}, \zeta)-\sum_{1\leqq s_1\leqq k} v_{s_1}\,P(\mathfrak{a}\mathfrak{p}_{s_1}, \zeta\,;\,\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_{s_1-1})$$

$$=P(\mathfrak{a}, \zeta)-\sum_{1\leqq s_1\leqq k} v_{s_1}\,P(\mathfrak{a}\mathfrak{p}_{s_1}, \zeta)+\sum_{1\leqq s_2<s_1\leqq k} v_{s_1}v_{s_2}\,P(\mathfrak{a}\mathfrak{p}_{s_1}\mathfrak{p}_{s_2}, \zeta\,;\,\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_{s_2-1})\,.$$

Consequently, making use of Brun's method and taking $\mathfrak{a}=I$, we get

$$P(I, \zeta\,;\,\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k) \leqq P(I, \zeta) + \sum_{q=1}^{2t} (-1)^q \sum_{s_1, s_2, \cdots, s_q} v_{s_1} v_{s_2} \cdots v_{s_q} P(\mathfrak{p}_{s_1} \mathfrak{p}_{s_2} \cdots \mathfrak{p}_{s_q}, \zeta)\,.$$

where $s_q (q=1, 2, \cdots, 2t)$ runs over

$$0 < s_{2t} < s_{2t-1} < \cdots < s_2 < s_1\,,$$

$$s_q \leqq k_{[(q-1)/2]}$$

for suitably chosen

(7) $$0 = k_t < k_{t-1} < \cdots < k_1 < k_0 = k\,.$$

By the aid of (6), therefore, we obtain

(8) $$P(I, \zeta\,;\,\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k) = O\left(N(\zeta) \sum_{q=0}^{2t} (-1)^q \sum_{s_1, s_2, \cdots, s_q} \gamma_{s_1} \gamma_{s_2} \cdots \gamma_{s_q}\right)$$
$$+ O\left(N(\zeta)^{1-\frac{1}{n}} \prod_{r=0}^{t-1} (2k_r)^2\right)$$

with the abbreviation $\gamma_s = v_s / N\mathfrak{p}_s$.

Now we assume

$$11 \leqq N\mathfrak{p}_1 \leqq N\mathfrak{p}_2 \leqq \cdots \leqq N\mathfrak{p}_k\,.$$

Obviously

(9) $$\frac{4}{5} < \frac{9}{11} \leqq 1 - \gamma_s\,.$$

First we set $k_0 = k$. Next we choose k_r in (7) such that

(10) $$\frac{4}{5} \leqq \prod_{k_r < s \leqq k_{r-1}} (1 - \gamma_s) \qquad (r = 1, 2, \cdots, t)\,,$$
$$\prod_{k_r \leqq s \leqq k_{r-1}} (1 - \gamma_s) < \frac{4}{5} \qquad (r = 1, 2, \cdots, t-1)\,.$$

Denoting by T_r the right hand side of the first inequality in (10) and using Theorem 79 of [3], we obtain

$$\left| \sum_{q=0}^{2t} (-1)^q \sum_{\substack{s_q < \cdots < s_2 < s_1 \\ s_q \leqq k_{[(q-1)/2]}}} \gamma_{s_1} \gamma_{s_2} \cdots \gamma_{s_q} \right| < 2 \prod_{r=1}^{t} T_r\,.$$

Inserting this in (8), we get

$$P(I, \zeta\,;\, \mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k) = O\left(N(\zeta) \prod_{r=1}^{t} T_r \right) + O\left(N(\zeta)^{1-\frac{1}{n}} \prod_{r=0}^{t-1} (2k_r) \right). \tag{11}$$

For brevity we write

$$Q = \prod_{s=1}^{k} \left(1 - \frac{1}{N\mathfrak{p}_s}\right).$$

On account of (9) and (10),

$$\left(\prod_{s=1}^{k_r} \left(1 - \frac{1}{N\mathfrak{p}_s}\right)\right)^{-1} = \frac{1}{Q} \prod_{j=1}^{r} \prod_{k_j < s \leqq k_{j-1}} \left(1 - \frac{1}{N\mathfrak{p}_s}\right) \leqq \frac{1}{Q} \sqrt[3]{\prod_{j=1}^{r} T_j} \tag{12}$$

$$< \frac{1}{Q} \sqrt[3]{\prod_{j=1}^{r} \frac{4}{5} (1-\gamma_j)^{-1}} < \frac{1}{Q} \left(1 - \frac{1}{135}\right)^r \quad (r = 0, 1, \cdots, t-1).$$

Let $\pi(x)$ be the number of all prime ideals having norms not exceeding x. It is well known that

$$\pi(x) < c \frac{x}{\log x} \tag{13}$$

and

$$\frac{c}{\log x} < \sum_{N\mathfrak{p} \leqq x} \left(1 - \frac{1}{N\mathfrak{p}}\right) < \frac{c}{\log x}. \tag{14}$$

Now we take all prime ideals $\mathfrak{p}$ satisfying $11 \leqq N\mathfrak{p} \leqq \sqrt[c_2]{N(\zeta)}$ as $\mathfrak{p}_1$, $\mathfrak{p}_2, \cdots, \mathfrak{p}_k$, where c_2 is decided later. By (13),

$$s \leqq \pi(N\mathfrak{p}_s) < c \frac{N\mathfrak{p}_s}{\log N\mathfrak{p}_s} < c\, N\mathfrak{p}_s. \tag{15}$$

From (12) and (15), with the help of the second inequality of (14), we have

$$\log 2k_r < \frac{c}{Q} \left(1 - \frac{1}{135}\right)^r.$$

If follows from this that

$$\prod_{r=0}^{t-1}(2k_r)^2 < \exp\left(\frac{c}{Q}\right),$$

whereas, employing the first inequality, we have

$$\frac{1}{Q}=\left(\prod_{11\leqq N\mathfrak{p}\leqq {}^{c_2}\sqrt{N(\zeta)}}\left(1-\frac{1}{N\mathfrak{p}}\right)\right)^{-1} < c\log {}^{c_2}\sqrt{N(\zeta)}$$

Hence, by suitably chosen c_2, we obtain

$$\prod_{r=0}^{t-1}(2k_r)^2 < N(\zeta)^{\frac{1}{2n}}. \tag{16}$$

On the other hand,

$$\begin{aligned}\prod_{r=1}^{t} T_r &= \prod_{s=1}^{k}\left(1-\frac{v_s}{N\mathfrak{p}_s}\right) \leqq \prod_{s=1}^{k}\left(1-\frac{1}{N\mathfrak{p}_s}\right)^{v_s} \\ &= \prod_{s=1}^{k}\left(1-\frac{1}{N\mathfrak{p}_s}\right)^2 \prod_{\zeta\in\mathfrak{p}}\left(1-\frac{1}{N\mathfrak{p}_s}\right)^{-1} \leqq Q^2 \prod_{\substack{\zeta\in\mathfrak{p} \\ 11\leqq N\mathfrak{p}\leqq {}^{c_2}\sqrt{N(\zeta)}}}\left(1-\frac{1}{N\mathfrak{p}}\right)^{-1} \\ &= Q^2\prod_{\zeta\in\mathfrak{p}}\left(1-\frac{1}{N\mathfrak{p}^2}\right)^{-1}\prod_{\zeta\in\mathfrak{p}}\left(1+\frac{1}{N\mathfrak{p}}\right) < c\,Q^2\prod_{\zeta\in\mathfrak{a}}\frac{1}{N\mathfrak{a}} \\ &< \frac{c}{\log^2 N(\zeta)}\sum_{\zeta\in\mathfrak{a}}\frac{1}{N\mathfrak{a}},\end{aligned} \tag{17}$$

by the second inequality of (14). Inserting (16) and (17) in (11), we get

$$P(I,\zeta\,;\mathfrak{p}_1,\cdots,\mathfrak{p}_k)=O\left(\frac{N(\zeta)}{\log^2 N(\zeta)}\sum_{\zeta\in\mathfrak{a}}\frac{1}{N\mathfrak{a}}\right). \tag{18}$$

Let $P(\zeta)$ be the number of solutions of

$$\zeta=\lambda+\mu, \qquad \|\lambda\|\leqq c_1\|\zeta\|, \qquad \|\mu\|\leqq c_1\|\zeta\| \tag{19}$$

where λ and μ are primes with

$$\|\lambda\|\leqq c_0\sqrt[n]{N(\lambda)}, \qquad \|\mu\|\leqq c_0\sqrt[n]{N(\lambda)}. \tag{20}$$

If $N(\lambda)$ and $N(\mu) > {}^{c_2}\sqrt{N(\zeta)}$ in (19), then neither λ nor μ is divided by any prime ideal satisfying $11\leqq N\mathfrak{p}\leqq {}^{c_2}\sqrt{N(\zeta)}$. By (18), the number of solutions in this case is

$$O\left(\frac{N(\zeta)}{\log^2 N(\zeta)} \sum_{\zeta\in\mathfrak{a}} \frac{1}{N\mathfrak{a}}\right).$$

On the other hand, in the case of $N(\lambda)$ or $N(\mu) \leqq \sqrt[c_2]{N(\zeta)}$, the number of solutions is

$$O(\pi(\sqrt[c_2]{N(\zeta)}\,)) = O\left(\frac{N(\zeta)}{\log^2 N(\zeta)}\right),$$

in virtue of (20) and $c_2 > 1$.

Collecting these results, we get

THEOREM 1. *Let ζ be an algebraic integer in K with*

$$\|\zeta\| \leqq c_1 \sqrt[n]{N(\zeta)}.$$

Let $P(\zeta)$ be the number of solutions of

$$\zeta = \lambda + \mu,$$

where λ and μ are primes fulfilling the conditions

$$\|\lambda\| \leqq c_0 \sqrt[n]{N(\lambda)}, \quad c_1 \|\zeta\|; \quad \|\mu\| \leqq c_0 \sqrt[n]{N(\mu)}, \quad c_1 \|\zeta\|.$$

Then

$$P(\zeta) = O\left(\frac{N(\zeta)}{\log^2 N(\zeta)} \sum_{\zeta\in\mathfrak{a}} \frac{1}{N\mathfrak{a}}\right).$$

§ 2. On the density of the set consisting of integers in K represented by a sum of two primes.

From Theorem 1, we obtain

$$\sum_{N(\zeta)\leqq y} P^2(\zeta) = O\left(\sum_{N(\zeta)\leqq y} \frac{N(\zeta)^2}{\log^4 N(\zeta)} \sum_{\zeta\in\mathfrak{a}} \frac{1}{N\mathfrak{a}} \sum_{\zeta\in\mathfrak{b}} \frac{1}{N\mathfrak{b}}\right)$$

$$= O\left(\frac{y^2}{\log^4 y} \sum_{N(\zeta)\leqq y} \left(\sum_{\zeta\in\mathfrak{a}} \frac{1}{N\mathfrak{a}} \sum_{\zeta\in\mathfrak{b}} \frac{1}{N\mathfrak{b}}\right)\right)$$

$$= O\left(\frac{y^2}{\log^4 y} \sum_{\substack{N\mathfrak{a}\leqq y\\ N\mathfrak{b}\leqq y}} \frac{1}{N\mathfrak{a}\, N\mathfrak{b}} \sum_{\substack{N(\zeta)\leqq y\\ \zeta\in\{\mathfrak{a},\mathfrak{b}\}}} 1\right),$$

where $\{\mathfrak{a}, \mathfrak{b}\}$ is the least common multiple of $\mathfrak{a}$ and $\mathfrak{b}$, so that $\sqrt{N\mathfrak{a}\,N\mathfrak{b}} \leqq N\{\mathfrak{a}, \mathfrak{b}\}$. Since

$$\sum_{\substack{N(\zeta)\leqq y \\ \zeta\in\{\mathfrak{a},\mathfrak{b}\}}} 1 = O\left(\frac{y}{N\{\mathfrak{a}, \mathfrak{b}\}}\right) = O\left(\frac{y}{\sqrt{N\mathfrak{a}\,N\mathfrak{b}}}\right),$$

we obtain

$$\sum_{N(\zeta)\leqq y} P^2(\zeta) = O\left(\frac{y^3}{\log^4 y}\right) \qquad (2 \leqq y). \tag{21}$$

Now we consider the sets

$$L = \left\{\lambda : \left(\frac{1}{2c_0}\right)^n y \leqq N(\lambda) \leqq \left(\frac{2}{3c_0}\right)^n y, \quad \|\lambda\| \leqq c_0 \sqrt[n]{N(\lambda)}, \ \lambda \text{ prime}\right\}$$

and

$$M = \left\{\mu : \left(\frac{1}{6c_0^{n+1}}\right)^n y \leqq N(\mu) \leqq \left(\frac{1}{3c_0^{n+1}}\right)^n y, \ \|\mu\| \leqq c_0 \sqrt[n]{N(\mu)}, \ \mu \text{ prime}\right\}$$

for $2\cdot 6^n c_0^{n(n+1)} \leqq y$. If we put

$$\zeta = \lambda + \mu,$$

then

$$\|\zeta\| = \|\lambda + \mu\| \leqq c_0 \left(\sqrt[n]{N(\lambda)} + \sqrt[n]{N(\mu)}\right)$$

$$\leqq c_0 \left(\frac{2}{3c_0} + \frac{1}{3c_0^{n+1}}\right) \sqrt[n]{y} \leqq \sqrt[n]{y} \tag{22}$$

and

$$N(\zeta) \leqq y. \tag{23}$$

It is easy to see

$$\frac{1}{c_0^{n-1}} \sqrt[n]{N(\lambda)} \leqq \|\lambda\|,$$

whence follows

$$\frac{1}{6c_0^n}\sqrt[n]{y}=\frac{1}{c_0^{n-1}}\frac{1}{2c_0}\sqrt[n]{y}-c_0\frac{1}{3c_0^{n+1}}\sqrt[n]{y}\leqq\frac{1}{c_0^{n-1}}\sqrt[n]{N(\lambda)}-c_0\sqrt[n]{N(\mu)}$$

(24)

$$\leqq\|\lambda\|-\|\mu\|\leqq\|\zeta\|.$$

By (22) and (24),

$$\frac{1}{6c_0^n}\|\zeta\|\leqq\frac{1}{6c_0^n}\sqrt[n]{y}\leqq\sqrt[n]{N(\zeta)}$$

or

(25) $$\|\zeta\|\leqq c_1\sqrt[n]{N(\zeta)}$$

by (4). Because of (24),

$$\frac{1}{4c_0^n}\|\lambda\|\leqq\frac{1}{4c_0^{n-1}}\sqrt[n]{N(\lambda)}\leqq\frac{1}{4c_0^{n-1}}\frac{2}{3c_0}\sqrt[n]{y}\leqq\frac{1}{6c_0^n}\sqrt[n]{y}\leqq\|\zeta\|$$

whence follows

(26) $$\|\lambda\|\leqq c_1\|\zeta\|$$

by (4). Moreover, by (24),

$$\frac{1}{2}\|\mu\|\leqq\frac{c_0}{2}\sqrt[n]{N(\mu)}\leqq\frac{c_0}{2}\frac{1}{3c_0^{n+1}}\sqrt[n]{y}=\frac{1}{6c_0^n}\sqrt[n]{y}\leqq\|\zeta\|$$

so that

(27) $$\|\mu\|\leqq c_1\|\zeta\|.$$

We see that the conditions in Theorem 1 are satisfied by (25), (26) and (27). In view of (23), we can deduce that

(28) $$\text{the number of } \{\lambda+\mu:\ \lambda\in L,\ \mu\in M\}\leqq\sum_{N(\zeta)\leqq y}P(\zeta).$$

Let $\pi(x, H)$ be the number of principal prime ideals whose norms do not exceed x. Then the left hand side of (28) is greater than

$$c(\pi(c_3y, H)-\pi(c_4y, H))\,(\pi(c_5y, H)-\pi(c_6y, H))$$

with abbreviations

$$c_3=\left(\frac{2}{3c_0}\right)^n,\quad c_4=\left(\frac{1}{2c_0}\right)^n,\quad c_5=\left(\frac{1}{3c_0^{n+1}}\right)^n,\quad c_6=\left(\frac{1}{6c_0^{n+1}}\right)^n.$$

If we use Hecke's prime ideal theorem [2], then

$$O(1)=\sum_{\substack{c_4y\leqq N\mathfrak{p}\leqq c_3y\\ \mathfrak{p}\in H}}\frac{\log N\mathfrak{p}}{N\mathfrak{p}}\leqq(\pi(c_3y,H)-\pi(c_4y,H))\,\frac{\log c_4y}{c_4y}$$

and

$$O(1)=\sum_{\substack{c_6y\leqq N\mathfrak{p}\leqq c_5y\\ \mathfrak{p}\in H}}\frac{\log N\mathfrak{p}}{N\mathfrak{p}}\leqq(\pi(c_5y,H)-\pi(c_6y,H))\,\frac{\log c_6y}{c_6y}$$

whence follows

$$c\frac{y^2}{\log^2 y}<\sum_{N(\zeta)\leqq y}P(\zeta)\qquad(2\cdot 6^n c_0^{n(n+1)}\leqq y) \tag{29}$$

by (28).

Now we denote by $U(y)$ the number of

$$\{\zeta: N(\zeta)\leqq y;\ \zeta=\lambda+\mu;\ \lambda,\mu \text{ primes};\ ||\zeta||\leqq c_1\sqrt[n]{N(\zeta)};$$
$$||\lambda||\leqq c_0\sqrt[n]{N(\lambda)},\quad c_1||\zeta||;\ ||\mu||\leqq c_0\sqrt[n]{N(\mu)},\quad c_1||\zeta||\}. \tag{30}$$

Then, by (21) and (29),

$$c\,\frac{y^4}{\log^4 y}<\Big(\sum_{N(\zeta)\leqq y}P(\zeta)\Big)^2\leqq U(y)\sum_{N(\zeta)\leqq y}P^2(\zeta)<U(y)\,c\,\frac{y^3}{\log^4 y}.$$

This gives

$$cy<U(y) \tag{31}$$

for sufficiently large y.

We write

(32)
$$A(x)=\text{the number of }\{\zeta: ||\zeta||\leqq x;\ \zeta\in I\}$$
$$E(x)=\text{the number of }\{\zeta: ||\zeta||\leqq x;\ \zeta=\lambda+\mu;\ \lambda,\mu \text{ primes};$$
$$||\lambda||,\ ||\mu||\leqq c_1x\}.$$

Let $\omega_1,\omega_2,\cdots,\omega_n$ be an integral basis of K. Then $A(x)$ is the number of lattice points $(x_1,x_2,\cdots,x_n)$ in S satisfying

$$|x_1\omega_1^{(i)}+x_2\omega_2^{(i)}+\cdots+x_n\omega_n^{(i)}|\leqq x\qquad(i=1,2,\cdots,n),$$

$$x_j \in \Gamma \qquad (j=1, 2, \cdots, n).$$

Hence we can deduce as in (6) that

$$A(x) \sim cx^n. \tag{33}$$

Replacing y by $\frac{x^n}{c_1^n}$ in (30), we see

$$U\left(\frac{x^n}{c_1^n}\right) \leqq E(x). \tag{34}$$

From (31), (33) and (34), we have

$$cA(x) < E(x)$$

for sufficiently large x.

Hence we have

THEOREM 2. *If we define* $A(x)$ *and* $E(x)$ *as in* (32), *then*

$$cA(x) < E(x)$$

for sufficiently large x.

§ 3. On a density theorem in an algebraic number field.

For the sets $\mathfrak{A}, \mathfrak{B}, \cdots$ which consist of elements in I, we define

$$\mathfrak{A} + \mathfrak{B} = \{\gamma : \gamma = \alpha, \beta \text{ or} \alpha + \beta ; \alpha \in \mathfrak{A}, \beta \in \mathfrak{B}\},$$

$$-\mathfrak{B} = \{-\beta : \beta \in \mathfrak{B}\},$$

$$\mathfrak{A} - \mathfrak{B} = \mathfrak{A} + (-\mathfrak{B}).$$

Now we take

$$\mathfrak{A} = I,$$

$$\mathfrak{E} = \{\zeta : \zeta = \lambda + \mu ; \lambda, \mu \text{ primes}\},$$

and write

$$\mathfrak{A}[x] = \{\zeta : \zeta \in \mathfrak{A}, ||\zeta|| \leqq x\},$$

$$\mathfrak{E}[x] = \{\zeta : \zeta \in \mathfrak{E}, ||\zeta|| \leqq x ; ||\lambda|| \, ||\mu|| \leqq c_1 x\}.$$

Then the number of elements of $\mathfrak{A}[x]$ and $\mathfrak{E}[x]$ are $A[x]$ and $E[x]$ in (32). Moreover, we define

$$\mathfrak{B}[x]=\{\zeta:\zeta=x_1\omega_1+x_2\omega_2+\cdots+x_n\omega_n,\ -c_7x\leqq x_j\leqq c_7x,\ x_j\in\Gamma\}$$

such that

$$\mathfrak{A}[x]\subset\mathfrak{B}[x] \tag{35}$$

by taking c_7 sufficiently large. Since the number of elements of $\mathfrak{B}[x]$, say $B[x]$, is of order cx^n, we obtain, for sufficiently large x

$$cB(x)<E(x) \tag{36}$$

by Theorem 2 and (33) whereas

$$\mathfrak{E}[x]\subset\mathfrak{B}[x] \tag{37}$$

by (35).

From (36) and (37), we know that more than cx^k elements in the form

$$x_1\omega_1+x_2\omega_2+\cdots+x_k\omega_k+x_{k+1}\omega_{k+1}+\cdots+x_n\omega_n\,,$$
$$-c_7x\leqq x_j\leqq c_7x,\qquad x_j\in\Gamma\qquad (j=1,2,\cdots,k)\,,$$

for suitably fixed $x_{k+1}, x_{k+2},\cdots, x_n$ are contained in $\mathfrak{E}[x]$ for every k $(k=1,2,\cdots,n)$. Hence, $\mathfrak{E}-\mathfrak{E}$ contains more than cx^k elements in the form

$$\begin{aligned}&y_{k1}\omega_1+y_{k2}\omega_2+\cdots+y_{kk}\omega_k\,,\\ &y_{kj}\in\Gamma,\ -2c_7x\leqq y_{kj}\leqq 2c_7x\end{aligned} \tag{38}$$

for every k. Consequently, if we write

$$\mathfrak{C}_k[x]=\{y_{kk}:y_{k1}\omega_1+y_{k2}\omega_2+\cdots+y_{kk}\omega_k\in\mathfrak{E}-\mathfrak{E},\ |y_{kk}|\leqq x\}\,,$$

$$C_k(x)=\text{the number of elements of }\mathfrak{C}_k[x]\,,$$

then

$$cx<C_k(x) \tag{39}$$

for every k, if x is sufficiently large.

Since $\mathfrak{E}-\mathfrak{E}$ is contained in J, J contains elements in the form (38). In view of (39), we denote by a_{kk} the smallest positive integer y_{kk} such that

$$y_{k1}\omega_1+y_{k2}\omega_2+\cdots+y_{kk}\omega_k\in J\,,\quad y_{kj}\in\Gamma\,,$$

and write such an element in J which satisfies $y_{kk}=a_{kk}$ as follows.

$$\zeta_k=a_{k1}\omega_1+a_{k2}\omega_2+\cdots+a_{kk}\omega_k\,,\quad a_{kj}\in\Gamma\,.$$

Then we can easily deduce that $\zeta_1,\zeta_2,\cdots,\zeta_n$ form a basis of the additive group J, and

$$[I:J]=a_{11}a_{22}\cdots a_{nn}$$

is finite.

The elements in (38) can be expressed in the form

$$y_{k1}\omega_1+y_{k2}\omega_2+\cdots+y_{kk}\omega_k=z_{k1}\zeta_1+z_{k2}\zeta_2+\cdots+z_{kk}\zeta_k$$

with

$$-c_8x\leqq z_{kj}\leqq c_8x\,,\qquad z_{kj}\in\Gamma\,.$$

If we write

$$\mathfrak{D}_k[x]=\{z_{kk}:|z_{kk}|\leqq x\}\,,$$

$$D_k(x)=\text{the number of elements of }\mathfrak{D}_k[x]\,,$$

then

$$cx<D_k(x) \tag{38}$$

for every k, if x is sufficiently large. Let $\mathfrak{F}$ be the set consisting of all elements in $\mathfrak{E}-\mathfrak{E}$ and $\pm\zeta_1,\pm\zeta_2,\cdots,\pm\zeta_n$. Then $\mathfrak{F}$ is a subset of J. Now consider all elements in $\mathfrak{F}$ such form as

$$f_{k1}\zeta_1+f_{k2}\zeta_2+\cdots+f_{kk}\zeta_k\,,\qquad f_{kj}\in\Gamma\,.$$

If we write

$$\mathfrak{F}_k[x]=\{f_{kk}:|f_{kk}|\leqq x\}$$

$$F_k(x)=\text{the number of elements of }\mathfrak{F}_k[x]$$

then, by (38),

$$cx < F_k(x)$$

for every k and $1 \leqq x$. Hence, by the density theorem of Schnirelmann [1], we can deduce that a finite sums of $\mathfrak{F}$ contains such an element as

$$\xi_k = q_{k1}\zeta_1 + q_{k2}\zeta_2 + \cdots + q_{kk}\zeta_k, \qquad q_{kj} \in \Gamma$$

for any given $q_{kk} \in \Gamma$. Hence, every element ξ in J can be expressed in the form

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n.$$

Hence we get the desired result.

THEOREM 3. *J is of finite index in I and every element in J can be expressed as at most c sums of primes.*

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