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Inner endomorphisms of an associative algebra.

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The inner automorphisms of an algebra are operations of fundamental importance defined on the algebra, where the adjective "inner" implies that these automorphisms are defined by some elements of the algebra by means of a certain canonical procedure which enables one to compute these automorphisms. Not only automorphisms, but homomorphisms of the algebra into itself i. e. endomorphisms of the algebra play some rôles in the structure theory of the algebra. In this connection, an attempt will be made in the following lines to generalize the notion of the inner automorphisms so as to include some endomorphisms which may be called inner in the above sense.

§1. Throughout this paper, A will denote an associative algebra with an identity and of a finite dimension, say n, over a ground field K. Set $K_0=K(X_1,\cdots,X_n)$ and $K_1=K(X_1,\cdots\cdots,X_n, Y_1,\cdots, Y_n)$, where $X_1,\cdots, X_n, Y_1,\cdots, Y_n$ are independent variables over K; an element $f(X_1,\cdots,X_n)$ of K_0, or an element $g(X_1,\cdots,X_n, Y_1,\cdots, Y_n)$ of K_1 will be written in the simplified form $f(X)$ or $g(X, Y)$, respectively. Construct an auxiliary algebra A_1 by extending the ground field K of A to K_1. A K-basis (u_i) of A serves also as a K_1-basis of A_1. Let the multiplication table of (u_i) be $u_i u_j=\sum \gamma_{ijk}u_k$, $\gamma_{ijk}\in K$. If $a=\sum f_i u_i$, $f_i\in K_1$, is an element of A_1, we shall denote by $M(a, u)$ the n-rowed matrix $(\sum_i f_i \gamma_{ijk})_{jk}$. Then a is regular if and only if $\det M(a,u)\neq 0$. Since $\det M(1, u)$ does not vanish certainly, we have also $\det M(S, u)\neq 0$ for the general element $S=\sum X_i u_i$, which has therefore an inverse element. Put

$$(1) \qquad (\sum X_i u_i)(\sum Y_i u_i)(\sum X_i u_i)^{-1}=\sum Y_i' u_i ,$$

where $Y_i'=Y_i'(X, Y)$ is linear in Y_i, and we can write

$$(2) \qquad Y_i'=\sum R_{ij}(X)Y_j \qquad \text{with} \qquad R_{ij}(X)\in K_0 .$$

Then the inner automorphism by S is represented by the matrix $R_u(S) = (R_{ij}(X))$. Now, an element $a=\sum \alpha_i u_i$, $\alpha_i \in K$, of A will be called *left-semiregular*, or simply *semiregular* in §§ 1, 2, if every $R_{ij}(X)$ can be so expressed in the form $R_{ij}(X)=P_{ij}(X)/Q_{ij}(X)$, P_{ij}, $Q_{ij} \in K[X_1,\cdots,X_n]$ that Q_{ij} does not vanish at $X_1=\alpha_1,\cdots, X_n=\alpha_n$. If this is the case, the value $R_{ij}(\alpha)=P_{ij}(\alpha)/Q_{ij}(\alpha)$ is uniquely determined, independent of the choice of the expression of R_{ij} subjected to the condition above. Denoting the matrix $(R_{ij}(\alpha))$ by $R_u(a)$, we shall say in this case that *$R_u(S)$ is defined at a.* We shall define an operation I_a, for such a semiregular element a, as follows:

$$(3) \qquad I_a: \quad \sum \beta_i u_i \to \sum Y'_i(\alpha, \beta)\, u_i\,, \qquad \beta_i \in K\,.$$

Since I_a is obtained by the inner automorphism $b \to SbS^{-1}$ followed by the specialization $S \to a$, I_a is clearly a homomorphism of A into A, and we shall call I_a the *inner endomorphism* by a.

Let $(\tilde{u}_i)$ be another K-basis of A. Put

$$(\sum X_i \tilde{u}_i)(\sum Y_i \tilde{u}_i)(\sum X_i \tilde{u}_i)^{-1} = \sum Y''_i(X, Y)\, \tilde{u}_i\,.$$

On the other hand, if $\sum f_i u_i = \sum \tilde{f}_i \tilde{u}_i$, we have by (1)

$$(\sum \tilde{X}_i \tilde{u}_i)(\sum \tilde{Y}_i \tilde{u}_i)(\sum \tilde{Y}_i \tilde{u}_i)^{-1} = \sum \tilde{Y}'_i \tilde{u}_i\,,$$

whence $Y''_i(\tilde{X}, \tilde{Y}) = \tilde{Y}'_i$. But if $a=\sum \alpha_i u_i = \sum \tilde{\alpha}_i \tilde{u}_i$ is semiregular with respect to (u_i), then Y'_i, and hence $\tilde{Y}'_i$, which is a linear combination of $Y'_1,\cdots, Y'_n$, can be defined at $X_i=\alpha_i$; this implies that Y''_i is defined at $X_i=\tilde{\alpha}_i$, and hence the notion of semiregularity is independent of the choice of the K-basis (u_i). A similar argument shows that the inner endomorphism by a is also independent of the choice of (u_i).

EXAMPLES. 1) A regular element of A is obviously semiregular.

2) If A is commutative, every element of A is semiregular, and induces the identity automorphism of A.—This is clear since we have $Y'_i = Y_i$ in (1). Cf. § 2, v).

3) Let A be the subalgebra of the 3-rowed matrix algebra over K consisting of linear combinations $a=\alpha c_{11} + \beta_{23} + \gamma c_{33} + \delta c_{21}$, $\alpha, \beta, \gamma, \delta \in K$, where c's denote the matrix units. Then a is semiregular if and only if $\alpha \neq 0$. The kernel of the inner endomorphism by c_{11} is the radical $\{c_{21}\}$, while the inner endomorphism by $c_{11}+c_{22}$ is actually the identity automorphism in spite of the non-regularity of $c_{11}+c_{22}$.

§ 2. The following two facts are immediate consequences of the definition of the semiregularity.

i) *An element* a_1+a_2 *of the direct sum* A_1+A_2, *where* $a_1\in A_1$, $a_2\in A_2$, *is semiregular if and only if* a_1 *as well as* a_2 *is semiregular in* A_1 *and* A_2, *respectively.*

ii) *Let* L *be an extension field of* K, *then* $a\in A$ *is semiregular if and only if* a *is semiregular in* $A_L=A\otimes L$.

Now we shall prove:

iii) *Let* $\mathfrak{a}$ *be a two-sided ideal of* A, $\mathfrak{a}\neq A$. *If* a *is semiregular in* A, *the residue class* $\bar{a}$ *of* a *modulo* $\mathfrak{a}$ *is semiregular in* $A/\mathfrak{a}$.

Take a K-basis (u_i) of A such that $u_{k+1},\cdots,u_n$, $0<k<n$, span $\mathfrak{a}$. Then the residue classes $\bar{u}_i$ of u_i, $i=1,\cdots,k$, modulo $\mathfrak{a}$ constitute a basis of $A/\mathfrak{a}$. Since $u_iu_j\in\mathfrak{a}$ if i or $j>k$, it is a trivial matter to see that

$$\Big(\sum_{i=1}^{k} X_i\bar{u}_i\Big)\Big(\sum_{i=2}^{k} Y_i\,\bar{u}_i\Big)\Big(\sum_{i=1}^{k} X_i\,\bar{u}_i\Big)^{-1}=\sum_{i=1}^{k} Y'_i\,\bar{u}_i\,,$$

whence the proposition follows.

iv) *If* a *and* b *are semiregular, then* ab *is also semiregular, and we have* $I_a\circ I_b=I_{ab}$.

This is clear since $R_u(S)$ is defined at ab and $R_u(ab)=R_u(a)\,R_u(b)$.

v) *The zero element is semiregular if and only if* A *is commutative.*

The "if" part is shown in Example 2). Since $M(S,u)$ (cf. § 1) is homogeneous in X's, 0 is semiregular only if every $R_{ij}(X)$ in (2) is a polynomial. Since the numerator of $R_{ij}(X)$ has the same degree as the denominator, $R_{ij}(X)$ must be a constant in this case. Then Y'_i in (1) is independent of X's, $i=1,\cdots,n$. Specialize $\sum X_iu_i$ to the identity, and we have $Y'_i=Y_i$, which implies the commutativity of A.

From iv) and v) follows at once:

vi) *A semiregular element of a non-commutative algebra is not nilpotent.*

The kernel of I_a is a two-sided ideal of A, which we shall denote by $J(a)$. Since we have $J(a)\subset J(a^2)\subset\cdots$, there exists some integer k, for which $J(a^k)=J(a^{k+1})$. Let $a^k=b$, then we have $J(b)=J(b^2)$. Now we have

vii) *If* $J(b)=J(b^2)$ *for a semiregular element* b, A *is the direct sum of* $J(b)$ *and* $I_b(A)$.

Indeed, since $J(b)=J(b^2)$, we have $J(b)\cap I_b(A)=(0)$. The proposition follows as we have $\big(I_b(A):K\big)=\big(A/J(b):K\big)=(A:K)-\big(J(b):K\big)$.

viii) *A semiregular element of a simple algebra A is a regular element or the zero element.*

For, let a be semiregular. Since we have $ax=I_a(x)a$ for every $x\in A$, Aa is a two-sided ideal of A. If $Aa=A$, then a is regular, and if $Aa=(0)$, then $a=0$.

ix) *Every semiregular element of a non-commutative primary algebra A is a regular element.*

Let N be the radical of A. If a is semiregular in A, the residue class $\bar{a}$ of a modulo N is semiregular in $\bar{A}=A/N$ by iii). By viii), $\bar{a}$ is regular or $\bar{a}=0$. If $\bar{a}$ is regular in $\bar{A}$, then a itself is regular in A. If $\bar{a}=0$, a is nilpotent, but this is not the case in view of vi).

§ 3. The inner endomorphism by a left-semiregular element is a generalization of the mapping of the form $x\rightarrow axa^{-1}$. But, we can equally consider the mapping $x\rightarrow a^{-1}xa$. Let A' be an anti-isomorphic copy of A, with a fixed anti-isomorphism $a\longleftrightarrow a'$. Then $a\in A$ will be called *right-semiregular* if a' is left-semiregular in A'. A right-semiregular element a defines an *inner endomorphism* $x\rightarrow I_a^*$ where $\big(I_a^*(x)\big)'=I_{a'}(x')$ for every $x\in A$. It is evident that i)—ix) hold for right-semiregular elements with the trivial modification $I_a^*\circ I_b^*=I_{ba}^*$ in iv).

In general, the left-semiregularity and the right-semiregularity are independent of each other.

Example. In the algebra A of Example 3) of § 1, $a=\alpha c_{11}+\beta c_{22}+\gamma c_{33}+\delta c_{21}$ is right-semiregular if and only if $\beta\neq 0$.

x) *For $a\in A$, the following two conditions are equivalent:*

I) *a is both left and right-semiregular.*

II) *a is left-semiregular and the inner endomorphism I_a by a is an automorphism.*

If one of these (equivalent) conditions is satisfied, I_a^* is defined and is the inverse of I_a.

Proof. For the general element S, we have obviously $I_S\circ I_S^*=$ identity. Since I_a and I_a^*, if they can be defined, are given by the specialization from I_S and I_S^* respectively, we have also $I_a\circ I_a^*=$identity, which implies that I_a and I_a^* are automorphisms inverse of each other. Next, let us assume II). The coefficients of the matrix $R_u^{-1}(S)$ of

$I_S^* = I_S^{-1}$ are given by the cofactors of $R_u(S)$ devided by the determinant of $R_u(S)$. On the other hand, if I_a is an automorphism, the determinant of $R_u(a)$ is non-vanishing. Hence $R_u^{-1}(S)$ is defined at a, and a is right-semiregular.

Finally, we shall indicate a further possibility of extending the notion of the inner automorphism. Take, as an example, the algebra of all 3-rowed matrices over K, whose right upper halves are vanishing. Such a matrix (α_{ij}), $\alpha_{ij} \in K$, is left-semiregular if and only if $\alpha_{11} \neq 0$, $\alpha_{22} \neq 0$. Now, if we substitute in the matrix $R_u(S)$, where S is the general element $S = \sum X_{ij} u_{ij}$, μX_{22} and νX_{22}, μ, $\nu \in K$, for the variables X_{32} and X_{33}, respectively, the resulting matrix is defined at every element of A, such that $\alpha_{11} \neq 0$, $\alpha_{32} = \mu \alpha_{22}$, $\alpha_{33} = \nu \alpha_{22}$. In this manner, we get a very wide class of *semiregular* elements and of *inner endomorphisms*. A remarkable fact is that a semiregular element in this sense may induce various endomorphisms different from each other, according to the ways of specialization. In particular, the zero element may define several endomorphisms. Thus, those semiregular elements may provide us with some endomorphisms or automorphisms different from the usual inner automorphisms. But, these remarks suggest us on the other hand, that we should have some difficulties in treating this subject along these lines.

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