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Supplements and corrections to my paper: On the automorphisms of a real semi-simple Lie algebra.

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In the present paper we give supplements and corrections to the paper mentioned in the title. We abbreviate this paper by [A].

Supplements.

In [A] we had to depend essentially on the theorems of Gantmacher which allows us to describe the situation of a real semi-simple Lie algebra in its complex form by making use of the concept of a particular rotation. We have introduced in [A] p. 112 this latter concept in an apparently different way from the one originally given by Gantmacher. But the equivalency of both concepts is justified by another theorem of Gantmacher, so that we could avail ourselves of his theorems. However, he obtained these theorems after long considerations on the automorphisms of a complex semi-simple Lie algebra. In these circumstances, we shall prove here anew, starting from our definition of the particular rotation, the required results. This will make us free from the theorems of Gantmacher used in the previous paper [A].

Let $\tilde{\mathfrak{G}}$ be a complex semi-simple Lie algebra with the canonical basis

$$h_1, \cdots, h_l, e_\alpha, e_{-\alpha}, \cdots$$

and $\mathfrak{G}_u$ the unitary restriction of $\tilde{\mathfrak{G}}$ with respect to this basis (see [A].

p. 109). The vectors $h_1, \cdots, h_l$ span a Cartan subalgebra $\tilde{\mathfrak{H}}$ and the elements of $\tilde{\mathfrak{H}}$ whose adjoint mappings in $\tilde{\mathfrak{G}}$ have only real eigenvalues form the real part $\mathfrak{H}$ of $\tilde{\mathfrak{H}}$. $\mathfrak{H}$ contains the roots $\alpha=-[e_\alpha, e_{-\alpha}]$ and has the inner product induced by the fundamental bilinear form of $\tilde{\mathfrak{G}}$. A rotation is defined as an orthogonal transformation in $\mathfrak{H}$ which permutes the roots among themselves. Therefore, it permutes clearly the hyperplanes $E_\alpha=\{\lambda\,;\ (\alpha, \lambda)=0\}$ in $\mathfrak{H}$ among themselves and hence it permutes the connected components of the set $\mathfrak{H}-\bigcup_\alpha E_\alpha$ among themselves. Then, according to our definition, a particular rotation is a rotation which maps one of these connected components onto itself. On the other hand we know that if such a connected component is restricted by the hyperplanes $E_{\alpha_1}, \cdots, E_{\alpha_l}$ and is defined by the inequalities $(\alpha_1, \lambda)>0, \cdots, (\alpha_l, \lambda)>0$ then the roots $\alpha_1, \cdots, \alpha_l$ constitute a fundamental basis in the rootsystem, that is, they form a basis in $\mathfrak{H}$ with the property that each root α is expressed in the form $\alpha=p_1\alpha_1+\cdots+p_l\alpha_l$ where p_i's are integers either all $\geqq 0$ or all $\leqq 0$.[1] From this fact we may see that a particular rotation is a rotation which permutes the roots in a fundamental basis among themselves.

PROPOSITION 1. *A rotation is a particular rotation if and only if it leaves fixed a regular element of* $\tilde{\mathfrak{G}}$ *contained in* $\mathfrak{H}$.

PROOF. A regular element of $\tilde{\mathfrak{G}}$ contained in $\mathfrak{H}$ is an element for which all roots do not vanish, i.e., $(\alpha, \lambda)\neq 0$ for all roots α. Now, if a rotation is particular, it permutes the roots $\alpha_1, \cdots, \alpha_l$ in a fundamental basis among themselves and hence it leaves fixed the element λ determined by the equations $(\alpha_1, \lambda)=\cdots=(\alpha_l, \lambda)=c$, c being a real number $\neq 0$. By the property of a fundamental basis λ is surely a regular element. Conversely, if a rotation leaves fixed a regular element, then it maps the connected component containing this element of the set $\mathfrak{H}-\bigcup_\alpha E_\alpha$ onto itself. By definition the rotation is a particular rotation.

Our next step is to build up relations between an involutive automorphism of the unitary restriction $\mathfrak{G}_u$ and a particular rotation. For an involutive automorphism S of $\mathfrak{G}_u$ we set $\mathfrak{G}_1=\{x+Sx\,;\ x\in\mathfrak{G}_u\}$ and $\mathfrak{G}_{-1}=\{x-Sx\,;\ x\in\mathfrak{G}_u\}$. Then the real subspace $\mathfrak{G}=\mathfrak{G}_1+\sqrt{-1}\ \mathfrak{G}_{-1}$ of $\tilde{\mathfrak{G}}$ is a real semi-simple Lie algebra. We have studied in [A] a real semi-simple Lie algebra in this form, according to a theorem of Cartan.

Further, in order to research the structure of $\mathfrak{G}$, we had to impose there some assumptions on S. In what follows we do not distinguish an automorphism of $\mathfrak{G}_u$ and its unique extension onto $\tilde{\mathfrak{G}}$. Then, the assumption which we had to set in [A] p. 112 is as follows: S leaves invariant $\tilde{\mathfrak{H}}$ and induces a particular rotation in $\mathfrak{H}$. This assumption was assured there by theorems of Gantmacher, but it can be also justified by the following proposition because any automorphism conjugate to S determines a Lie algebra isomorphic to $\mathfrak{G}$. Here and in the following two automorphisms are said to be *conjugate* if there exists an inner automorphism of $\mathfrak{G}_u$ by which one of them is transformed into the other.

PROPOSITION 2.*) *An involutive automorphism S of $\mathfrak{G}_u$ is conjugate to an automorphism which leaves invariant the Cartan subalgebra $\tilde{\mathfrak{H}}$ and induces a particular rotation in its real part $\mathfrak{H}$.*

PROOF. As is well known, $\mathfrak{G}_u$ is the Lie algebra of the compact connected Lie group G_u^0, the so-called adjoint group of $\mathfrak{G}_u$, and any maximal abelian subalgebra contains a regular element of $\mathfrak{G}_u$ so that this subalgebra coincides with its normalizer. Now we take a maximal abelian subalgebra $\mathfrak{A}_1$ of the subalgebra $\mathfrak{G}_1$ in $\mathfrak{G}_u$ and let $\mathfrak{A}$ be a maximal abelian subalgebra of $\mathfrak{G}_u$ containing $\mathfrak{A}_1$ For an element x in $\mathfrak{A}$, $x+Sx$ belongs to $\mathfrak{G}_1$ and it commutes with any element of $\mathfrak{A}_1$, which shows that $x+Sx$ belongs to $\mathfrak{A}_1$. It follows further that Sx belongs to $\mathfrak{A}$ as well as x and therefore $\mathfrak{A}$ is left invariant under S. By these facts we show that $\mathfrak{A}$ is uniquely determined by $\mathfrak{A}_1$. Let $\mathfrak{A}'$ be another maximal abelian subalgebra of $\mathfrak{G}_u$ containing $\mathfrak{A}_1$ and y an element of $\mathfrak{A}'$. Then $y'=(y+Sy)/2$ belongs to $\mathfrak{A}_1$ a fortiori to $\mathfrak{A}$, so that if we see that $y-y'$ belongs to $\mathfrak{A}$ y itself belongs to $\mathfrak{A}$ and our assertion will be proved. To see this, as $y-y'$ belongs to $\mathfrak{A}'$ and $S(y-y')=-(y-y')$, it is sufficient that an element y of $\mathfrak{A}'$ with the property $Sy=-y$ belongs also to $\mathfrak{A}$. Let x be an element contained in $\mathfrak{A}$. Setting $x'=(x+Sx)/2$ and $x''=x-x'$, x' belongs to $\mathfrak{A}_1$ and $Sx''=-x''$. Therefore, $[y,x]=[y,x'+x'']=[y,x'']$ and this element belongs to $\mathfrak{G}_1$. Moreover by the identity $[z,[y,x]]=[[z,y],x]+[y,[z,x]]$ we see that $[y,x]$ commutes with any element z of $\mathfrak{A}_1$. Thus $[y,x]$ must belong to $\mathfrak{A}_1$ and so to $\mathfrak{A}$. x being an arbitrary element of the maximal abelian subalgebra $\mathfrak{A}$, this shows that y belongs to the normalizer of $\mathfrak{A}$. Hence y belongs to $\mathfrak{A}$. The uniqueness of the maximal abelian subalgebra $\mathfrak{A}$

containing $\mathfrak{A}_1$ is proved.

It follows that $\mathfrak{A}_1$ contains a regular element of $\mathfrak{G}_u$. Indeed, if $\mathfrak{A}_1$ contains no regular elements, every element of the abelian subgroup A_1 in G_u^0 generated by $\mathfrak{A}_1$ has non commutative centralizer and therefore it is contained in at least two maximal abelian subgroups of G_u^0. As A_1 is a maximal abelian subgroup of the closed connected subgroup K^0 corresponding to the subalgebra $\mathfrak{G}_1$ (see [A] p. 108), it is a closed abelian subgroup and hence it has a generator. At least two maximal abelian subgroups of G_u^0 containing this generator contain the subgroup A_1 itself. This implies the existence of at least two maximal abelian subalgebras of $\mathfrak{G}_u$ containing $\mathfrak{A}_1$, which is a contradiction.

Now the subspace $\sqrt{-1}\ \mathfrak{H}$ of $\mathfrak{H}$ is contained in $\mathfrak{G}_u$ and is a maximal abelian subalgebra of $\mathfrak{G}_u$. By the conjugateness of maximal abelian subgroups in the compact group G_u^0 we may find an inner automorphism P of $\mathfrak{G}_u$ which maps $\sqrt{-1}\ \mathfrak{H}$ onto $\mathfrak{A}$. Set $S'=P^{-1}\ SP$. Then, since S leaves invariant $\mathfrak{A}$, S' leaves invariant $\sqrt{-1}\ \mathfrak{H}$. Extended over $\mathfrak{G}$, S' leaves invariant the Cartan subalgebra $\tilde{\mathfrak{H}}$, as $\tilde{\mathfrak{H}}$ is spanned by $\sqrt{-1}\ \mathfrak{H}$ over complex numbers. As such, the automorphism S' induces in the real part $\mathfrak{H}$ of $\tilde{\mathfrak{H}}$ a rotation ρ [2)]. While we have proved above that $\mathfrak{A}_1$ contains a regular element x of $\mathfrak{G}_u$ which is naturally regular in $\mathfrak{G}$. The regular element $\sqrt{-1}\ P^{-1}x$ is surely contained in $\mathfrak{H}$ and is left invariant under ρ. By Proposition 1, ρ is then a particular rotation. This completes the proof.

Note that in the last part of this proof we obtain incidentally a proof of Lemma 7 in [A] p. 114; the element $P^{-1}x$ is a regular element contained in $\sqrt{-1}\ \mathfrak{H}_1$ by the notation used there.

Let S be an involutive automorphism which leaves invariant the Cartan subalgebra $\tilde{\mathfrak{H}}$ and induces a particular rotation in its real part $\mathfrak{H}$. As is mentioned before Proposition 2 allows us to confine our attentions to such an automorphism S in studying a real semi-simple Lie algebra. However for the actual determination of this Lie algebra we may further impose upon S the assumption that it has a canonical representation in connection with its inducing particular rotation. We have used this restriction on S in [A] p. 127 again resting on the theorems of Gantmacher. In the following we prove this fact from our standpoint.

Assume that a particular rotation ρ is involutive, which is the case of the one induced by an involutive automorphism of $\mathfrak{G}_u$. For brevity, we denote $\rho(\lambda)$ by λ^*, λ being an element of $\mathfrak{H}$, We may designate by $\alpha_1, \cdots, \alpha_s, \xi_1, \xi_1^*, \cdots, \xi_r, \xi_r^*$, where $\alpha_i=\alpha_i^*$, the roots in a fundamental basis which are permuted by ρ among themselves. We denote by $\mathfrak{H}_1$ the subspace of $\mathfrak{H}$ consisting of elements λ such that $\lambda^*=\lambda$. It is easily seen that $\mathfrak{H}_1$ is the linear subspace of $\mathfrak{H}$ defined by the equations $(\xi_1, \lambda)=(\xi_1^*, \lambda), \cdots, (\xi_r, \lambda)=(\xi_r^*, \lambda)$. While, according to a theorem of Gantmacher[2] ρ determines uniquely the automorphism S_0 defined as follows: S_0 induces the particular rotation ρ in $\mathfrak{H}$ which defines its behavior in $\mathfrak{H}$, and we set

$$S_0 e_\alpha = \mu_\alpha e_{\alpha^*} \tag{1}$$

where the numbers μ_α are equal to 1 for $\alpha=\alpha_1, \cdots, \alpha_s, \xi_1, \xi_1^*, \cdots, \xi_r, \xi_r^*$ and are determined for other roots α by the following principle. In general an automorphism which leaves invariant $\mathfrak{H}$ induces a rotation ρ in $\mathfrak{H}$ and takes the form (1)[2]. Then the numbers μ_α are uniquely determined by their values for roots α in a fundamental basis according to the formulae

$$\mu_\alpha \mu_{-\alpha}=1\ ;\ \mu_{\alpha+\beta}=\frac{N_{\alpha^*,\beta^*}}{N_{\alpha,\beta}}\mu_\alpha\mu_\beta \tag{2}$$

by virtue of Lemma 4 in [A] p. 110. In other words, an automorphim of this type is uniquely determined by the rotation and by its effects on the eigenvectors e_α for roots in a fundamental basis. We may see also by this principle that S_0 is an involutive automorphism.

Our purpose stated above is now furnished by the following

PROPOSITION 3. *An involutive automorphism S of $\mathfrak{G}_u$ is conjugate to an automorphism of the form*

$$S_0 \exp(\mathrm{ad}\sqrt{-1}\lambda_0)$$

where S_0 is the automorphism uniquely determined by an involutive particular rotation ρ and λ_0 is an element of $\mathfrak{H}$ fixed under ρ.

PROOF. By Proposition 2, we may assume that S leaves invariant the Cartan subalgebra $\mathfrak{H}$ and induces an involutive particular rotation ρ in its real part $\mathfrak{H}$. We apply the notions defined above for this particular rotation ρ. We may set

$$Se_{\alpha}=\nu_{\alpha}\, e_{\alpha^*}\,.$$

Then the numbers ν_{α} satisfy the relations obtained from (2) by replacing μ with ν. Moreover, since S leaves invariant the unitary restriction $\mathfrak{G}_u$, $\nu_{-\alpha}=\bar{\nu}_{\alpha}$ and consequently $|\nu_{\alpha}|=1$. On the other hand, since S is involutive $\nu_{\alpha}\nu_{\alpha^*}=1$ and $\nu_{\alpha^*}=\bar{\nu}_{\alpha}$. Therefore $\nu_{\alpha}=\pm 1$ for the roots α with the property $\alpha^*=\alpha$, especially for $\alpha=\alpha_1, \cdots, \alpha_s$. We show in advance that we may assume the equalities $\nu_{\xi_j}=\nu_{\xi_j^*}=1$ $(1\leqq j\leqq r)$. In fact, since the vectors $\xi_j-\xi_j^*$ are linearly independent and since $|\nu_{\xi_j}|=1$ we can find an element λ in $\mathfrak{H}$ satisfying the equations

$$\exp(\sqrt{-1}\,(\xi_j^*-\xi_j\,,\lambda))=\nu_{\xi_j} \qquad (1\leqq j\leqq r)\,.$$

Taking the complex conjugate of both sides we have then

$$\exp(\sqrt{-1}\,(\xi_j-\xi_j^*\,,\lambda))=\nu_{\xi_j^*} \qquad (1\leqq j\leqq r)\,.$$

Consider the inner automorphism $U=\exp(\mathrm{ad}\sqrt{-1}\,\lambda)$ of $\mathfrak{G}_u$. The automorphism $U^{-1}SU$ leaves invariant $\tilde{\mathfrak{H}}$ and induces the particular rotation ρ in $\mathfrak{H}$ as well as S. By the choice of λ and by the relations among ν_{α}'s we see further that the numbers ν_{ξ_j} and $\nu_{\xi_j^*}$ for S are all replaced by 1 for $U^{-1}SU$. Transferring to $U^{-1}SU$ if necessary, we may assume from the beginning that $\nu_{\xi_j}=\nu_{\xi_j^*}=1$ $(1\leqq j\leqq r)$ for S. We prove under such restrictions that S has the canonical representation required in the proposition.

S and S_0 are commutative. To observe this, consider the automorphisms SS_0 and S_0S. Both of them reduce to the identical rotation ρ^2 in $\mathfrak{H}$ and hence they fix each element of $\tilde{\mathfrak{H}}$. Outside of $\tilde{\mathfrak{H}}$

$$SS_0\, e_{\alpha}=\nu_{\alpha^*}\,\mu_{\alpha}\, e_{\alpha}\,; \qquad S_0\, Se_{\alpha}=\nu_{\alpha}\,\mu_{\alpha^*}\, e_{\alpha}\,.$$

The numbers μ_{α} for the roots in the fundamental basis being equal to 1 by the definition of S_0, under the conditions $\nu_{\xi_j}=\nu_{\xi_j^*}=1$ $(1\leqq j\leqq r)$ we see that $\nu_{\alpha^*}\,\mu_{\alpha}=\nu_{\alpha}\,\mu_{\alpha^*}$ for every roots α in the fundamental basis. Then, as we have remarked before, two automorphisms SS_0 and S_0S coincide: $SS_0=S_0S$. An immediate consequence of this relation is that S_0 maps the 1-eigenspace $\mathfrak{G}_1$ of S into itself. Therefore SS_0 maps $\mathfrak{G}_1$ into itself, too.

Now the proposition will be proved if we show that the automorphism $S_0^{-1}S$, or what amounts to the same, its inverse automorphism SS_0 takes the form $\exp(\mathrm{ad}\sqrt{-1}\,\lambda_0)$ with an suitable element λ_0 in $\mathfrak{H}_1$. By the results just obtained on SS_0 this last assertion is assured as soon as we prove the following

LEMMA. *Let A be an automorphism of $\mathfrak{G}_u$. If A leaves fixed each element of $\tilde{\mathfrak{H}}$ and if A maps $\mathfrak{G}_1$ into itself, A has the form* $\exp(\mathrm{ad}\sqrt{-1}\,\lambda_0)$ *with an element λ_0 in $\mathfrak{H}_1$.*

PROOF. Since A leaves fixed each element of $\tilde{\mathfrak{H}}$, we may set

$$Ae_\alpha = \kappa_\alpha e_\alpha . \tag{3}$$

The complexification of $\mathfrak{G}_1$, that is, the linear subspace $\tilde{\mathfrak{G}}_1$ of $\tilde{\mathfrak{G}}$ spanned by $\mathfrak{G}_1$ over complex numbers, obviously consists of elements $x+Sx$, $x\in\tilde{\mathfrak{G}}$, and therefore it is spanned by $\mathfrak{H}_1$ and by the linearly independent vectors e_α and $e_\xi+\nu_\xi e_{\xi^*}$ where α are the roots with the properties that $\alpha^*=\alpha$ and $\nu_\alpha=1$ and ξ the representatives of each pair ξ, ξ^* $(\xi\neq\xi^*)$. Since A leaves invariant $\tilde{\mathfrak{G}}_1$ as well as $\mathfrak{G}_1$, $A(e_\xi+\nu_\xi e_{\xi^*}) = \kappa_\xi e_\xi+\kappa_{\xi^*}\nu_\xi e_{\xi^*}$ belongs to $\tilde{\mathfrak{G}}_1$ and it must be a scalar multiple of $e_\xi+\nu_\xi e_{\xi^*}$. Therefore $\kappa_\xi=\kappa_{\xi^*}$ and especially $\kappa_{\xi_j}=\kappa_{\xi_j^*}$ $(1\leq j\leq r)$. On the other hand, the numbers κ_α in (3) have the analogous properties as the numbers ν_α for S. For example, $|\kappa_\alpha|=1$. Then we can find an element λ_0 in $\mathfrak{H}$ satisfying the equations

$$\begin{aligned} &\exp(\sqrt{-1}(\alpha_i,\lambda_0))=\kappa_{\alpha_i} && (1\leq i\leq s),\\ &\exp(\sqrt{-1}(\xi_j,\lambda_0))=\kappa_{\xi_j}, && \\ &\exp(\sqrt{-1}(\xi_j^*,\lambda_0))=\kappa_{\xi_j^*} && (1\leq j\leq r). \end{aligned}$$

By $\kappa_{\xi_j}=\kappa_{\xi_j^*}$ we may assume here that $(\xi_j,\lambda_0)=(\xi_j^*,\lambda_0)$ $(1\leq j\leq r)$. Then λ_0 belongs to $\mathfrak{H}_1$, as is remarked earlier. The automorphism $\exp(\mathrm{ad}\sqrt{-1}\,\lambda_0)$ of $\mathfrak{G}_u$ leaves fixed each element of $\tilde{\mathfrak{H}}$ as well as A and by the choice of λ_0 it gives the same effects as A to the eigenvectors e_α for the roots in the fundamental basis. Hence $A=\exp(\mathrm{ad}\sqrt{-1}\,\lambda_0)$ by the same reason as before, which proves the lemma.

Corrections.

The following corrections should be made in the paper [A].

1. The proof of Lemma 12 on p. 117 is not complete. However, as we have remarked there, this lemma is a weaker form of the lemma proved above in the present paper.

2. In § 4, we have applied our results to compute the automorphisms of each real form of the complex simple Lie algebra A_n. There it should be assumed that $n \geq 2$. In the case $n=1$ the particular rotation ρ_0 defined on p. 127 is the identical transformation in $\mathfrak{H}$ and we need some trivial modifications. We may see that there are essentially two real forms of A_1, the Lie algebras of the unimodular unitary group and of the real unimodular group both of degree 2, which have respectively no or one typical outer automorphism.

3. Errata; p. 103, line 3 from below, read "Cartan [2]" instead of "Cartan [3]"; p. 110, line 11, read "(α, α)" instead of "(β, β)"; p. 116, line 22, read "$\sum_3$" instead of "$\sum_2$"; p. 130, line 9, read "$n=2f-1$" instead of "$n=2f+1$".

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Notes

1) See H. Weyl: The structure and representation of continuous groups. Lectures at the Institute for Advanced Study, Princeton, 1934-1935, p. 166.

2) See F. Gantmacher: Canonical representation of automorphisms of a complex semi-simple Lie group. Rec. Math. N.S., vol. 5 (1934), 101-144; p. 129, Theorem 20; p. 130, Theorem 21. We note that these theorems are direct consequences of the structure theory for a complex semi-simple Lie algebra and are outside of our present considerations, though the theorems of this type profit us largely in the previous paper [A].

*) The writer thanks to Mr. I. Satake who kindly remarks him that Prop. 2 follows from the original definition of a particular rotation of Gantmacher also in a simple manner.