Tohoku Math. J.
60 (2008), 303–328

NONCONSTANT SELFSIMILAR BLOW-UP PROFILE FOR THE EXPONENTIAL REACTION-DIFFUSION EQUATION

MAREK FILA AND AAPPO PULKKINEN

(Received October 25, 2006, revised October 19, 2007)

Abstract. We study the blow-up profile of radial solutions of a semilinear heat equation with an exponential source term. Our main aim is to show that solutions which can be continued beyond blow-up possess a nonconstant selfsimilar blow-up profile. For some particular solutions we determine this profile precisely.

1. Introduction.

We consider the following problem

$$\begin{cases} u_t = \Delta u + f(u)\,, & x \in \Omega\,, \quad t > 0\,, \\ u = 0\,, & x \in \partial\Omega\,, \quad t > 0\,, \\ u(x,0) = u_0(x) \geq 0\,, & x \in \Omega\,, \end{cases} \tag{1.1}$$

where $\Omega = B(R) = \{x \in \boldsymbol{R}^n\,;\ |x| < R\}$. Throughout the paper, we assume that the initial condition $u_0 \in C^1(\bar{\Omega})$ is radially symmetric. In the first part of the paper, we shall assume that

$$f \in C^1\,, \quad f(\cdot) \geq 0 \ \text{ in } \ [0,\infty) \quad \text{and} \quad \lim_{u\to\infty} e^{-u} f(u) = 1\,. \tag{1.2}$$

We shall study solutions that blow up in finite time, by which we mean that there is $T = T(u_0) \in (0,\infty)$ such that

$$\lim_{t\nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty\,.$$

Our first result is the following

THEOREM 1.1. *Let $n \in [3,9]$. Assume that* (1.2) *holds and that u is a solution of* (1.1) *which blows up in a finite time T and satisfies $u(0,t) = \max_\Omega u(\cdot,t)$ for all t close to T. Then there exists a constant $K < \infty$ such that*

$$\log(T-t) + \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq K \quad \text{for all} \quad t \in [0,T)\,. \tag{1.3}$$

The blow-up rate (1.3) for solutions of (1.1) with $f(u) = e^u$ was only known before under the assumption that $u_t \geq 0$, see [16]. In this paper, we are interested mainly in solutions which can be continued beyond blow-up as L^1-solutions (see the definition below), and such solutions cannot be nondecreasing in time, since $u_t \geq 0$ implies complete blow-up, see [1].

To formulate our next result we introduce the definition of L^1-solutions of Problem (1.1).

2000 *Mathematics Subject Classification.* Primary 35K57; Secondary 35B40.

Key words and phrases. Semilinear parabolic equation, blow-up profile, selfsimilar solutions.

DEFINITION 1.1. By an L^1-*solution* of (1.1) on $[0, \mathcal{T}]$ we mean a function $u \in C([0, \mathcal{T}]; L^1(\Omega))$ such that $f(u) \in L^1(Q_{\mathcal{T}})$, $Q_{\mathcal{T}} := \Omega \times (0, \mathcal{T})$ and the equality

$$\int_\Omega [u\Psi]_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_\Omega u\Psi_t dx dt = \int_{t_1}^{t_2} \int_\Omega (u\Delta\Psi + f(u)\Psi) dx dt$$

holds for any $0 \le t_1 < t_2 \le \mathcal{T}$ and $\Psi \in C^2(\bar{Q}_{\mathcal{T}})$, $\Psi = 0$ on $\partial\Omega \times [0, \mathcal{T}]$. By a *global* L^1-*solution* we mean an L^1-solution which exists on $[0, \mathcal{T}]$ for every $\mathcal{T} > 0$.

The existence of global unbounded L^1-solutions of (1.1) with $f(u) = \lambda e^u$, $n \ge 3$, was shown in [23] for $\lambda > 0$ small enough. If $3 \le n \le 9$, then these global unbounded L^1-solutions blow up in finite time, see [14].

THEOREM 1.2. *Let $f(u) = e^u$, $n \in [3, 9]$, and assume that the initial function u_0 is radially nonincreasing. Suppose u is an L^1-solution of* (1.1) *on* $[0, \mathcal{T}]$ *which blows up at $t = T < \mathcal{T}$. Then*

$$\lim_{t\to T} [\log(T - t) + u(y\sqrt{T - t}, t)] = \varphi(|y|), \qquad y \in \boldsymbol{R}^n,$$

where φ satisfies

$$\begin{cases} \varphi_{\eta\eta} + \left(\dfrac{n-1}{\eta} - \dfrac{\eta}{2}\right)\varphi_\eta + e^\varphi - 1 = 0, & \eta > 0, \\ \varphi(0) = \mu, \quad \varphi_\eta(0) = 0, \end{cases} \tag{1.4}$$

and

$$\lim_{\eta\to\infty} [\varphi(\eta) + 2\log\eta] = C_\mu \tag{1.5}$$

for some $\mu > 0$ and $C_\mu \in \boldsymbol{R}$.

In the case $n = 1, 2$, there is no solution of (1.4) satisfying

$$\lim_{\eta\to\infty} \left(1 + \frac{\eta}{2}\varphi_\eta(\eta)\right) = 0, \tag{1.6}$$

see [9, 3]. On the other hand, for $3 \le n \le 9$, there exists an increasing sequence $\{\mu_i\}_{i=0}^\infty$, $\mu_i \to \infty$, such that the solution φ_i of (1.4) with $\mu = \mu_i$ satisfies (1.6), see [10]. Lacey and Tzanetis proved in [23] that for $3 \le n \le 9$ the solution ϕ_0 of (1.4) with $\mu = \mu_0$ satisfies

$$\lim_{\eta\to\infty} \left(\varphi_0(\eta) + \log\frac{\eta^2}{2(n-2)}\right) = -c_0, \qquad c_0 > 0, \tag{1.7}$$

and the equation

$$\varphi_0(\eta) + \log\frac{\eta^2}{2(n-2)} = 0 \tag{1.8}$$

has two roots.

For some particular solutions u (the L^1-connections from a stationary solution ϕ_2 to another stationary solution ϕ_0, see Proposition 4.3) we show (see Theorem 4.4) that

$$\lim_{t\to T} [\log(T - t) + u(y\sqrt{T - t}, t)] = \varphi_0(|y|), \qquad y \in \boldsymbol{R}^n,$$

where φ_0 satisfies (1.4), (1.7) and (1.8) has two roots. As far as we know, this is the first example of a solution of (1.1) with a precisely determined nonconstant selfsimilar blow-up profile. The existence of a class of solutions of (1.1) with nonconstant selfsimilar blow-up profiles was known before for $f(u) = u^p$ and some $p > (n+2)/(n-2)$, $n > 2$, see [24]. But no characterization of the limit selfsimilar profile for any such solution was given in [24].

The paper is organized as follows. In Sections 2 and 3 we prove Theorems 1.1 and 1.2. Section 4 is devoted to determining the exact profile of some special solutions mentioned above.

2. Blow-up rate. In this section we prove Theorem 1.1. We shall use the method from [7] that has to be modified and combined with an estimate from [16] because the rescalings employed here and in [7] are different. In particular, the present rescaling does not preserve positivity. This fact is also a reason why the arguments from [24] do not seem to apply easily to Problem (1.1) with a nonlinearity like $f(u) = e^u$.

In the following lemma we will consider the equation

$$v_{rr} + \frac{n-1}{r} v_r + f(v) = 0\,, \quad v_r \le 0 < v\,, \quad \text{in} \quad (0, \varepsilon)\,, \tag{2.1}$$

where $n \ge 3$ and $\varepsilon > 0$ is small.

LEMMA 2.1. *Assume that $f \in C(\boldsymbol{R})$ and $\lim_{u\to\infty} e^{-u} f(u) = 1$ and $n \ge 3$. Then there exists a singular solution $v = v^*$ of* (2.1) *satisfying*

$$\lim_{r\to 0} (v^*(r) + \log r^2) = \log(2(n-2))\,. \tag{2.2}$$

PROOF. The proof of the lemma is similar to the proof of an analogous lemma in [7] and so further details can be found there. Set $s = \log r$ and $W(s) = v(r) - \phi^*(r)$, where $\phi^*(r) = \log(2(n-2)r^{-2})$. Then v is a solution to (2.1) if and only if W satisfies

$$W_{ss} + (n-2)W_s + 2(n-2)W + h = 0 \quad \text{in} \quad (-\infty, \log\varepsilon)\,,$$

where the nonlinearity $h = h(s, W) = h_1(W) + h_2(s, W)$ and

$$h_1(W) = 2(n-2)(e^W - 1 - W)\,, \quad h_2(s, W) = e^{2s} f(W + \phi^*) - 2(n-2)e^W\,.$$

Moreover, v verifies the asymptotic behavior (2.2) if and only if $W(s) \to 0$ as $s \to -\infty$. If the solution W exists, it can be written by the variation of constants as

$$W(s) = \int_{-\infty}^{s} \frac{e^{\lambda_1(s-\tau)} - e^{\lambda_2(s-\tau)}}{\lambda_1 - \lambda_2} h(\tau, W(\tau))d\tau\,,$$

where λ_1 and λ_2 are the two roots to the characteristic equation $\lambda^2 + (n-2)\lambda + 2(n-2) = 0$.

The existence of a solution can now be proved using Schauder's fixed point theorem. Therefore, define

$$\mathcal{X} = \{\phi \in C((-\infty, \log\varepsilon)\,;\ \|\phi\|_{\mathcal{X}} = \sup_{s<\log\varepsilon} |\phi(s)| < \infty\}\,.$$

Let $B(\delta)$ be the closed ball of radius δ centered at 0 in $\mathcal{X}$, and let

$$T_i\phi(s) = \int_{-\infty}^{s} \frac{e^{\lambda_1(s-\tau)} - e^{\lambda_2(s-\tau)}}{\lambda_1 - \lambda_2} h_i(\tau, \phi(\tau))d\tau$$

for $i = 1, 2$. We need to show that the operator $(I - T_1)^{-1}T_2$ is well defined and that it has a fixed point.

Since, for every $|W_1|, |W_2| \le \delta$ and for some $\eta \in (W_1, W_2)$, we have

$$\begin{aligned}|h_1(W_1) - h_1(W_2)| &= 2(n-2)|e^{W_1} - e^{W_2} + W_2 - W_1| \\ &= 2(n-2)(e^{\eta} - 1)|W_1 - W_2| \le C\delta|W_1 - W_2|\,,\end{aligned}$$

we know that $\|T_1\phi\| \le (1/2)\|\phi\|$, for δ small enough, and hence the operator $(1 - T_1)^{-1}$: $B(\delta/2) \to B(\delta)$ exists with $\|(I - T_1)^{-1}\phi\| \le 2\|\phi\|$.

Define then a nonnegative and nondecreasing function

$$\omega(s) = \sup_{u \ge -s} \left| \frac{f(u)}{e^u} - 1 \right|.$$

So for any $W \in B(\delta)$, we have

$$|h_2(s, W(s))| = 2(n-2)e^{W(s)}\left(\frac{f(W(s) - 2s + \log(2(n-2))}{e^{W(s)-2s+\log(2(n-2))}} - 1\right) \le 2(n-2)e^{\delta}\omega(s)$$

and also $|T_2W(s)| \le C_1\omega(s)$ and $|dT_2W(s)/ds| \le C_2\omega(s)$. It can easily be seen that T_2 is continuous. Therefore, $T_2B(\delta) \subset \hat{B} = \{\phi \in \mathcal{X}\,;\ |\phi(s)| + |\phi'(s)| \le (C_1 + C_2)\omega(s)$ for every $s \le \log\varepsilon\}$. Taking ε small enough, we get that $\hat{B}$ is a compact subset of $B(\delta)$, and so $(I - T_1)^{-1}T_2$ is continuous operator from $B(\delta)$ to itself, and by Schauder's fixed point theorem it has a fixed point $W \in B(\delta)$. Showing that $|W(s)| \to 0$ as $s \to -\infty$, we can finish the proof. □

The following result is already known. For the proof we refer to [21].

PROPOSITION 2.2. *Assume that* $3 \le n \le 9$. *Then there is a unique solution* ϕ *to*

$$\begin{cases} \phi_{rr} + \dfrac{n-1}{r}\phi_r + e^{\phi} = 0\,, & r \in (0, \infty)\,, \\ \phi_r(0) = 0\,, \\ \phi(0) = 0\,. \end{cases}$$

The solution satisfies $\phi_r < 0$ *in* $(0, \infty)$ *and for* $\phi^*(r) = \log(2(n-2)r^{-2})$, *there are infinitely many roots of the equation* $\phi - \phi^* = 0$.

We will also need an estimate for the gradient of the solution u of (1.1). This lemma can be found in [16].

LEMMA 2.3. *Assume that* f *satisfies* (1.2), *and that the solution* u *of* (1.1) *blows up at* $t = T$. *Then, for* $u_M(t) = \max_{x\in\Omega} u(x, t)$ *and* t_0 *close to* T, *we have that*

$$\frac{1}{2}|\nabla u(x, t)|^2 \le \int_{u(x,t)}^{u_M(t_0)} f(u)du$$

for every $t < t_0$ *and* $x \in \Omega$.

Now that we have the above preliminary results, we are ready to prove Theorem 1.1, which gives the blow-up rate of the solution u. The proof is a modified version of that in [7]. Notice that by integrating the inequality $u_t(0,t) \le e^{u(0,t)}$ from t to T, we have

$$\log(T-t) + u(0,t) \ge 0\,. \tag{2.3}$$

PROOF OF THEOREM 1.1. Let v^* be as in Lemma 2.1, extended to its maximum existence interval $(0, \varepsilon^*]$, and define $R^* = \min\{\varepsilon^*, R\}$. By the zero number diminishing property (see [8]), it can be verified that both $\mathcal{Z}_{[0,R]}(u_t(\cdot,t))$ and $\mathcal{Z}_{[0,R^*]}(u(\cdot,t) - v^*(\cdot))$ are nonincreasing in $t \in [0,T)$ so that they are constant for all $t \in [T_1, T)$ and for some $T_1 \in [0,T)$. Here we used the usual notation

$$\mathcal{Z}_I(g) = \#\{r \in I\,;\; g(r) = 0\} \tag{2.4}$$

defined for an arbitrary interval I and a function $g \in C(I)$. Let now $\mathcal{Z}_{[0,R^*]}(u(\cdot,t) - v^*) = N^*$, for $t \in [T_1, T)$.

We will set

$$M(t) = u(0,t) \quad \text{and} \quad \delta = \liminf_{t\to T} \frac{u_t(0,t)}{e^{u(0,t)}} = \liminf_{t\to T} \frac{M'(t)}{e^{M(t)}}\,,$$

and claim that $\delta > 0$.

By contradiction, assume that $\delta = 0$. Then there exists a sequence $t_i \to T$ as $i \to \infty$ such that $\lim_{i\to\infty} M'(t_i)e^{-M(t_i)} = 0$. Moreover, we may assume that

$$\frac{f(u(0,t))}{e^{u(0,t)}} \in (1/2, 2)$$

for every $t \ge t_0$. Define

$$R_i = e^{-u(0,t_i)/2} \quad \text{and} \quad w_i(\rho,\tau) = u(R_i\rho, R_i^2\tau + t_i) + 2\log R_i\,.$$

Then w_i satisfies

$$w_{i\tau} - \Delta w_i = R_i^2 f(w_i - 2\log R_i) \quad \text{in} \quad B(R/R_i) \times (-t_i R_i^{-2}, 1/4)\,.$$

Moreover, we have that

$$w_{i\tau}(0,0) = R_i^2 u_t(0,t_i) = \frac{M'(t_i)}{e^{M(t_i)}} \to 0 \quad \text{as} \quad i \to \infty\,.$$

By Lemma 2.3, we get that $u_r(r,t)^2 \le 2f(u(0,t_i))(u(0,t_i) - u(r,t))$ for every $t < t_i$ and $r \in [0,R)$, assuming that $u(0,t_0)$ is large enough. Therefore, by integrating the inequality

$$|u_r(r,t)|(u(0,t_i) - u(r,t))^{-1/2} \le \sqrt{2f(u(0,t_i))}$$

from 0 to r, we have

$$u(0,t_i) - u(r,t) \le 4f(u(0,t_i))r^2 \le 8e^{u(0,t_i)}r^2 \tag{2.5}$$

for every $t \le t_i$ and $r \in [0,R]$. With the above estimate we can write

$$w_i(\rho,\tau) = u(R_i\rho, R_i^2\tau + t_i) - u(0,t_i) \ge -8e^{u(0,t_i)}R_i^2\rho^2 = -8\rho^2 \ge -8C\,,$$

whenever $\rho \leq \sqrt{C}$. Since clearly $w_i(\rho, \tau) \leq 0$ for every $\tau < 0$ and $r \in [0, R/R_i]$, we know that the family $\{w_i\}_i$ is uniformly bounded in $L^\infty([0, \sqrt{C}] \times (-t_i R_i^{-2}, 0))$.

Because of the assumption that u attains the maximum at the origin, we know that $u_t(0, t) \leq f(u(0, t)) \leq 2e^{u(0,t)}$ for every $t > t_0$. Integrating this inequality with respect to t from t_i to $t_i + \tau R_i^2$ (where $\tau > 0$), we obtain

$$-(e^{-u(0,t_i+\tau R_i^2)} - e^{-u(0,t_i)}) \leq 2\tau R_i^2 = 2\tau e^{-u(0,t_i)},$$

which then yields

$$u(0, t_i + \tau R_i^2) \leq u(0, t_i) + \log \frac{1}{1 - 2\tau} \leq u(0, t_i) + \log 2, \tag{2.6}$$

for every $\tau \in [0, 1/4]$. Hence we have that $w_i(0, \tau) = u(0, t_i + \tau R_i^2) - u(0, t_i) \leq \log 2$ for every $\tau \in [0, 1/4]$.

By using the inequalities (2.5) and (2.6), we get that for $\tau \in [0, 1/4]$ and $\rho \in [0, \sqrt{C}]$:

$$\begin{aligned} w_i(\rho, \tau) &= u(R_i\rho, R_i^2\tau + t_i) - u(0, t_i) \\ &= u(R_i\rho, R_i^2\tau + t_i) - u(0, R_i^2\tau + t_i) + u(0, R_i^2\tau + t_i) - u(0, t_i) \\ &\geq u(R_i\rho, R_i^2\tau + t_i) - u(0, R_i^2\tau + t_i) \geq -8e^{u(0,R_i^2\tau+t_i)} R_i^2\rho^2 \\ &= -8e^{u(0,R_i^2\tau+t_i)-u(0,t_i)}\rho^2 \geq -16\rho^2 \geq -16C\,. \end{aligned}$$

Therefore we now know that $w_i(\rho, \tau) \leq w_i(0, \tau) \leq \log 2$ and $w_i(\rho, \tau) \geq -16C$ for every $\rho \in [0, \sqrt{C}]$ and $\tau \in [0, 1/4]$. Altogether we have that $\{w_i\}_i$ is uniformly bounded in $L^\infty([0, \sqrt{C}] \times [-t_i R_i^{-2}, 1/4])$.

It follows from the parabolic estimates that $\{w_i\}_i$ is a uniformly bounded family in $C^{2,1}$. Therefore, along a subsequence, it converges uniformly in any compact subset of $B(\sqrt{C}) \times (-\infty, 1/4)$ to a radially symmetric limit w. Because

$$\lim_{i\to\infty} R_i^2 f(w_i - 2\log R_i) = \lim_{i\to\infty} e^{-w_i+2\log R_i} f(w_i - 2\log R_i)e^{w_i} = e^w,$$

we have that w satisfies

$$\begin{cases} w_\tau - \Delta w = e^w & \text{in } \ B(\sqrt{C}) \times (-\infty, 1/4)\,, \\ w(0,0) = 0\,, \quad w_\tau(0,0) = 0\,. \end{cases}$$

Exactly the same arguments as in [7] show that actually $w_\tau \equiv 0$ and so $w(\cdot, \tau) = \phi(\cdot)$, where ϕ is the unique solution to the problem in Proposition 2.2. Taking now ρ^* large, we can assume that $\mathcal{Z}_{[0,\rho^*]}(\phi - \phi^*) = N^* + 1$, where $\phi^*(r) = \log[2(n-2)r^{-2}]$. Taking then C such that $\sqrt{C} \geq \rho^*$, we can show, in the same manner as in [7], that $\mathcal{Z}_{[0,R^*]}(u(\cdot, t_i) - v^*(\cdot)) \geq N^* + 1$, which is a contradiction and therefore $\delta > 0$.

Now we know that there exists $T_2 \in [T_1, T)$ such that

$$\frac{M'(t)}{e^{M(T)}} \geq \frac{\delta}{2}$$

for every $t \in [T_2, T)$. By integrating this inequality over the interval (t, T), we obtain the claim. $\square$

Combining the techniques of the proofs of Theorem 1.1 above and Theorem 1 in [7], it is straightforward to prove the following theorem.

THEOREM 2.4. *Assume that* (1.2) *holds and* $n \in [3, 9]$. *If* u *is a global classical solution of* (1.1), *then* u *is uniformly bounded.*

3. Convergence to a backward selfsimilar solution. The aim of this section is to prove Theorem 1.2. Most of the work is needed to show the following:

THEOREM 3.1. *Let* $f(u) = e^u$ *and assume that the initial function* u_0 *is radially nonincreasing. If* u *is a solution of* (1.1) *that blows up at* $t = T$, *and*

$$\lim_{t\to T}[\log(T-t) + u(y\sqrt{T-t}, t)] = 0 \tag{3.1}$$

uniformly for y *in compact sets, then*

$$u(x, T) = -2\log|x| + \log|\log|x|| + \log 8 \quad \text{as} \quad x \to 0\,. \tag{3.2}$$

It was shown in [2] that (3.2) holds for solutions of

$$\begin{cases} u_t = \Delta u + e^u\,, & x \in \boldsymbol{R}^n\,, \quad t > 0\,,\\ u(x,0) = u_0(x) \geq 0\,, & x \in \boldsymbol{R}^n\,, \end{cases} \tag{3.3}$$

provided u is radially symmetric, $u_r \leq 0$, $u_t \geq 0$. In [20] it was proved that either (3.2) or

$$u(x, T) = -m\log|x| + C_m \quad \text{as} \quad x \to 0 \tag{3.4}$$

holds for some integer $m \geq 4$ and $C_m \in \boldsymbol{R}$ for solutions of (3.3) under the assumptions that $n = 1$, u_0 is continuous, bounded, it has a single maximum and $x = 0$ is the blow-up point. The existence of solutions of (1.1) which blow up at $x = 0 \in \Omega$, $t = T$, and have the profile (3.2) was established in [4] when Ω is convex. The existence of initial data such that (3.4) occurs with $m = 4$ was shown in [20] for Problem (3.3) with $n = 1$, for any integer $m \geq 4$ see [5]. In our case the profile (3.4) does not occur since we assume that u is radially decreasing. This follows from [3], where it is shown that if $u_r \leq 0$, then

$$u(x, t) \leq -2\log|x| + \log|\log|x|| + C \tag{3.5}$$

for some constant C and for any $t \in (0, T)$ and $x \in B(R)$.

As in [26], the first thing we will have to do, is to extend the solution u to the whole space $\boldsymbol{R}^n$ in order to be able to use semigroup methods in appropriate weighted L^2 spaces. We will also derive some useful estimates for the new nonlinearity and discuss the functional analytic framework.

Throughout this section we will adopt the assumptions of Theorem 3.1. Take $\zeta \in C^\infty(\boldsymbol{R}^n)$ such that $\zeta(x) = 1$ for $|x| \leq R_1$, $\zeta(x) \in (0, 1)$ for $|x| \in (R_1, R_2)$ and $\zeta(x) = 0$ for $|x| \geq R_2$, where $0 < R_1, R_2 < R$. Then define

$$\tilde{u}(x, t) = \zeta(x)u(x, t) - (\log(T-t) + 1)(1 - \zeta(x)) \tag{3.6}$$

for $x \in \boldsymbol{R}^n$ and $t \in [0, T)$. This gives us that the new extended function satisfies

$$\tilde{u}_t = \Delta\tilde{u} + f\,, \quad x \in \boldsymbol{R}^n\,,\ t \in (0, T)\,,$$

where

$$f = f(x,t) = (T-t)^{-1}(1-\zeta) - (1+\log(T-t)+u)\Delta\zeta - 2\nabla\zeta\cdot\nabla u + \zeta e^u\,.$$

Notice that Theorem 1.1 and Lemma 2.3, now applied to $f(u) = e^u$, imply that

$$|(T-t)f(x,t)| \le C \tag{3.7}$$

for every $(x,t) \in \boldsymbol{R}^n \times [0,T)$ and for some constant depending only on the choice of ζ and the constant appearing in Theorem 1.1. As above, we henceforth denote by C a generic constant possibly changing from line to line and depending only on some fixed functions or parameters like u_0 or the dimension of the space.

Following the usual method, we use the similarity variables to define the rescaled function

$$\tilde{w}(y,s) = \log(T-t) + \tilde{u}(x,t)\,,$$

where $y = (T-t)^{-1/2}x$ and $s = -\log(T-t)$. Then $\tilde{w}$ satisfies

$$\tilde{w}_t = \Delta\tilde{w} - \frac{1}{2}y\cdot\nabla\tilde{w} + (T-t)f - 1 = A\tilde{w} + h\,, \quad y \in \boldsymbol{R}^n,\ s > -\log T\,, \tag{3.8}$$

where $A = \Delta - y/2\cdot\nabla + I$ and $h(y,s) = (T-t)f(x,t) - 1 - \tilde{w}(y,s)$. Using Lemma 2.3 and Theorem 1.1, it is easy to verify that $|\nabla\tilde{w}| \le C$ and hence (2.3) implies that

$$|\tilde{w}| \le C(1+|y|)\,. \tag{3.9}$$

In what follows, we will give some estimates for the function h. Assume first that $|y| \le e^{s/2}R_1$. Then $\tilde{w} = \log(T-t) + u$ and $h = e^{\tilde{w}} - 1 - \tilde{w}$. Therefore

$$|h| \le e^{|\tilde{w}|}|\tilde{w}|^2 \le e^K|\tilde{w}|^2\,,$$

where K is the constant appearing in Theorem 1.1. We can also argue that either $-1 \le \tilde{w} \le K$, which implies that $|h| \le e^K K|\tilde{w}|$, or $\tilde{w} \le -1$, in which case $|h| = |e^{\tilde{w}} - 1 - \tilde{w}| \le 2 + |\tilde{w}| \le 3|\tilde{w}|$.

Assume then that $|y| \in (e^{s/2}R_1, e^{s/2}R_2)$. Because $u(x,t) \le C$ for every $|x| \in (R_1,R_2)$ and $t \in [0,T)$, there exists $s_0 > 0$ such that

$$\tilde{w} = -1 + \zeta(u + \log(T-t) + 1) \le -1$$

for every $s \ge s_0$. Therefore we can estimate

$$|h| \le |(T-t)f| + 1 + |\tilde{w}| \le C + |\tilde{w}| \le (C+1)|\tilde{w}| \le (C+1)|\tilde{w}|^2$$

in $\boldsymbol{R}^n \times [s_0,\infty)$, where we used the estimate (3.7).

Since, for $|y| > e^{s/2}R_2$, it holds that $h = -\tilde{w}$ and $\tilde{w} = -1$, we can collect the above estimates together to obtain that

$$|h| \le C_1|\tilde{w}| \quad \text{and} \quad |h| \le C_2|\tilde{w}|^2 \quad \text{in} \quad \boldsymbol{R}^n \times [s_0,\infty) \tag{3.10}$$

for some constants C_1 and C_2. In a similar way we can also show that

$$\left|h - \frac{1}{2}\tilde{w}^2\right| \le C_3|\tilde{w}|^3 \quad \text{in} \quad \boldsymbol{R}^n \times [s_0,\infty)\,. \tag{3.11}$$

We will next discuss the operator A. A convenient space to work in is the weighted space

$$L^2_\rho(\boldsymbol{R}^n) = \left\{ f \in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)\,;\; \int_{\boldsymbol{R}^n} |f(y)|^2 e^{-|y|^2/4} dy < \infty \right\}.$$

It is well-known that A is a self-adjoint operator in $L^2_\rho(\boldsymbol{R}^n)$ with domain $H^2_\rho(\boldsymbol{R}^n)$ and it has a complete family of orthogonal eigenfunctions $\{H_\alpha\}_{\alpha\in\boldsymbol{N}^n}$ with the corresponding eigenvalues $\lambda_\alpha = 1 - |\alpha|/2$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The eigenfunctions can be written as $H_\alpha(y) = \prod_{i=1}^n H_{\alpha_i}(y_i)$, where H_m is the standard Hermite polynomial of order $m \in \boldsymbol{N}$. We will denote by $\{S(s)\}_s$ the semigroup generated by A.

Since u, and so also $\tilde{w}$, is assumed to be radially symmetric, we only need to consider radially symmetric eigenfunctions. The first ones are $h_0(y) = 1 \in \mathrm{span}\{H_0\}$ corresponding to the eigenvalue $\lambda_0 = 1$ and $h_2(y) = |y|^2 - 2n \in \mathrm{span}\{H_\alpha\,;\; |\alpha| = 2,\ \alpha_i \text{ even}\}$ corresponding to the eigenvalue $\lambda_2 = 0$. Therefore we can decompose

$$\tilde{w} = \pi_+\tilde{w} + \pi_c\tilde{w} + \pi_-\tilde{w} = a(s) + b(s)(|y|^2 - 2n) + \theta(y,s)\,, \tag{3.12}$$

where $\pi_+\tilde{w}$ and $\pi_c\tilde{w}$ are the projections to the eigenspaces spanned by h_0 and h_2, and $\pi_-\tilde{w} = \tilde{w} - \pi_+\tilde{w} - \pi_c\tilde{w} \in \overline{\mathrm{span}\{H_\alpha\,;\; |\alpha| > 2\}}$.

A well-known fact is the regularizing property of the semigroup (see [30]), namely, for every $p, q \in (1,\infty)$ there exists $R = R(p,q)$ and $C = C(R)$ such that

$$\|S(R)\phi\|_{L^p_\rho} \le C\|\phi\|_{L^q_\rho} \quad \text{for every} \quad \phi \in L^p_\rho(\boldsymbol{R}^n)\,, \tag{3.13}$$

where the definition of $L^p_\rho(\boldsymbol{R}^n)$ is analogous to that of $L^2_\rho(\boldsymbol{R}^n)$. Using the first inequality in (3.10) and applying the above inequality to $\tilde{w}$, we obtain

$$\|\tilde{w}(\cdot,s)\|_{L^p_\rho} \le e^{C_1R}\|S(R)\tilde{w}(\cdot,s-R)\|_{L^p_\rho} \le e^{C_1R}C\|\tilde{w}(\cdot,s-R)\|_{L^q_\rho}\,. \tag{3.14}$$

Also, the reversed inequality is known in L^2_ρ. Assuming that there exists a constant $\beta > 0$ such that $a(s)^2 + \|\theta(\cdot,s)\|^2 \le \beta\, b(s)^2$, we can use Lemma 3.1 in [19] to obtain that

$$\|\tilde{w}(\cdot,s)\| \le C(R,\beta)\|\tilde{w}(\cdot,s+R)\|\,, \tag{3.15}$$

where we used the notation $\|\cdot\| = \|\cdot\|_{L^2_\rho}$.

The assumption (3.1) implies that

$$\lim_{s\to\infty} \tilde{w}(y,s) = 0 \tag{3.16}$$

uniformly for y in compact sets. In the following Lemma and two Propositions, we will assume that the convergence (3.16) is not exponential in rate, that is, we assume that for every $C, \varepsilon > 0$ we have

$$\|\tilde{w}(\cdot,s)\| > Ce^{-\varepsilon s} \tag{3.17}$$

for some $s > -\log T$.

The following lemma is proved in the case of $f(u) = u^p$ in [15] and it states that the unstable and stable part of the solution $\tilde{w}$ are dominated by the center part of it. The proof in our case is almost the same as in [15] and therefore we do not repeat it here. The only difference is that [15] assumes the boundedness of $\tilde{w}$, and we use the inequality $|h| \leq C_1|\tilde{w}|$ whenever the boundedness is needed.

LEMMA 3.2. *Let $\tilde{w}$ satisfy* (3.16) and (3.17). *Then for every $\varepsilon > 0$ there exists s_0 such that*

$$\|\pi_-\tilde{w}\|_{L^2_\rho} + \|\pi_+\tilde{w}\|_{L^2_\rho} \leq \varepsilon\|\pi_c\tilde{w}\|_{L^2_\rho}$$

for any $s \geq s_0$.

In what follows, we will derive differential equations for the functions a and b appearing in the expansion (3.12). Inserting (3.12) in Equation (3.8), and projecting to the unstable subspace, we have

$$\|1\|^2_{L^2_\rho}a'(s) = \|1\|^2_{L^2_\rho}a(s) + P_+h\,,$$

where we use the notation $(P_+h)h_0 = \pi_+h$. We can write $h = (\pi_+\tilde{w} + \pi_c\tilde{w})^2/2 + g$, where

$$g = (\pi_+\tilde{w} + \pi_c\tilde{w})\pi_-\tilde{w} + \frac{1}{2}(\pi_-\tilde{w})^2 + h - \frac{1}{2}\tilde{w}^2\,.$$

Using Lemma 3.2 and inequalities (3.11), (3.14) and (3.15), we can estimate

$$\begin{aligned}|P_+g| &\leq (\varepsilon^2 + \varepsilon)\|\pi_c\tilde{w}\|^2 + \frac{1}{2}\varepsilon^2\|\pi_c\tilde{w}\|^2 + C_3\|\tilde{w}^3\| \leq 2\varepsilon\,\|\pi_c\tilde{w}\|^2 + C\|\tilde{w}(\cdot, s-R)\|^3\\ &\leq 2\varepsilon\,\|\pi_c\tilde{w}\|^2 + C\|\tilde{w}(\cdot, s)\|^3 \leq 2\varepsilon\,\|\pi_c\tilde{w}\|^2 + C\|\pi_c\tilde{w}\|^3 = 2\varepsilon\, b^2 + Cb^3\end{aligned}$$

for s large enough. Therefore, a satisfies

$$a'(s) = a(s) + \frac{\|1\|^{-2}_{L^2_\rho}}{2}P_+(\pi_+w + \pi_cw)^2 + P_+g\,.$$

Since we know that $|\tilde{w}(y,s)| \leq C(1+|y|)$ and $\tilde{w}(y,s) \to 0$ as $s \to \infty$ pointwise for every y, it follows from the Lebesgue dominated convergence theorem that $\tilde{w}(\cdot, s) \to 0$ as $s \to \infty$ also in $L^2_\rho(\boldsymbol{R}^n)$. Hence $a(s) \to 0$ and $b(s) \to 0$ as $s \to \infty$, and we can write for $s \geq s_0$

$$a'(s) = a(s) + \frac{1}{2}(a(s)^2 + 8nb(s)^2) + \varepsilon O(b(s)^2)\,,$$

where the second term on the right is easily obtained from $P_+(\pi_+w + \pi_cw)^2$ by simple integration. In the same way, we can prove that b satisfies

$$b'(s) = a(s)b(s) + 4b(s)^2 + \varepsilon O(b(s)^2)$$

for $s \geq s_0$.

Using now Lemma 3.2 and the above differential equations for the functions a and b, we can repeat the arguments used in Theorem 2.6 in [2] and so we obtain the following result.

PROPOSITION 3.3. *Let $\tilde{w}$ satisfy* (3.16) and (3.17). *Then*

$$\tilde{w}(y,s) = -\frac{1}{4s}(|y|^2 - 2n) + o\left(\frac{1}{s}\right) \quad in \quad L^2_\rho(\boldsymbol{R}^n)\,.$$

By the regularizing effect of the semigroup $\{S(s)\}_s$, we can conclude that the above convergence holds also uniformly on compact sets. However, we need to consider the convergence in larger sets, namely, when $|y| \le \sqrt{s}\, R$. This is done in the proposition below, which follows [26, 30].

PROPOSITION 3.4. *Let $\tilde{w}$ satisfy* (3.16) and (3.17). *Then it holds that*

$$\lim_{t\to T}[\log(T-t)+\tilde{u}(\xi(T-t)^{1/2}|\log(T-t)|^{1/2},t)] = -\log\left(1+\frac{|\xi|^2}{4}\right) \tag{3.18}$$

uniformly for $|\xi| \le R$.

PROOF. To get started, define

$$G(\xi) = -\log\left(1+\frac{|\xi|^2}{4}\right)$$

and

$$\bar{\phi}(y,s) = G\left(\frac{y}{\sqrt{s}}\right) + \frac{n}{2s}\,.$$

Then $G(\xi) = -|\xi|^2/4 + R(\xi)$, where $|R(\xi)| \le C|\xi|^4$. Therefore we have that

$$\begin{aligned}
&\|\tilde{w}(\cdot,s) - \bar{\phi}(\cdot,s)\|_{L^2_\rho} \\
&\quad\le o\left(\frac{1}{s}\right) + \left\{\int_{\boldsymbol{R}^n}\left|-\frac{1}{4s}(|y|^2-2n)+\frac{|y|^2}{4s}-R\left(\frac{y}{\sqrt{s}}\right)-\frac{n}{2s}\right|^2 e^{-|y|^2/4}dy\right\}^{1/2} \\
&\quad\le o\left(\frac{1}{s}\right) + C\left\{\int_{\boldsymbol{R}^n}\frac{|y|^8}{s^4}e^{-|y|^2/4}dy\right\}^{1/2} = o\left(\frac{1}{s}\right).
\end{aligned}$$

Defining $W = \tilde{w} - \bar{\phi}$ and using the equations

$$\bar{\phi}_s(y,s) = -\frac{\xi}{2s}\cdot\nabla G(\xi) - \frac{n}{2s^2}$$

and

$$-\frac{\xi}{2}\cdot\nabla G(\xi) = 1 - e^G\,,$$

we get that W satisfies

$$W_s = AW + g + \frac{\xi}{2s}\cdot\nabla G + \frac{n}{2s^2} + L\,, \tag{3.19}$$

where

$$g = h + 1 + \bar{\phi} - e^{\bar{\phi}} \quad\text{and}\quad L = \frac{\Delta G}{s} + e^{\bar{\phi}} - e^G\,,$$

and h is as in (3.8). Multiplying the above equation (3.19) by $\operatorname{sgn}(W)$, defining $Z = |W|$ and using Kato's inequality, we get that

$$\begin{aligned}
Z_s &\le AZ + \operatorname{sgn}(W)g + \operatorname{sgn}(W)\left(\frac{\xi}{2s}\cdot\nabla G + \frac{n}{2s^2}\right) + \operatorname{sgn}(W)L \\
&\le AZ + \operatorname{sgn}(W)g + C\left(\frac{|\xi|^2}{s} + \frac{1}{s^2}\right) + \operatorname{sgn}(W)L\,.
\end{aligned} \tag{3.20}$$

Next, we want to get estimates for the terms in the right hand side of (3.20). Because $|\Delta G(\xi) - \Delta G(0)| \le C|\xi|^2$, we get that

$$|L(y,s)| = \left|\frac{\Delta G(\xi) - \Delta G(0)}{s} + e^{G+n/2s} - e^G - \frac{n}{2s}\right|$$
$$\le C\frac{|\xi|^2}{s} + \frac{1}{1+|\xi|^2/4}\left(\frac{n}{2s} + O(s^{-2}) - \frac{n}{2s}(1+|\xi|^2/4)\right) \le C\frac{|\xi|^2}{s} + O\left(\frac{1}{s^2}\right).$$

To estimate the function g, consider first the subset $|y| \le e^{s/2}R_1$. Then $(T-t)f = e^{\tilde{w}}$ and we have by the mean value theorem that for some $\Theta \in (0, W)$

$$\mathrm{sgn}(W)g = \mathrm{sgn}(W)(e^{\bar{\phi}+W} - W - e^{\bar{\phi}}) = \mathrm{sgn}(W)\left(e^{\bar{\phi}} + (e^{\bar{\phi}} - 1)W + \frac{1}{2}e^{\bar{\phi}+\Theta}W^2 - e^{\bar{\phi}}\right)$$
$$= \left(-\frac{|\xi|^2/4}{1+|\xi|^2/4} + \frac{e^{n/2s}-1}{1+|\xi|^2/4}\right)Z + \frac{1}{2}e^{\bar{\phi}+\Theta}Z^2 \le \frac{n}{2s}Z + CZ^2\,,$$

since clearly $e^{\bar{\phi}+\Theta} \le e^K$. Notice that we also have

$$|g| = |e^{\bar{\phi}} + (e^{\bar{\phi}+\Theta} - 1)W - e^{\bar{\phi}}| \le CZ\,.$$

Assume then that $|y| \in (e^{s/2}R_1, e^{s/2}R_2)$. Because $(T-t)f(x,t)$ and $e^{\bar{\phi}}$ are uniformly bounded, we have that

$$\mathrm{sgn}(W)g = \mathrm{sgn}(W)(T-t)f(x,t) - Z + \mathrm{sgn}(W)e^{\bar{\phi}} \le C \le C(Z^2+1)\,.$$

Clearly, we also have that $\mathrm{sgn}(W)g \le C(Z+1)$.

Finally, for $|y| \ge e^{s/2}R_2$, we have that $(T-t)f(x,t) = 1$ and $\tilde{w} = -1$. Therefore $W \ge -1 + \log(1 + e^s R_2^2/4s) - n/2s > 1$ for s large enough, and we get

$$\mathrm{sgn}(W)g \le C \le CZ \le CZ^2\,.$$

Collecting the above results, we know that Z satisfies the differential inequalities

(3.21) $$Z_s \le AZ + C\left(\frac{|y|^2+1}{s^2} + Z^2 + \frac{Z}{s} + \chi\right) \quad \text{in} \quad [s_0, \infty) \times \boldsymbol{R}^n$$

and

(3.22) $$Z_s \le AZ + C\left(\frac{|y|^2+1}{s^2} + Z + \chi\right) \quad \text{in} \quad [s_0, \infty) \times \boldsymbol{R}^n\,,$$

where $\chi = \chi(y,s) = 1$ if $|y| \in (e^{s/2}R_1, e^{s/2}R_2)$ and $\chi = 0$ otherwise, and s_0 is large enough.

The proof can now be finished by using the above inequalities and proceeding as in [30, Proposition 2.3]. □

In what follows, we shall handle the case where the convergence (3.16) is exponential. Therefore we shall assume that

(3.23) $$\|\tilde{w}(\cdot, s)\| = o(e^{-\varepsilon s})$$

for some $\varepsilon > 0$. The proof of the following proposition is the same as in [29].

PROPOSITION 3.5. *Assume that* (3.23) *holds. Then either there exists $m \geq 3$ and constants C_α, not all equal to zero, such that*

$$\tilde{w}(y,s) = -e^{(1-m/2)s} \sum_{|\alpha|=m} C_\alpha H_\alpha(y) + o(e^{(1-m/2)s}) \quad in \quad L^2_\rho(\boldsymbol{R}^n)\,,$$

or $\tilde{w}$ is the trivial solution $\tilde{w}(\cdot, s) = 0$.

Notice that the term $\sum_{|\alpha|=m} C_\alpha H_\alpha$ has to be radially symmetric, and so m is actually even. Since $H_\alpha(y) = \prod_{i=1}^n H_{\alpha_i}(y_i)$ and $H_{\alpha_i}(y_i) = \sum_{k=0}^{\alpha_i/2} c_{2k}(\alpha_i) y_i^{2k}$ for some constants $c_k(\alpha_i)$ and α_i even, we have that

$$|H_\alpha(y) - \bar{c}_\alpha y^\alpha| \leq C(1 + |y|^{m-2})\,, \tag{3.24}$$

where $\bar{c}_\alpha = \sum_{i=1}^n c_{\alpha_i}(\alpha_i)$. Moreover, it has to hold that $\sum_{|\alpha|=m} C_\alpha H_\alpha \to \infty$ as $|y| \to \infty$ and therefore $\sum_{|\alpha|=m} a_\alpha y^\alpha > 0$ for every $y \neq 0$, where $a_\alpha = C_\alpha \bar{c}_\alpha$.

Following [29], we shall next prove an analogue of Proposition 3.4 and extend the convergence to larger sets.

PROPOSITION 3.6. *Let $\tilde{w}$ and $m \geq 4$ be as in Proposition* 3.5. *Then*

$$\lim_{t\to T} [\log(T-t) + \tilde{u}(\xi(T-t)^{1/m}, t)] = -\log\left(1 + \sum_{|\alpha|=m} a_\alpha \xi^\alpha\right) \tag{3.25}$$

uniformly for $|\xi| \leq R$, where the constants $a_\alpha = C_\alpha \bar{c}_\alpha$ are as above.

PROOF. Define

$$G(\xi) = -\log\left(1 + \sum_{|\alpha|=m} a_\alpha \xi^\alpha\right), \quad \xi = e^{(1/m-1/2)s} y\,,$$

and

$$\bar{\phi}(y,s) = G(\xi) - e^{(1-m/2)s} \sum_{|\alpha|=m} C_\alpha [H_\alpha(y) - \bar{c}_\alpha y^\alpha] = G - L\,.$$

Then it is easily seen that

$$\|\tilde{w} - \bar{\phi}\|_{L^2_\rho} = o(e^{(1-m/2)s})\,.$$

Since

$$\frac{\xi \cdot \nabla G}{m} = e^G - 1\,,$$

we get, by defining $W = \tilde{w} - \bar{\phi}$, that

$$\begin{aligned}
W_s &= \Delta W - \frac{y}{2}\nabla W + W + h - \bar{\phi}_s + \Delta\bar{\phi} - \frac{y}{2}\nabla\bar{\phi} + \bar{\phi} \\
&= AW + h - \left\{\left(\frac{1}{m} - \frac{1}{2}\right)\xi\nabla_\xi G - \left(1 - \frac{m}{2}\right)L\right\} \\
&\quad + \left\{e^{(2/m-1)s}\Delta_\xi G - \Delta L\right\} - \left\{\frac{\xi}{2}\nabla_\xi G - \frac{y}{2}\nabla L\right\} + G - L \\
&= AW + (T-t)f - \tilde{w} - e^G + G + e^{(2/m-1)s}\Delta_\xi G - \Delta L + \frac{y}{2}\nabla L - \frac{m}{2}L\,.
\end{aligned}$$

Using now the facts that $\Delta H_\alpha - (y/2)\nabla H_\alpha = -(|\alpha|/2)H_\alpha$ and $(y/2)\nabla y^\alpha = (|\alpha|/2)y^\alpha$, we get that

$$\Delta L - \frac{y}{2}\nabla L = -\frac{m}{2}L - e^{(1-m/2)s}\sum_{|\alpha|=m} a_\alpha \Delta y^\alpha .$$

Writing then $Z = |W|$ and

$$\Delta G = -\frac{\sum_{|\alpha|=m} a_\alpha \Delta_\xi \xi^\alpha}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha} + \left(\frac{\sum_{|\alpha|=m} a_\alpha \alpha_i \xi^{\alpha-1_i}}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha}\right)^2 = (\Delta G)_1 + (\Delta G)_2 ,$$

where we use the notation $\alpha - 1_i = (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n)$, we have that

$$Z_s \le AZ + \operatorname{sgn}(W)K_1 + |K_2| + |e^{(2/m-1)s}(\Delta G)_2| ,$$

where

$$K_1 = (T-t)f - \tilde{w} - e^G + G$$

and

$$K_2 = e^{(2/m-1)s}(\Delta G)_1 + e^{(1-m/2)s}\sum_{|\alpha|=m} a_\alpha \Delta y^\alpha .$$

Clearly, it holds that

$$\begin{aligned} e^{(2/m-1)s}(\Delta G)_2 &= e^{(2/m-1)s}\left(\frac{\sum_{|\alpha|=m} a_\alpha \alpha_i e^{(1/m-1/2)(m-1)s} y^{\alpha-1_i}}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha}\right)^2 \\ &\le e^{2(1-m/2)s}|y|^{2m-2} . \end{aligned}$$

Estimating then K_2 using the equality $e^{(2/m-1)s}\Delta_\xi \xi^\alpha = e^{(1-m/2)s}\Delta y^\alpha$, we obtain

$$\begin{aligned} |K_2| &= e^{(1-m/2)s}\sum_{i=1}^{n}\left|\frac{\left(\sum_{|\alpha|=m} a_\alpha \xi^\alpha\right)\left(\sum_{|\alpha|=m} a_\alpha \Delta y^\alpha\right)}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha}\right| \\ &= e^{2(1-m/2)s}\sum_{i=1}^{n}\left|\frac{\left(\sum_{|\alpha|=m} a_\alpha y^\alpha\right)\left(\sum_{|\alpha|=m} a_\alpha \alpha_i(\alpha_i-1) y^{\alpha-2_i}\right)}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha}\right| \\ &\le Ce^{2(1-m/2)s}(1+|y|^{2m-2}) . \end{aligned}$$

To give some estimates for K_1, define $\Omega_1(s) = \{y\,;\,|y|^{m-2}e^{(1-m/2)s} \le R_1\}$ and $\Omega_2(s) = \{y\,;\,|y|^m e^{(1-m/2)s} \le \tilde{R}\}$, where $\tilde{R}$ is large enough such that

$$\frac{e^{C_L(1+R_1)}}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha} < 1$$

for every $|\xi|^m = |y|^m e^{(1-m/2)s} > \tilde{R}$, and $C_L = C\sum_{|\alpha|=m} C_\alpha$ with C as in (3.24). Then $\Omega_2(s) \subset \Omega_1(s)$ for s large enough, and we have that

$$|L| \le C_L e^{(1-m/2)s}(1+|y|^{m-2}) \le C_L(1+R_1) \quad \text{for} \quad y \in \Omega_1(s) ,$$

and

$$|L| \le C_L e^{(2/m-1)s}(1+[e^{(1/m-1/2)s}|y|]^{m-2}) \le C_L e^{(2/m-1)s}(1+\tilde{R}^{(m-2)/m}) \le \frac{C}{s}$$

for $y \in \Omega_2(s)$, and

$$e^{\bar{\phi}} = \frac{e^{-L}}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha} < 1 \quad \text{for} \quad y \in \Omega_1(s) \setminus \Omega_2(s) .$$

Consider $y \in \Omega_1(s)$. In this domain, we have that

$$K_1 = e^{\tilde{w}} - \tilde{w} - e^G + G = e^{W+\bar{\phi}} - W - \bar{\phi} - e^G + G = (e^{\bar{\phi}}-1)W + \frac{1}{2}e^{\bar{\phi}+\Theta}W^2 + e^{G-L} + L - e^G$$

for some $\Theta \in (0, W)$. Now

$$\operatorname{sgn}(W)(e^{\bar{\phi}}-1)W = -\frac{\sum_{|\alpha|=m} a_\alpha \xi^\alpha}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha} Z + \frac{e^{-L}-1}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha} Z < \frac{C}{s} Z$$

for $y \in \Omega_2(s)$ and

$$\operatorname{sgn}(W)(e^{\bar{\phi}}-1)W \leq 0 \leq \frac{C}{s} Z$$

for $y \in \Omega_1(s) \setminus \Omega_2(s)$.

For $\Theta \in (0, L)$ and $y \in \Omega_1(s)$, we also have the estimate $e^{G-\Theta} \leq e^{G+|L|} \leq C$, and so

$$\begin{aligned}
|e^{G-L} + L - e^G| &= \left|(1-e^G)L + \frac{1}{2}e^{G-\Theta}L^2\right| \\
&\leq \frac{\sum_{|\alpha|=m} a_\alpha \xi^\alpha}{1+\sum_{|\alpha|=m} a_\alpha \xi^\alpha} C_L e^{(1-m/2)s}(1+|y|^{m-2}) + Ce^{2(1-m/2)s}(1+|y|^{m-2})^2 \\
&\leq Ce^{2(1-m/2)s}(1+|y|^{2m-2}) .
\end{aligned}$$

Hence we have

$$\operatorname{sgn}(W)K_1 \leq \frac{C}{s}Z + CZ^2 + Ce^{2(1-m/2)s}(1+|y|^{2m-2}) \qquad \text{for} \quad y \in \Omega_1(s) .$$

Consider then $y \in \{|y| \leq e^{s/2}R_2\} \setminus \Omega_1(s)$. It yields that $|\xi| \in (e^{s/m}R_1^{1/(m-2)}, e^{s/m}R_2)$ and we can easily estimate

$$|K_1| = \left|(T-t)f + s - \tilde{u}(x,t) - s - \log\left(e^{-s} + \sum_{|\alpha|=m} a_\alpha (e^{-s/m}\xi)^\alpha\right) - e^G\right| \leq C .$$

Finally, let $y \in \boldsymbol{R}^n \setminus \{|y| \geq e^{s/2}R_2\}$. In this domain we have that $(T-t)f = 1$ and $\tilde{w} = -1$ and therefore

$$|K_1| = |2 - e^G + G| \leq 1 + |e^G - 1 - G| \leq 1 + \frac{1}{2}e^{\Theta}G^2$$

for some $\Theta \in (0, G)$. Since $|L| \leq C_L e^{(1-m/2)s}(1+|y|^{m-2})$ and $G = \tilde{w} + L - W$, we get

$$G^2 \leq C(W^2 + (L-1)^2) \leq C(W^2 + L^2 + 1) \leq C(W^2 + 1 + e^{2(1-m/2)s}(1+|y|^{2m-2})) .$$

Altogether we have obtained that

$$Z_s \leq AZ + \frac{C}{s}Z + CZ^2 + Ce^{2(1-m/2)s}(1+|y|^{2m-2}) + C\chi ,$$

where $\chi = \chi(y,s) = 1$, for $|y| \geq e^{s/2}R_2$ and $\chi = 0$ otherwise. Now we can finish the proof exactly as in [30]. □

In what follows, our aim is to describe the asymptotic blow-up profile of u. In other words, we want to show that either (3.2) or (3.4) holds. To that end, define for $\tau \in [0, T]$

$$\psi_\tau(x,t) = \log(T-\tau) + \tilde{u}(\lambda(\tau)\xi + x\sqrt{T-\tau}, \tau + (T-\tau)t)\,,$$

where $\lambda(\tau) = \sqrt{T-\tau}|\log(T-\tau)|^{1/2}$ if the case as in Proposition 3.4 occurs and $\lambda(\tau) = (T-\tau)^{1/m}$ if the convergence of u is as in Proposition 3.6. Here ξ is fixed, $x \in \boldsymbol{R}^n$ and $t \in [0,1]$. Moreover, let

$$\phi_\tau(y,s) = \log(1-t) + \psi_\tau(x,t)\,,$$

where $y = (1-t)^{-1/2}x$ and $s = -\log(1-t)$. Then we have that

$$(\psi_\tau)_t = \Delta\psi_\tau + (T-\tau)f\,, \quad x \in \boldsymbol{R}^n\,,\ t \in (0,1)$$

and

$$(\phi_\tau)_s = \Delta\phi_\tau - \frac{y}{2}\cdot\nabla\phi_\tau + h_\tau\,, \quad y \in \boldsymbol{R}^n\,,\ s > 0\,,$$

where $h_\tau(y,s) = (T-\tau)(1-t)f(\lambda(\tau)\xi + x\sqrt{T-\tau}, \tau + (T-\tau)t) - 1$. By the above Propositions 3.4 and 3.6, we know that

$$\begin{aligned}\phi_\tau(y,0) &= \psi_\tau(x,0) = \log(T-\tau) + \tilde{u}\big(\lambda(\tau)\xi + x\sqrt{T-\tau}, \tau\big)\\ &= -\log\left(1 + \sum_{|\alpha|=m} a_\alpha\left(\xi + \frac{x\sqrt{T-\tau}}{\lambda(\tau)}\right)^\alpha\right) + \gamma_\tau(x)\,,\end{aligned}$$

where $m \geq 2$ and $\sum_{|\alpha|=m} a_\alpha\xi^\alpha = |\xi|^2/4$ if (3.18) holds, and otherwise $m \geq 3$ and the constants a_α are as in Proposition 3.6. Above $|\gamma_\tau(x)| \to 0$ uniformly for $|x| \leq C(T-\tau)^{-1/2}\lambda(\tau)$ as $\tau \to T$. Therefore

$$\lim_{\tau\to T}\psi_\tau(x,0) = -\log\left(1 + \sum_{|\alpha|=m} a_\alpha\xi^\alpha\right),$$

pointwise for every $x \in \boldsymbol{R}^n$. Because of Propositions 3.4 and 3.6, we also know that $|\psi_\tau(0,0)| \leq C$ as $\tau \to T$ and therefore Proposition 2.3 yields that $\psi_\tau(x,0) \leq C + |\nabla\psi_\tau||x| \leq C(1+|x|) \in L^2_\rho(\boldsymbol{R}^n)$. By the dominated convergence theorem we then obtain that

$$\left\|\psi_\tau(\cdot,0) + \log\left(1 + \sum_{|\alpha|=m} a_\alpha\xi^\alpha\right)\right\|_{L^2_\rho} \leq \gamma_\tau \to 0 \tag{3.26}$$

as $\tau \to T$.

Define also

$$\tilde{\phi}(s) = \log(1-t) - \log\left(1 - t + \sum_{|\alpha|=m} a_\alpha\xi^\alpha\right), \quad W_\tau = \phi_\tau - \tilde{\phi} \quad \text{and} \quad Z_\tau = |W_\tau|\,,$$

where $m \geq 2$. Then W_τ verifies the equation

$$(W_\tau)_s = \Delta W_\tau - \frac{y}{2}\cdot\nabla W_\tau + h_\tau + 1 - e^{\tilde{\phi}} = \tilde{A}W_\tau + e^{\tilde{\phi}}W_\tau + f_\tau\,, \tag{3.27}$$

where $\tilde{A} = \Delta - (y/2)\cdot\nabla$ and

$$f_\tau(y,s) = (T-\tau)(1-t)f(\lambda(\tau)\xi + x\sqrt{T-\tau}, \tau + (T-\tau)\,t) - e^{\tilde{\phi}}W_\tau - e^{\tilde{\phi}}\,,$$

and so Z_τ satisfies respectively the equation

$$(Z_\tau)_s \le \tilde{A} Z_\tau + e^{\tilde{\phi}} Z_\tau + |f_\tau| \tag{3.28}$$

and by (3.26) also

$$\|Z_\tau(\cdot, 0)\|_{L^2_\rho} \le \gamma_\tau \to 0 \quad \text{as} \quad \tau \to T\,. \tag{3.29}$$

Now, for $|\lambda(\tau)\xi + x\sqrt{T-\tau}| \le R_1$, we have $f_\tau = e^{\phi_\tau} - e^{\tilde{\phi}} W_\tau - e^{\tilde{\phi}}$ and so we have for some $\Theta_\tau = \Theta_\tau(y, s) \in [0, W_\tau(y, s)]$ that

$$f_\tau = e^{\tilde{\phi}}(e^{W_\tau} - W_\tau - 1) = \frac{1}{2} e^{\tilde{\phi} + \Theta_\tau} W_\tau^2\,. \tag{3.30}$$

Clearly, $\tilde{\phi} + \Theta_\tau \le \tilde{\phi} + \max\{0, W_\tau\} \le \max\{\tilde{\phi}, \phi_\tau\} \le K$ and so the inequality

$$|f_\tau| \le C Z_\tau \tag{3.31}$$

holds as well.

For $|\lambda(\tau)\xi + x\sqrt{T-\tau}| > R_1$, we have that $W_\tau \le -1$, at least for τ close to T, and therefore the uniform bound (3.7) gives us that $|f_\tau| \le C \le C Z_\tau \le C Z_\tau^2$. Thus the inequality (3.31) holds for every $s > 0$, $y \in \boldsymbol{R}^n$ and τ close to T with some constant C depending only on the constant appearing in Theorem 1.1 and the choice of ζ.

In the forthcoming statements and proofs C denotes again a generic constant, possibly changing from line to line, depending only on the solution u, our choice of ζ and $\xi \in \boldsymbol{R}^n$ and the dimension n.

LEMMA 3.7. *Let f_τ be as above and assume that* $\sup_{s \le \bar{s}} \|Z_\tau(\cdot, s)\| \le \varepsilon_\tau$, *where* $\varepsilon_\tau \to 0$ *as* $\tau \to T$. *Then there exist a constant* $C' > 0$ *such that*

$$\|f_\tau(\cdot, s)\|_{L^2_\rho} \le C' e^{-s} \varepsilon_\tau$$

for every $s \le \bar{s}$.

PROOF. We will first estimate the part of the norm where $|y|$ is large. Recall that, using the regularizing effect of the semigroup together with the inequalities (3.28) and (3.31), we know that there exists a constant $R > 0$ depending only on $p \ge 1$ and the dimension of the space such that

$$\|Z_\tau(\cdot, s)\|_{L^p_\rho} \le \|e^{CR} \tilde{S}(R) Z_\tau(\cdot, s-R)\|_{L^p_\rho} \le C \|Z_\tau(\cdot, s-R)\|_{L^2_\rho}\,. \tag{3.32}$$

Then define $\Omega_1(s, \tau) = \{y \in \boldsymbol{R}^n\,;\ |y| > e^{s/2}\lambda(\tau)|\xi|/2\sqrt{T-\tau}\}$ and use the inequality (3.31) together with Hölder's inequality and the above inequality (3.32) to obtain

$$\begin{aligned}
&\int_{\Omega_1(s,\tau)} |f_\tau(y, s)|^2 e^{-|y|^2/4} dy \\
&\qquad \le \left\{ \int_{\Omega_1(s,\tau)} |f_\tau(y, s)|^4 e^{-|y|^2/4} dy \right\}^{1/2} \left\{ \int_{|y| \ge e^{s/2}} e^{-|y|^2/4} dy \right\}^{1/2} \\
&\qquad \le C \|Z_\tau(\cdot, s)\|^2_{L^4_\rho} e^{-e^s} \le C e^{-2s} \|Z_\tau(\cdot, s-R)\|^2_{L^2_\rho} \le C e^{-2s} \varepsilon_\tau^2
\end{aligned}$$

for $s \le \bar{s}$ and τ close to T. Here we used the fact that

$$\int_{|y|\ge R} e^{-|y|^2} dy \le Ce^{-R^2}.$$

In what follows, we consider the part of the integral where $y \in \Omega_2(s,\tau) = \boldsymbol{R}^n \setminus \Omega_1(s,\tau)$ and notice that then $f_\tau = e^{\tilde{\phi}}(e^{W_\tau} - 1 - W_\tau) = e^{\tilde{\phi}+\Theta_\tau} Z_\tau^2/2$ for $\Theta_\tau \in (0, W_\tau)$ and τ sufficiently close to T. By taking τ close to T and y in $\Omega_2(s,\tau)$, we have that $|\lambda(\tau)\xi + \sqrt{T-\tau}x| > \lambda(\tau)|\xi|/2$ and $(T-\tau)|\log(\lambda(\tau))|/\lambda(\tau)^2 \le 1$. By using the estimate (3.5), we then get

$$\begin{aligned} W_\tau(y,s) &= \log(T-\tau) + \tilde{u}\big(\lambda(\tau)\xi + \sqrt{T-\tau}x, \tau + (T-\tau)t\big) \\ &\quad + \log\big(1 - t + \textstyle\sum_{|\alpha|=m} a_\alpha \xi^\alpha\big) \\ &\le \log\left(\frac{2(T-\tau)|\log(\lambda(\tau)|\xi|)|}{(\lambda(\tau)|\xi|)^2}\right) + \overline{C} \le C. \end{aligned}$$

Therefore $f_\tau \le e^{\tilde{\phi}+C} Z_\tau^2$ and

$$\begin{aligned} \int_{\Omega_2(s,\tau)} |f_\tau(y,s)|^2 e^{-|y|^2/4} dy &\le e^{2(\tilde{\phi}+C)} \int_{\Omega_2(s,\tau)} |Z_\tau(y,s)|^4 e^{-|y|^2/4} dy \\ &\le Ce^{-2s}\|Z(\cdot,s)^2\|^2 \le C(R)e^{-2s}\|Z(\cdot,s-R)\|^4 \le Ce^{-2s}\varepsilon_\tau^4, \end{aligned}$$

which finishes the proof. □

Now we are ready to prove that the norm of Z_τ stays small forever if it is initially small enough, using an idea from [29]. This will then allow us to pass to the limit as $s \to \infty$ and complete the proof concerning the blow-up profile.

PROPOSITION 3.8. *Let Z_τ be as above. Then there exists a constant $C > 0$ independent of s such that*

$$\|Z_\tau(\cdot,s)\|_{L^2_\rho} \le C\gamma_\tau$$

and

$$\sup_{|y|\le R} Z(y,s) \le C\gamma_\tau. \tag{3.33}$$

PROOF. Let τ be close to T and s_0 be large enough so that all the above estimates hold. Let now $\{\tilde{S}(s)\}_s$ be the semigroup generated by $\tilde{A}$. It is clear that because of (3.28), (3.29) and (3.31), we have that

$$\|Z_\tau(\cdot,s_0)\|_{L^2_\rho} \le e^{Cs_0}\|\tilde{S}(s_0)Z_\tau(\cdot,0)\|_{L^2_\rho} \le e^{Cs_0}\gamma_\tau \tag{3.34}$$

for some constant $C > 0$. Define

$$\bar{s} = \sup\{s\,;\ \|Z_\tau(\cdot,s)\|_{L^2_\rho} \le 4e^{Cs_0}\gamma_\tau\}$$

and assume that $\bar{s} < \infty$. Take then s_0 large enough so that both

$$2C'e^{-s_0} < \frac{1}{4} \quad \text{and} \quad \frac{e^{-s_0} + \sum_{|\alpha|=m} a_\alpha \xi^\alpha}{\sum_{|\alpha|=m} a_\alpha \xi^\alpha} < 2, \tag{3.35}$$

where C' is the constant appearing in Lemma 3.7.

Using Lemma 3.7, the previous inequalities (3.34) and (3.35) together with the definition of $\bar{s}$ and the variation of constants formula, we obtain

$$\begin{aligned}\|Z(\cdot,\bar{s})\|_{L^2_\rho} &\le \Big(\|\tilde{S}(\bar{s}-s_0)Z_\tau(\cdot,s_0)\|_{L^2_\rho} + \int_{s_0}^{\bar{s}} \|\tilde{S}(\bar{s}-t)f_\tau(\cdot,t)\|_{L^2_\rho}dt\Big)\exp\Big(\int_{s_0}^{s} e^{\tilde{\phi}(t)}dt\Big)\\ &\le \Big(\|Z_\tau(\cdot,s_0)\|_{L^2_\rho} + C'\int_{s_0}^{\bar{s}} e^{-t}(4e^{Cs_0}\gamma_\tau)dt\Big)\frac{e^{-s_0}+\sum_{|\alpha|=m}a_\alpha\xi^\alpha}{e^{-s}+\sum_{|\alpha|=m}a_\alpha\xi^\alpha}\\ &\le 2\big(e^{Cs_0}\gamma_\tau + C'(e^{-s_0}-e^{-\bar{s}})4e^{Cs_0}\gamma_\tau\big) < \frac{3}{4}\cdot 4e^{Cs_0}\gamma_\tau\,,\end{aligned}$$

which contradicts the choice of $\bar{s}$. Therefore it has to hold that $\bar{s}=\infty$, which yields the first part of the claim.

Because of the estimate (3.31), we obtain also the second part of the claim by

$$\begin{aligned}\sup_{|y|\le R} Z_\tau(y,s) &\le \sup_{|y|\le R} |e^{CL}\tilde{S}(L)Z_\tau(y,s-L)|\\ &\le C\sup_{|y|\le R}\frac{e^{CL}}{(1-e^{-L})^{n/2}}\int_{\boldsymbol{R}^n}\exp\Big(-\frac{(ye^{-L/2}-\lambda)^2}{4(1-e^{-L})}\Big)Z_\tau(\lambda,s-L)d\lambda\\ &\le C\sup_{|y|\le R}\Big\{\int_{\boldsymbol{R}^n}\exp\Big(-\frac{(ye^{-L/2}-\lambda)^2}{2(1-e^{-L})}\Big)e^{|\lambda|^2/4}d\lambda\Big\}^{1/2}\\ &\quad\cdot\Big\{\int_{\boldsymbol{R}^n} Z_\tau(\lambda,s-L)^2e^{-|\lambda|^2/4}\Big\}^{1/2}\\ &\le C\|Z_\tau(\cdot,s-L)\|_{L^2_\rho} \le C\gamma_\tau\,,\end{aligned}$$

and the proof is complete. □

PROOF OF THEOREM 3.1. Passing to the limit as $s\to\infty$ in (3.33), which corresponds to taking $t=1$ and $x=0$, we have

$$\log(T-\tau)+\tilde{u}(\lambda(\tau),T)+\log\Big(\sum_{|\alpha|=m}a_\alpha\xi^\alpha\Big)\le C\gamma_\tau\to 0$$

as $\tau\to\infty$. Set $x=\lambda(\tau)\xi$ and follow the estimates in [2], for instance, to notice that the above convergence implies that

$$\lim_{|x|\to 0}[u(x,T)+2\log|x|-\log|\log|x||-\log 8]=0$$

if (3.18) holds, and

$$\lim_{|x|\to 0}\Big[u(x,T)+m\log|x|+\log\Big(\sum_{|\alpha|=m}a_\alpha\hat{\xi}^\alpha\Big)\Big]=0$$

if (3.25) holds, where $\hat{\xi} = x/|x|$. However, the latter convergence is impossible because of the estimate (3.5), so the claim follows. □

PROOF OF THEOREM 1.2. We shall first prove that if $\tilde{w}(\cdot, s_n) \to \varphi(\cdot)$ uniformly on compact sets for some sequence $s_n \to \infty$, then φ is a stationary solution of the corresponding rescaled equation, that is, it satisfies (1.4) with $0 \le \mu < \infty$. The argument is similar to that in [13] (see also [18]).

Because of the inequality (3.9) and parabolic regularization, we know that $\tilde{w}$ is contained in a compact subset of $C^{2,1}(B_M(0) \times [s_0, \infty))$ with uniformly Hölder continuous derivatives, where $M > 0$ is arbitrary. Using then the inequality

$$\int_{s_0}^{s} \int_{B_{R_1 e^{t/2}}(0)} |\tilde{w}_s(y,t)|^2 e^{-|y|^2/4} dy dt \le E[w](0) - E[w](s),$$

where

$$E[w](s) := \int_0^{R_1 e^{s/2}} \left(\frac{1}{2} w_y^2 - e^w + w \right) e^{-|y|^2/4}\, dy$$

is the energy functional corresponding to the rescaled equation, and proving that $E[w](s)$ is bounded from below, we obtain that $\tilde{w}_s(y,s)$ converges to zero uniformly on compact sets and hence φ is a stationary solution. Clearly $\varphi_\eta(0) = 0$ and since $\tilde{w}(0,s) \ge 0$ by (2.3), we also have that $\mu \ge 0$.

Following then [25], it is straightforward to show that such φ exists and $\tilde{w}(\cdot, s) \to \varphi(\cdot)$ uniformly on compact sets for $s \to \infty$. In the proof one first argues that the set of possible φ can be written as

$$\omega(\tilde{w}) = \bigcap_s \overline{\bigcup_{\sigma \ge s} \{\tilde{w}(\cdot, s)\}}$$

in a suitable topology. Then it is fairly simple to see that the above set is nonempty, compact and connected. Taking then φ as above and using the zero number property, we can see that $\tilde{w}(0,s) - \varphi(0)$ never changes sign for s large enough. Assuming then that $\omega(\tilde{w})$ contains at least three solutions of (1.4), denoted by ψ_i, $i \in \{1,2,3\}$, it has to hold that $\tilde{w}(0,s) \in (\psi_i(0), \psi_{i+1}(0))$ for i equal to 1 or 2 and s large enough, which contradicts the fact that $\tilde{w}(\cdot, s) \to \psi_j(\cdot)$, for $j \notin \{i, i+1\}$.

Theorem 3.1 enables us to conclude that $\mu > 0$ by applying the following proposition [28, Theorem 3.6].

PROPOSITION 3.9. *There exists a constant $C > 0$ such that there is no nonnegative L^1-solution of* (1.1) *with $f(u) = e^u$ and*

$$u_0(|x|) \ge -2\log|x| + \log(2(n-2)) + C$$

for $|x|$ close to 0.

Namely, if $\varphi \equiv 0$, then u cannot be continued beyond $t = T$ as an L^1-solution.

It is known, see [3], that if φ is a nontrivial solution of (1.4), then either $\varphi(\eta) = -2\log\eta + C + o(1)$ or $\varphi(\eta) = -C\eta^{-n}e^{\eta^2/4} + o(1)$ as $\eta \to \infty$. Since (3.9) holds, φ cannot have the exponential decay at infinity and the claim is proved. $\square$

4. Profile of L^1-connections. In this section we consider the problem

$$\begin{cases} u_t = u_{rr} + \dfrac{n-1}{r}u_r + \lambda e^u\,, & r \in (0,1)\,, \quad t > 0\,, \\ u_r(0,t) = u(1,t) = 0\,, & \qquad\qquad\quad t > 0\,, \\ u(r,0) = u_0(r) \geq 0\,, & r \in [0,1]\,, \end{cases} \tag{4.1}$$

where $\lambda > 0$ and $n \in [3, 9]$.

We first recall some known properties of equilibria of (4.1). The stationary problem corresponding to (4.1) is:

$$\begin{cases} \phi_{rr} + \dfrac{n-1}{r}\phi_r + \lambda e^{\phi} = 0\,, & r \in (0,1)\,, \\ \phi_r(0) = 0\,, \qquad \phi(1) = 0\,. \end{cases} \tag{4.2}$$

PROPOSITION 4.1 ([17, 21], see Figure 1). *Denote by S the solution set of the parameterized problem* (4.2):

$$S = \{(\phi, \lambda)\,;\, \lambda \in \boldsymbol{R}^+ \text{ and } \phi \text{ is a solution of } (4.2)\}\,.$$

Then there exists a smooth curve

$$\begin{array}{ccc} \boldsymbol{R}^+ & \to & C([0,1]) \times \boldsymbol{R}^+ \\ \Psi & & \Psi \\ s & \mapsto & (\phi(s), \lambda(s)) \end{array}$$

such that $S = \{(\phi(s), \lambda(s))\,;\, s > 0\}$ and that

$$\sup_{x \in B_1(0)} \phi(s)(x) = \phi(s)(0) = s\,.$$

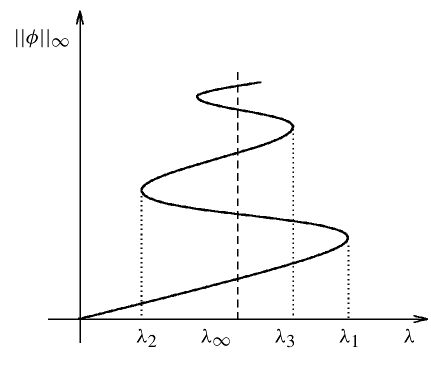

FIGURE 1.

Moreover, the following holds:

(i) $\lim_{s\to 0}\lambda(s) = 0$, $\lim_{s\to\infty}\lambda(s) = \lambda_\infty := 2(n-2)$.

(ii) *The set of all zeros of* $\lambda'(\cdot)$ *is given by a sequence* $0 < s_1 < s_2 < s_3 < \cdots \to \infty$ *and the critical values* $\lambda_j = \lambda(s_j)$, $j = 1, 2, 3, \ldots$, *satisfy*

$$\lambda_1 > \lambda_3 > \cdots > \lambda_{2j+1} \searrow \lambda_\infty, \quad \lambda_2 < \lambda_4 < \cdots < \lambda_{2j+2} \nearrow \lambda_\infty .$$

(iii) *For each* $\lambda \le \lambda_1$ *define*

$$\phi_i^\lambda = \phi(\tilde{s}_i), \quad i = 0, 1, \ldots,$$

where $\tilde{s}_0 < \tilde{s}_1 < \cdots$ *is the sequence of all points* s *with* $\lambda(s) = \lambda$. *This sequence is finite if* $\lambda \neq \lambda_\infty$ *and infinite if* $\lambda = \lambda_\infty$. *In the latter case we have*

$$\phi_i^\lambda(r) \to \phi_\infty^\lambda(r) := \log r \frac{2(n-2)}{\lambda r^2} \quad in\ C^1_{\mathrm{loc}}((0,1]) .$$

For the number of intersections of two equilibria and of equilibria with ϕ_∞^λ the following holds.

PROPOSITION 4.2.

(i) *If* $\lambda < \lambda_1$ *and* $k > j$ *are such that* ϕ_k^λ *and* ϕ_j^λ *are both defined, then* $\phi_k^\lambda - \phi_j^\lambda$ *has exactly* $j+1$ *zeros in* $[0,1]$, *all of them simple.*

(ii) *If* $\lambda = \lambda_\infty$ *and* $j \ge 0$, *then* $\phi_\infty^\lambda - \phi_j^\lambda$ *has* $j+1$ *zeros in* $[0,1]$.

(iii) *If* $\lambda < \lambda_\infty$ *and* $j \ge 0$ *are such that* ϕ_j^λ *is defined, then* $\phi_\infty^\lambda - \phi_j^\lambda$ *has* $j+1$ *zeros in* $[0,1]$ *when* j *is odd, and* j *zeros in* $[0,1]$ *when* j *is even.*

(iv) *If* $\lambda_\infty < \lambda \le \lambda_1$ *and* $j \ge 0$ *are such that* ϕ_j^λ *is defined, then* $\phi_\infty^\lambda - \phi_j^\lambda$ *has* j *zeros in* $[0,1]$ *when* j *is odd, and* $j+1$ *zeros in* $[0,1]$ *when* j *is even.*

All of the zeros of $\phi_\infty^\lambda - \phi_j^\lambda$ *are simple.*

PROOF. For the proof of (i) we refer to [14]. From (i) and Proposition 4.1 (iii) it follows that (ii) holds. To prove (iii) and (iv) one can then use the bifurcation diagram (Figure 1), the simplicity of zeros and continuation of ϕ_j^λ, taking into account that the zero of $\phi_\infty^\lambda - \phi_j^\lambda$ at $r = 1$, $\lambda = \lambda_\infty$, either moves inside or disappears when $\lambda \neq \lambda_\infty$ and λ is close to λ_∞. □

Next we recall the existence of a special blow-up solution which can be continued globally as an L^1-solution.

PROPOSITION 4.3. *For any* $\lambda \in (\lambda_2, \lambda_3]$ *and* $T > 0$ *there is* u_0 *such that the solution* $u(\cdot, t)$ *of* (4.1) *has the following properties*:

(i) $u(\cdot, t)$ *blows up at* $t = T$.

(ii) $u(\cdot, t)$ *is a global* L^1*-solution.*

(iii) $u(\cdot, t)$ *is defined* (*as a classical solution of* (4.1)) *on the interval* $(-\infty, T)$ *and* $u(\cdot, t) \to \phi_2^\lambda$ *in* $C^1([0,1])$ *as* $t \to -\infty$.

(iv) $u(\cdot, t)$ *is a classical solution of* (4.1) *on the interval* (T, ∞) *and* $u(\cdot, t) \to \phi_0^\lambda$ *in* $C^1([0,1])$ *as* $t \to \infty$.

(v) *There is a sequence* $\{u_i\}$ *of classical connections from* ϕ_2^λ *to* ϕ_0^λ *such that* $u_n(r,t) \nearrow u(r,t)$ *pointwise for* $(r,t) \in [0,1]\times\boldsymbol{R}$. *Here a classical connection from* ϕ_2^λ *to* ϕ_0^λ *is a classical solution of* (4.1) *on the interval* $(-\infty,\infty)$ *such that* $u(\cdot,t) \to \phi_2^\lambda$ *in* $C^1([0,1])$ *as* $t \to -\infty$, *and* $u(\cdot,t) \to \phi_0^\lambda$ *in* $C^1([0,1])$ *as* $t \to \infty$.

We call the solution u an L^1-*connection* from ϕ_2^λ to ϕ_0^λ.

For the proofs see Theorem 3.4 in [14] and Section 6 in [13].

THEOREM 4.4. *Let* $\lambda \in (\lambda_2, \lambda_3]$. *Suppose* u *is an* L^1-*connection from* ϕ_2^λ *to* ϕ_0^λ *as in Proposition* 4.3. *Then*

$$\lim_{t\to T}[\log(T-t) + u(\eta\sqrt{T-t},t)] = \varphi_0(\eta), \quad \eta \in [0,\infty)\,,$$

where φ_0 *satisfies*

$$\begin{cases} \varphi_{\eta\eta} + \left(\dfrac{n-1}{\eta} - \dfrac{\eta}{2}\right)\varphi_\eta + \lambda e^{\varphi} - 1 = 0\,, \quad \eta > 0\,,\\ \varphi(0) = \mu_0\,, \quad \varphi_\eta(0) = 0 \end{cases}$$

for some $\mu_0 > 0$ *and*

$$\lim_{\eta\to\infty}(\varphi_0(\eta) - \phi_\infty^\lambda(\eta)) = -c_0$$

for some $c_0 > 0$. *Moreover, the equation*

$$\varphi_0(\eta) - \phi_\infty^\lambda(\eta) = 0$$

has two roots.

For the proof we shall need the following lemma.

LEMMA 4.5 ([27]). *Let* $\lambda_\infty < \lambda \le \lambda_3$. *Denote the three zeros of* $\phi_\infty^\lambda - \phi_2^\lambda$ *by* $0 < r_1 < r_2 < r_3 < 1$. *Let* u *be an* L^1-*connection from* ϕ_2^λ *to* ϕ_0^λ *as in Proposition* 4.3. *Then* $u(\cdot,t) - \phi_\infty^\lambda$ *has at most two zeros in* $(0,r_1)$ *for* $t < T$.

PROOF. We use the notation (2.4). Since $u(\cdot,t) \to \phi_2^\lambda$ in C^1 as $t \to -\infty$ and $\mathcal{Z}_{(0,1)}(\phi_\infty^\lambda - \phi_2^\lambda) = 3$, it follows that there is $t_0 < 0$ such that $\mathcal{Z}_{(0,1)}(\phi_\infty^\lambda - u(\cdot,t)) = 3$ for $t < t_0$. Therefore, $\mathcal{Z}_{(0,1)}(\phi_\infty^\lambda - u(\cdot,t)) \le 3$ for $t < T$.

We now proceed by contradiction. Suppose there is $t_1 < T$ such that $\mathcal{Z}_{(0,r_1)}(\phi_\infty^\lambda - u(\cdot,t_1)) = 3$. Then there is a positive integer i and a classical connection u_i from ϕ_2^λ to ϕ_0^λ (cf. Proposition 4.3 (v)) such that $\mathcal{Z}_{(0,r_1)}(\phi_\infty^\lambda - u_i(\cdot,t_1)) = 3$. This means that

$$u_i(r,t_1) > \phi_\infty^\lambda(r)\,, \quad r \in [r_1,1]\,, \tag{4.3}$$

because $\mathcal{Z}_{(0,1)}(\phi_\infty^\lambda - u_i(\cdot,t)) \le 3$ for all $t \in \boldsymbol{R}$.

We claim that then

$$\mathcal{Z}_{(r_3,1)}(\phi_2^\lambda - u_i(\cdot,t_1)) = 1\,. \tag{4.4}$$

Indeed, otherwise either

$$\mathcal{Z}_{[0,1]}(\phi_2^\lambda - u_i(\cdot,t_1)) > 2 \tag{4.5}$$

or

$$\mathcal{Z}_{[0,1)}(\phi_2^\lambda - u_i(\cdot, t_1)) = 0\,. \tag{4.6}$$

Since $u_i(\cdot, t_1)$ belongs to the unstable manifold of ϕ_2^λ, we must have (cf. Theorem 2.1 in [6])

$$\mathcal{Z}_{[0,1]}(\phi_2^\lambda - u_i(\cdot, t)) \le 2\,, \quad t \in \boldsymbol{R}\,. \tag{4.7}$$

(We remark here that Theorem 2.1 in [6] concerns the zero number on the unstable manifold of an equilibrium of a semilinear parabolic equation in one space-dimension. But this result can be extended in a straightforward way to radially symmetric solutions in higher space-dimension using Theorem 2.1 from [8].) It follows from (4.7) that (4.5) cannot occur. On the other hand, (4.6) would imply that u_i blows up in a finite time (cf. [22]). Hence (4.4) holds. Therefore, we obtain that

$$u_i(r, t_1) > \phi_2^\lambda(r)\,, \quad r \in [0, r_3]\,. \tag{4.8}$$

We next show that

$$u_i(r, t) > \max\{\phi_2^\lambda(r), \phi_\infty^\lambda(r)\}\,, \quad (r, t) \in [r_1, r_3] \times [t_1, \infty)\,. \tag{4.9}$$

From (4.3) and (4.8) we have

$$u_i(r, t_1) > \max\{\phi_2^\lambda(r), \phi_\infty^\lambda(r)\}\,, \quad r \in [r_1, r_3]\,.$$

If (4.9) does not hold, then there is $t_2 > t_1$ such that

$$u_i(r, t) > \max\{\phi_2^\lambda(r), \phi_\infty^\lambda(r)\}\,, \quad (r, t) \in [r_1, r_3] \times [t_1, t_2)\,,$$

and either

$$u_i(r_1, t_2) = \phi_\infty^\lambda(r_1) (= \phi_2^\lambda(r_1))\,, \tag{4.10}$$

or

$$u_i(r_3, t_2) = \phi_\infty^\lambda(r_3) (= \phi_2^\lambda(r_3))\,. \tag{4.11}$$

Note that $\mathcal{Z}_{(r_3,1)}(\phi_2^\lambda - u_i(\cdot, t)) = 1$ for $t \in [t_1, t_2]$, so (4.10) is impossible because then

$$\mathcal{Z}_{[0,1]}(\phi_2^\lambda - u_i(\cdot, t_2)) = 3\,.$$

On the other hand, for $t \in [t_1, t_2]$ all intersections of ϕ_∞ and $u_i(\cdot, t)$ are contained in $[0, r_1]$. Thus (4.11) cannot occur.

Since $\phi_2^\lambda > \phi_0^\lambda$ in $[r_1, r_3]$, (4.9) yields a contradiction with the convergence of $u_i(\cdot, t)$ to ϕ_0^λ as $t \to \infty$. □

PROOF OF THEOREM 4.4. Consider first the case $\lambda_2 < \lambda \le \lambda_\infty$. Then $\mathcal{Z}_{(0,1)}(\phi_\infty^\lambda - \phi_2^\lambda) = 2$ and by the zero number diminishing property, it has to hold that $\mathcal{Z}_{[0,R]}(u(\cdot, t) - \phi_\infty^\lambda) \le 2$ for every $t \in (-\infty, T)$. After rescaling, we then get that $\mathcal{Z}_{[0,e^{s/2}]}(\tilde{w}(\cdot, s) - \phi_\infty^\lambda) \le 2$ for every $s \in (-\infty, \infty)$. Theorem 1.2 now states that $\tilde{w}(\cdot, s) \to \varphi$ uniformly on compact sets in y, where φ has the decay (1.5) and intersects ϕ_∞^λ at most twice. It follows then from [3] that φ has to intersect ϕ_∞^λ exactly twice.

If $\lambda_\infty < \lambda \leq \lambda_3$, then $\mathcal{Z}_{(0,1)}(\phi^\lambda_\infty - \phi^\lambda_2) = 3$, but Lemma 4.5 yields that $\mathcal{Z}_{(0,r_1)}(\phi^\lambda_\infty - u(\cdot,t)) \leq 2$ for $t < T$ and we can proceed as before. □

The existence of L^1-connections between two equilibria ϕ^λ_k and ϕ^λ_j was studied in [11, 12], and it was shown there that a singular L^1-connection from ϕ^λ_k to ϕ^λ_j exists if and only if $k \geq j+2$. By Theorem 1.1 any such L^1-connection blows up with the selfsimilar rate and by Theorem 1.2 it converges (after rescaling) to a nonconstant selfsimilar solution. It would be interesting to determine how this limit selfsimilar solution depends on k and j. Theorem 4.4 answers this question only for $k = 2$ and $j = 0$. To prove a more general result one has to be able to control the number of intersections with ϕ^λ_∞ that disappear at the moment of blow-up.

Acknowledgment. Part of this work was done while A. Pulkkinen was a predoc at the Comenius University supported by European Community's Human Potential Programme under contract HPRN-CT-2002-00274, Fronts - Singularities. M. Fila acknowledges also the support of the VEGA Grant 1/3021/06 and of the Finnish Mathematical Society International Visitor Programme. A. Pulkkinen was also supported by the Magnus Ehrnrooth Foundation during this work. The authors are indebted to the referee for helpful comments.

REFERENCES

[1] P. BARAS AND L. COHEN, Complete blow-up after $T_{\max}$ for the solution of a semilinear heat equation, J. Funct. Anal. 71 (1987), 142–174.

[2] J. BEBERNES AND S. BRICHER, Final time blow-up profiles for semilinear parabolic equations via center manifold theory, SIAM J. Math. Anal. 23 (1992), 852–869.

[3] J. BEBERNES AND D. EBERLY, Mathematical Problems from Combustion Theory, Springer-Verlag, New York, 1989.

[4] A. BRESSAN, Stable blow-up patterns, J. Differential Equations 98 (1992), 57–75.

[5] J. BRICMONT AND A. KUPIAINEN, Universality in blow-up for nonlinear heat equations, Nonlinearity 7 (1994), 539–575.

[6] P. BRUNOVSKÝ AND B. FIEDLER, Numbers of zeros on invariant manifolds in reaction-diffusion equations, Nonlinear Anal. 10 (1986), 179–193.

[7] X. CHEN, M. FILA AND J.-S. GUO, Boundedness of global solutions of a supercritical parabolic equation, Nonlinear Anal. 68 (2008), 621–628.

[8] X.-Y. CHEN AND P. POLÁČIK, Asymptotic periodicity of positive solutions of reaction-diffusion equations on a ball, J. Reine Angew. Math. 472 (1996), 17–51.

[9] D. EBERLY, On the nonexistence of solutions to the Kassoy problem in dimensions 1 and 2, J. Math. Anal. Appl. 129 (1988), 401–408.

[10] D. EBERLY AND W. TROY, On the existence of logarithmic-type solutions to the Kassoy-Kapila problem in dimensions $3 \leq N \leq 9$, J. Differential Equations 70 (1987), 309–324.

[11] M. FILA AND H. MATANO, Connecting equilibria by blow-up solutions, Discrete Contin. Dynam. Systems 6 (2000), 155–164.

[12] M. FILA, H. MATANO AND P. POLÁČIK, Existence of L^1-connections between equilibria of a semilinear parabolic equation, J. Dynam. Differential Equations 14 (2002), 463–491.

[13] M. FILA, H. MATANO AND P. POLÁČIK, Immediate regularization after blow-up, SIAM J. Math. Anal. 37 (2005), 752–776.

[14] M. FILA AND P. POLÁČIK, Global solutions of a semilinear parabolic equation, Adv. Differential Equations 4 (1999), 163–196.

[15] S. FILIPPAS AND R. KOHN, Refined asymptotics for the blow-up of $u_t - \Delta u = u^p$, Comm. Pure Appl. Math. 45 (1992), 821–869.
[16] A. FRIEDMAN AND B. MCLEOD, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. 34 (1985), 425–447.
[17] I. M. GELFAND, Some problems in the theory of quasilinear equations, Amer. Math. Soc. Transl. 29 (1963), 295–381.
[18] Y. GIGA AND R. KOHN, Asymptotically self-similar blow-up of semilinear heat equations, Comm. Pure Appl. Math. 38 (1985), 297–319.
[19] M. A. HERRERO AND J. J. L. VELÁZQUEZ, Blow-up behavior of one-dimensional semilinear parabolic equations, Ann. Inst. H. Poincaré, Anal. Non Linéaire 10 (1993), 131–189.
[20] M. A. HERRERO AND J. J. L. VELÁZQUEZ, Plane structures in thermal runaway, Israel J. Math. 81 (1993), 321–341.
[21] D. D. JOSEPH AND T. S. LUNDGREN, Quasilinear Dirichlet problems driven by positive sources, Arch. Ration. Mech. Anal. 49 (1972), 241–269.
[22] A. A. LACEY, Mathematical analysis of thermal runaway for spatially inhomogeneous reactions, SIAM J. Appl. Math. 43 (1983), 1350–1366.
[23] A. A. LACEY AND D. E. TZANETIS, Global, unbounded solutions to a parabolic equation, J. Differential Equations 101 (1993), 80–102.
[24] H. MATANO AND F. MERLE, On non-existence of type II blow-up for a supercritical nonlinear heat equation, Comm. Pure Appl. Math. 57 (2004), 1494–1541.
[25] J. MATOS, Convergence of blow-up solutions of nonlinear heat equations in the supercritical case, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999), 1197–1227.
[26] J. MATOS, Self-similar blow up patterns in supercritical heat equations, Commun. Appl. Anal. 5 (2001), 455–483.
[27] P. POLÁČIK, personal communication.
[28] J. L. VÁZQUEZ, Domain of existence and blowup for the exponential reaction-diffusion equation, Indiana Univ. Math. J. 48 (1999), 677–709.
[29] J. J. L. VELÁZQUEZ, Classification of singularities for blowing up solutions in higher dimensions, Trans. Amer. Math. Soc. 338 (1993), 441–464.
[30] J. J. L. VELÁZQUEZ, Higher-dimensional blow up for semilinear parabolic equations, Comm. Partial Differential Equations 17 (1992), 1567–1696.

DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS
COMENIUS UNIVERSITY
84248 BRATISLAVA
SLOVAKIA

E-mail address: fila@fmph.uniba.sk

INSTITUTE OF MATHEMATICS
HELSINKI UNIVERSITY OF TECHNOLOGY
02015 ESPOO
FINLAND

E-mail address: ampulkki@math.hut.fi