# ON THE SPACES WITH NORMAL CONFORMAL CONNEXIONS AND SOME IMBEDDING PROBLEM OF RIEMANNIAN SPACES, I.[*]

By

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An $n$-dimensional space with normal conformal connexion whose group of holonomy fixes a hypersphere $S_{n-1}$ is conformal with an Einstein space with constant scalar curvature which is negative, 0 or positive according as the sphere is real, point or imaginary.[1] If $S_{n-1}$ is real, the group of holonomy induces a group of Möbius' transformations on it. Therefore its image in the space, a hypersurface $F_{n-1}$, is an $(n-1)$-dimensional space with normal conformal connexion. The Riemannian metric induced in $F_{n-1}$ seems to us to be most general.

On the other hand, in J. E. Campbell's book[2] we find a theorem that any Riemannian space $V_{n-1}$ can be imbedded in some Einstein space $A_n$ as a hypersurface. Is the first conjecture true? What relations are there between above stated facts?

In the following paper we shall study these problems.

Article 1 (1—4) deals with such a space with normal conformal connexion whose group of holonomy fixes a hypersphere, and the image of the hypershere. In article 2 (5—12) attention is turned to solve the problem whether a given Riemannian space can be imbedded as the hypersurface of the image in a restricted sense or not. Finally article 3 (13—14) deals with the spaces whose groups of holonomy fix two hyperspheres.

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1) S. Sasaki, On the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere, I, II, III, Jap. J. of Math. 34 (1942) pp. 615—622, pp. 623—633, 35 (1943) pp. 791—795.

K. Yano, Conformal and concircular geometries in Einstein spaces. Proc. Imp. Acad. Japan, 20 (1944) pp. 45—53.

2) J. E. Campbell, A course of differential geometry, (1926).

## § 1.

**1.** According to Mr. E. Cartan,[3] let $R: (A_0, A_i, A_\infty)$ be frames of an $n$-dimensional space with conformal connexion satisfying the conditions

(1) $$A_0^2 = A_\infty^2 = A_0A_i = A_iA_\infty = 0,\quad A_0A_\infty = 1,\quad A_iA_j = \delta_{ij}\quad (i, j = 1, 2, \cdots, n).$$

We call these frames composed of $n+2$ hyperspheres as normal. Then the connexion is given by the following equations:

(2) $$\begin{cases} dA_0 = \omega_0^0 A_0 + \omega^i A_i, \\ dA_i = \omega_i^0 A_0 + \omega_i^k A_k - \omega^i A_\infty, \quad \omega_k^i + \omega_i^k = 0. \\ dA_\infty = -\omega_i^0 A_i - \omega_0^0 A_\infty, \end{cases}$$

where $(\omega_0^0, \omega^i, \omega_i^0, \omega_j^i)$ are Pfaffians.

If the space has no torsion and $\Omega_0^0 = 0$, the equations of structure are

(3) $$\begin{cases} (\omega_0^0)' = [\omega^k \omega_k^0], \\ (\omega^i)' = [\omega_0^0 \omega^i] + [\omega^k \omega_k^i], \\ (\omega_i^j)' = [\omega^i \omega_j^0] - [\omega^j \omega_i^0] + [\omega_i^k \omega_k^j] - \Omega_i^j, \\ (\omega_i^0)' = [\omega_i^0 \omega_0^0] + [\omega_i^k \omega_k^0] - \Omega_i^0. \end{cases}$$

Suppose the group of holonomy of this space fixes a hypersphere $S_{n-1}$. Let us now represent $S_{n-1}$ by

(4) $$X = x^0 A_0 + x^i A_i + x^\infty A_\infty,$$

then by virtue of (2), we get

$$dX = (dx^0 + x^0\omega_0^0 + x^k \omega_k^0) A_0 + (dx^i + x^0\omega^i + x^k\omega_k^i - x^\infty \omega_i^0) A_i + (dx^\infty - x^k\omega^k - x^\infty\omega_0^0) A_\infty.$$

Accordingly the system of Pfaff's equations

(5) $$\frac{dx^0 + x^0\omega_0^0 + x^k\omega_k^0}{x^0} = \frac{dx^i + x^0\omega^i + x^k\omega_k^i - x^\infty\omega_i^0}{x^i} = \frac{dx^\infty - x^k\omega^k - x^\infty\omega_0^0}{x^\infty}$$

must be integrable.

Let us now assume that the point $A_0$ does not lie on $S_{n-1}$, then $x^\infty \neq 0$. If we put

(6) $$y^0 = \frac{x^0}{x^\infty}, \qquad y^i = \frac{x^i}{x^\infty},$$

(5) becomes

(7) $$\begin{cases} dy^0 + y^0\omega_0^0 + y^k\omega_k^0 + y^0(y^k\omega^k + \omega_0^0) = 0, \\ dy^i + y^0\omega^i + y^k\omega_k^i - \omega_i^0 + y^i(y^k\omega^k + \omega_0^0) = 0. \end{cases}$$

Hence, for a change of secondary parameters, integrals $y^0, y^i$ have the

3) E. Cartan, Les espaces à connexion conforme. Ann. Soc. Pol. Math., 2 (1923) pp. 171—221., Les groupes d'holonomie des espaces généralisés. Acta Math., 48 (1926) pp. 1—42.

variations

$$\delta y^0 + 2y^0 e_0^0 + y^k e_k^0 = 0,$$
$$\delta y^i + y^k e_k^i - e_i^0 + y^i e_0^0 = 0,\ e_0^0 = \omega_0^0(\delta),\ e_i^0 = \omega_i^0(\delta),\ e_k^i = \omega_k^i(\delta).$$

The last equation shows that $y^i$ are transformed by the $n$-dimensional group of similarity. Accordingly we can choose, at each point, frames such that

(8) $$y^i = 0.$$

Then we have by (7)

(9) $$dy^0 + 2y^0 \omega_0^0 = 0,$$

(10) $$\omega_i^0 = y^0 \omega^i.$$

Consequently our spaces are classified into the following two cases I and II:

I. $$y^0 = 0.$$

In this case $S_{n-1}$ degenerates into the point-sphere $A_\infty$, and by (10) and (3) we have

(11) $$\omega_i^0 = 0,$$

(3′) $$\begin{cases} (\omega_0^0)' = 0,\ (\omega^i)' = [\omega_0^0 \omega^i] + [\omega^k \omega_k^i], \\ (\omega_i^j)' = [\omega_i^k \omega_k^j] - \Omega_i^j. \end{cases}$$

Therefore if we put

(12) $$\omega_1^1 = \omega_2^2 = \cdots\cdots = \omega_n^n = -\omega_0^0,$$

$(\omega^i, \omega_j^i)$ is a system of Pfaffians of the Weyl space in terms of normal frames.

II. $$y^0 \neq 0.$$

In this case, we can choose by (9), frames so that $y^0 = 1$ or $-1$, hence $S_{n-1}$ is represented analytically by $A_0 + A_\infty$(real hypersphere) or by $-A_0 + A_\infty$ (imaginary hypersphere). By virtue of (10) and (3) we have

(13) $$\omega_i^0 = \varepsilon \omega^i\ (\varepsilon = 1 \text{ or } -1),\ \omega_0^0 = 0,$$

(3) $$\begin{cases} (\omega^i)' = [\omega^k \omega_k^i], \\ (\omega_i^j)' = 2\varepsilon [\omega^i \omega^j] + [\omega_i^k \omega_k^j] - \Omega_i^j. \end{cases}$$

Therefore $(\omega^i, \omega_j^i)$ is a system of Pfaffians which defines a Riemannian space in terms of normal frames.

Now let us suppose that our space is normal, that is, it satisfies conformal conditions

(14) $$A_{ih} = A_i{}^k{}_{hk} = 0,$$

where $A_i{}^j{}_{hk}$ is defined by

(15) $$\Omega_j^i = \frac{1}{2} A_i{}^j{}_{hk} [\omega^h \omega^k].$$

In the case I we shall have a quantity $\varphi$ such that $\omega_0^0 = d\varphi$, therefore we may put $\varphi=$ constant. Then, making use of the relations

$$(16)\qquad -(\omega_i^j)'+[\omega_i^k\,\omega_k^j]=\frac{1}{2}R_i{}^j{}_{hk}[\omega^h\,\omega^k],$$

$$R_{ij}=R_i{}^h{}_{jh},\quad R=R_{ii},$$

where $R_i{}^j{}_{hk}$, $R_{ij}$ and $R$ are the components of Riemann tensor, Ricci tensor and Riemann scalar curvature respectively. As is well known, we shall have in both cases the following relation :

$$(17)\qquad \omega_i^0=-\frac{1}{n-2}R_{ik}\,\omega^k+\frac{1}{2(n-1)(n-2)}R\omega^i.$$

Accordingly, in the case I, we have by (11) and (17)

$$R_{ij}=\frac{1}{2(n-1)}R\delta_{ij},\ R=\frac{n}{2(n-1)}R.$$

As we are considering only the case $n>2$, we have $R=0$ from the latter and consequently we get

$$(18)\qquad R_{ij}=0.$$

Accordingly, the Riemannian space with line element $ds^2=\omega^i\,\omega^i$ is an Einstein space with scalar curvature 0.

In the second case, we have by (13) and (17)

$$\varepsilon\,\omega^i=-\frac{1}{n-2}R_{ik}\,\omega^k+\frac{1}{2(n-1)(n-2)}R\omega^i,$$

hence we get

$$R_{ij}=\left\{\frac{1}{2(n-1)}R-(n-2)\,\varepsilon\right\}\delta_{ij},$$

$$R=\frac{n}{2(n-1)}R-n(n-2)\,\varepsilon,$$

that is

$$(19)\qquad R=-2\varepsilon n(n-1),$$

$$(20)\qquad R_{ij}=-2\varepsilon(n-1)\,\delta_{ij}.$$

The last equation shows that the Riemannian space with line element $ds^2=\omega^i\,\omega^i$ is an Einstein space with scalar curvature $-2n(n-1)$ or $2n(n-1)$.

Thus we know that a space with normal conformal connexion whose group of holonomy fixes a hypersphere $S_{n-1}$ is conformal with an Einstein space whose Riemann scalar curvature is $<0$, $0$ or $>0$ according as the hypersphere $S_{n-1}$ is real, point or imaginary. The converse is evidently true.

**2.** In the following lines we shall investigate the space with normal conformal connexion whose group of holonomy fixes a real $S_{n-1}$ and the hypersurface $F_{n-1}$ which is the image of $S_{n-1}$ in the underlying manifold.

As the connexion is normal, we have by (2) and (3)

$$\omega_0^0 = 0, \quad \omega_i^0 = -\frac{1}{n-2} R_{ik}\,\omega^k + \frac{1}{2(n-1)(n-2)} R\omega^i, \tag{21}$$

$(\omega^i, \omega_j^i)$ is a system of Pfaffians which defines the connexion of a Riemannian space, and $R_i{}^j{}_{hk}$'s are components of its Riemann tensor.

Let us now restrict our consideration to a neighborhood of $F_{n-1}$. Then as $S_{n-1}$ is real, we shall have $x^\infty = 0$ and $x\,x > 0$ on $F_{n-1}$. Therefore we may assume $x^n \neq 0$ in that neighborhood. If we put

$$y^a = \frac{x^a}{x^n}, \; y^0 = \frac{x^0}{x^n}, \; y^\infty = \frac{x^\infty}{x^n} \quad (a = 1, 2, \cdots\cdots, n-1), \tag{22}$$

(5) becomes

$$\left\{\begin{aligned} & dy^0 + \omega_n^0 + y^a\,\omega_a^0 - y^0\,(y^0\omega^n + y^b\,\omega_b^n - y^\infty\,\omega_n^0) = 0, \\ & dy^a + \omega_n^a + y^0\,\omega^a + y^b\,\omega_b^a - y^\infty\,\omega_a^0 - y^a\,(y^0\,\omega^n + y^b\,\omega_b^n - y^\infty\,\omega_n^0) = 0, \\ & dy^\infty - \omega^n - y^b\,\omega^b - y^\infty\,(y^0\,\omega^n + y^b\,\omega_b^n - y^\infty\,\omega_n^0) = 0, \end{aligned}\right. \tag{23}$$

which are integrable on account of our hypothesis. For a change of secondary parameters, we have by (21)

$$\delta y^0 - y^0\,y^b\,e_b^n = 0, \qquad \delta y^\infty - y^\infty\,y^b\,e_b^n = 0,$$
$$\delta y^a + e_n^a + y^b\,e_b^a - y^a\,y^b\,e_b^n = 0.$$

Accordingly we can choose frames so that

$$y^a = 0. \tag{24}$$

Then (23) turns into

$$\left\{\begin{aligned} & dy^0 + \omega_n^0 - y^0\,(y^0\,\omega^n - y^\infty\,\omega_n^0) = 0, \\ & dy^\infty - \omega^n - y^\infty\,(y^0\,\omega^n - y^\infty\,\omega_n^0) = 0, \\ & \omega_n^a + y^0\,\omega^a - y^\infty\,\omega_a^0 = 0. \end{aligned}\right. \tag{25}$$

Thus our consideration is divided into two cases $y^0 = 0$ or $\neq 0$, that is the cases where $A_\infty$ lies on $S_{n-1}$ or not. Of course $y^0$ and $y^\infty$ are invariants. In this paper we shall treat with the case $y^0 = 0$. Accordingly in the following, we shall always mean by immersion of a given Riemannian space in an Einstein space immersion of the former as the hypersurface of the image of the hypersphere $S_{n-1}$ under the condition $y^0 = 0$.

**3.** Let us put $y^\infty = y$, then in our space (5) turns into the following system of Pfaffian equations:

$$\omega^n = dy, \tag{26}$$

$$\omega_n^a = y\Big(-\frac{1}{n-2} R_{ak}\,\omega^k + \frac{1}{2(n-1)(n-2)} R\omega^a\Big), \tag{27}$$

$$R_{nk}\,\omega^k = \frac{1}{2(n-1)} R\,dy. \tag{28}$$

Let us suppose that $x^1, x^2, \cdots, x^{n-1}$ be integrals of the system of Pfaffian equations

$$\omega^1 = \omega^2 = \cdots\cdots = \omega^{n-1} = 0,$$

then from (28) we shall obtain

$$R_{an} = 0, \tag{29}$$

$$R_{nn} = \frac{1}{2(n-1)} R. \tag{30}$$

Let us now again write the relations (26)—(30) by means of natural frames (repères naturelles). Putting

$$\omega^a = f^a_b\, dx^b, \tag{31}$$

and denoting natural frames by $R^*$ $(A^*_0, A^*_\lambda, A^*_n, A^*_\infty)$ $(\lambda = 1, 2, \cdots, n-1)$, we may assume that the connexion of our space is given by

$$\left\{\begin{aligned} dA^*_0 &= dx^\lambda A^*_\lambda + dy\, A^*_n, \\ dA^*_i &= \omega^{*0}_i A^*_0 + \omega^{*k}_i A^*_k + \omega^{*\infty}_i A^*_\infty, \\ dA^*_\infty &= \omega^{*i}_\infty A^*_i. \end{aligned}\right. \tag{32}$$

Then from $dA_0 = \omega^i A_i = f^a_\lambda dx^\lambda A_a + dy A_n = dx^\lambda A^*_\lambda + dy\, A^*_n$ we shall obtain

$$A^*_\lambda = f^a_\lambda A_a, \qquad A^*_n = A_n.$$

By virtue of (2) we get

$$\begin{aligned} dA^*_\lambda &= df^a_\lambda A_a + f^b_\lambda (\omega^0_b A_0 + \omega^i_b A_i - \omega^b A_\infty) \\ &= \omega^{*0}_\lambda A_0 + \omega^{*\mu}_\lambda f^a_\mu A_a + \omega^{*n}_\lambda A_n + \omega^{*\infty}_\lambda A^*_\infty, \end{aligned}$$

and hence we obtain

$$df^a_\lambda + f^b_\lambda \omega^a_b = \omega^{*\mu}_\lambda f^a_\mu, \; f^b_\lambda \omega^n_b = \omega^{*n}_\lambda, \; f^b_\lambda \omega^0_b = \omega^{*0}_\lambda. \tag{33}$$

Putting $A^*_\infty = -A_\infty$, we get also

$$f^a_\lambda \omega^a = \omega^{*\infty}_\lambda. \tag{34}$$

As $R^*_{\lambda\mu} = f^a_\lambda f^b_\mu R_{ab}$, (29) and (30) become

$$R^*_{an} = 0, \tag{29'}$$

$$R^*_{nn} = \frac{1}{2(n-1)} R^*. \tag{30'}$$

Therefore making use of new quantities $p^\mu_a$ such that $f^a_\lambda p^\mu_a = \delta^\mu_\lambda$, we obtain from (33)

$$\omega^a_n = -\omega^n_a = -p^\lambda_a \omega^{*n}_\lambda.$$

Thus (27) turns by (29) into

$$-p^\lambda_a \omega^{*n}_\lambda = y\Big(-\frac{1}{n-2} R_{ab} f^b_\mu dx^\mu + \frac{1}{2(n-1)(n-2)} R f^a_\mu dx^\mu\Big).$$

Now, if we put

$$\omega^{*i}_j = \Gamma^{*i}_{j\lambda} dx^\lambda + \Gamma^{*i}_{jn} dy, \tag{35}$$

it is evident that $\Gamma^{*i}_{jk}$'s are Christoffel's symbols given by the following fundamental tensor $g_{ij}$ :

(36) $$g_{\lambda\mu} = f^{\iota}_{\lambda} f^{\iota}_{\mu}, \quad g_{\lambda n} = 0, \quad g_{nn} = 1.$$

Therefore we obtain from the above equations

$$-p^{\rho}_{a}\Gamma^{*n}_{\rho\mu} = \frac{y}{n-2}\left(-R_{ab}f^{b}_{\mu} + \frac{1}{2(n-1)}Rf^{a}_{\mu}\right).$$

Multiplying $f^{\iota}_{\lambda}$ and contracting we have

(27′) $$-\Gamma^{*n}_{\lambda\mu} = \frac{y}{n-2}\left(-R^{*}_{\lambda\mu} + \frac{1}{2(n-1)}R^{*}g_{\lambda\mu}\right).$$

However, as

(37) $$\Gamma^{*n}_{\lambda\mu} = \Gamma^{*}_{\lambda n\mu} = -\frac{1}{2}\frac{\partial g_{\mu\lambda}}{\partial y},$$

we know that for our space the following relations hold good with respect to natural frames:

$$g_{\lambda\mu} = g_{\lambda\mu}(x.y), \quad g_{\lambda n} = 0, \quad g_{nn} = 1,$$

$$R^{*}_{\lambda n} = 0, \; R^{*}_{nn} = \frac{1}{2(n-1)}R^{*},$$

and

(38) $$\frac{\partial g_{\lambda\mu}}{\partial y} = \frac{y}{n-2}\left(-2R^{*}_{\lambda\mu} + \frac{1}{n-1}R^{*}g_{\lambda\mu}\right).$$

(27′) can also be obtained from $(\omega^{a})' - [\omega^{b}\omega^{a}_{b}] - [dy\,\omega^{a}_{n}] = 0$ taking account of the coefficient of the term $[dy\,dx^{\lambda}]$.

4. Let $V_{n-1}(y)$ be the Riemannian space induced on the hypersurface $F_{n-1}(y) : y =$ const. from the Riemannian space $V_n$ defined by the metric $ds^2 = g_{ab}\,dx^a dx^b + dy\,dy$, and $h_{ab}(x\,,\,y)$ be the second fundamental tensor of $F_{n-1}(y)$. Then making use of the well known "D-Symbolik" we obtain

(39) $$h_{ab} = -B^{i}_{a}B\,D^{j}_{bi}\,n_j = -D_a n_b = \Gamma^{*n}_{ab} = -\frac{1}{2}\frac{\partial g_{ab}}{\partial y}$$

where $(n_i) = (0, 0, \cdots, 0, 1)$ mean components of the normal unit vector and $(B^{i}_{a}) = (\underbrace{0, 0, \cdots, 1}_{a}, 0, \cdots, 0)$ $(a = 1, 2, \cdots, n-1)$ mean those of tangent vectors of $F_{n-1}(y)$.

In the following, let $R_{abcd}$ be the components of Riemann tensor of $V_{n-1}$, then by means of the formulas of Gauss-Codazzi we have

(40) $$R^{*}_{abcd} = R_{abcd} - h_{ac}h_{bd} + h_{ad}h_{bc},$$

(41) $$R^{*}_{nabc} = D_c h_{ab} - D_b h_{ac} = h_{ab,\,c} - h_{ac,\,b}.$$

On the other hand, we get by (36) and (34)

$$(42)\qquad \begin{cases} \Gamma^{*a}_{bc} = \Gamma^{a}_{bc}, \quad \Gamma^{*n}_{ab} = h_{ab}, \\ \Gamma^{*a}_{nc} = g^{ab}\,\Gamma^{*}_{nbc} = \dfrac{1}{2}\, g^{ab}\, \dfrac{\partial g_{bc}}{\partial y} = -\, g^{ab}\, h_{bc} = -\, h^{a}_{c}, \\ \Gamma^{*n}_{nb} = \Gamma^{*b}_{nn} = \Gamma^{*n}_{nn} = 0, \end{cases}$$

hence we obtain

$$(43)\qquad \begin{cases} R^{*n}_{acn} = \dfrac{\partial \Gamma^{*n}_{bc}}{\partial y} - \dfrac{\partial \Gamma^{*n}_{bn}}{\partial x^{c}} + \Gamma^{*i}_{bc}\,\Gamma^{*n}_{in} - \Gamma^{*i}_{bn}\,\Gamma^{*n}_{ic} = \dfrac{\partial\, h_{bc}}{\partial y} + h^{a}_{b}\, h_{ac}, \\ R^{*}_{nn} = R^{*a}_{n\,na} = \dfrac{\partial \Gamma^{*a}_{nn}}{\partial x^{a}} - \dfrac{\partial \Gamma^{*a}_{na}}{\partial y} + \Gamma^{*i}_{nn}\,\Gamma^{*a}_{ia} - \Gamma^{*i}_{na}\,\Gamma^{*a}_{in} = \dfrac{\partial h}{\partial y} - h^{b}_{a}\, h^{a}_{b}, \end{cases}$$

where we have put $h = h^{a}_{a}$. Accordingly, making use of

$$(44)\qquad \begin{aligned} R_{ac} &= g^{bd}\, R^{*}_{abcd} + R^{*}_{ancn} \\ &= R_{ac} - h\, h_{ac} + h^{b}_{a}\, h_{bc} + \frac{\partial\, h_{ac}}{\partial y} + h^{b}_{a}\, h_{bc} \\ &= \frac{\partial\, h_{ac}}{\partial y} + R_{ac} + 2h^{b}_{a} h_{bc} - h\, h_{ac} \end{aligned}$$

and the relation

$$(39')\qquad \frac{\partial g^{ab}}{\partial y} = 2h^{ab},$$

which is deduced by (39), we obtain

$$\begin{aligned} R^{*} &= g^{ac}\, R^{*}_{ac} + R^{*}_{nn} \\ &= R + g^{ac}\, \frac{\partial\, h_{ac}}{\partial y} + 2h^{b}_{a} h^{a}_{b} - h^{2} + \frac{\partial h}{\partial y} h^{a}_{b} - h^{a}_{b} \\ &= R + \frac{\partial h}{\partial y} - 2h^{ac}\, h_{ac} + 2h^{b}_{a}\, h^{a}_{b} - h^{2} + \frac{\partial h}{\partial y} - h^{b}_{a}\, h^{a}_{b} \end{aligned}$$

that is,

$$(45)\qquad R^{*} = R - h^{2} - h^{b}_{a}\, h^{a}_{b} + 2\frac{\partial h}{\partial y}.$$

By (41) we have

$$R^{*}_{an} = R^{*a}_{c\,nc} = g^{bc}\,(h_{cb,\,a} - h_{ca,\,b})$$

or

$$(46)\qquad R^{*}_{an} = h_{,\,a} - h^{b}_{a,\,b}\ ,$$

where comma denotes covariant differentiation.

Making use of the above relations we will now write (29′), (30′) and (38) by means of the quantities of $V_{n-1}$. First (29′) turns by (46) into

$$(47)\qquad h^{b}_{a,\,b} - h_{,\,a} = 0,$$

and (30′) turns, by (43) and (45), into

$$\frac{\partial h}{\partial y} - h^{b}_{a}\, h^{a}_{b} = \frac{1}{2(n-1)}\left(R - h^{2} - h^{b}_{a}\, h^{a}_{b} + 2\frac{\partial h}{\partial y}\right),$$

that is

$$(48)\qquad \frac{\partial h}{\partial y}=\frac{1}{2(n-2)}\{R-h^2+(2n-3)h_a^b h_b^a\}.$$

(38) turns by (39), (44) and (45) also into

$$\begin{aligned}
-2h_{\lambda\mu} &= \frac{y}{n-2}\Big\{-2\Big(\frac{\partial}{\partial y}h_{\lambda\mu}+R_{\lambda\mu}+2h_\lambda^a h_{a\mu}-h\,h_{\lambda\mu}\Big)\\
&\qquad+\frac{1}{n-1}g_{\lambda\mu}\Big(2\frac{\partial h}{\partial y}+R-h^2-h_a^b h_b^a\Big)\Big\}\\
&= \frac{y}{n-2}\Big[-2\Big(\frac{\partial}{\partial y}h_{\lambda\mu}+R_{\lambda\mu}+2h_\lambda^b h_{b\mu}-h\,h_{\lambda\mu}\Big)\\
&\qquad+\frac{1}{n-1}g_{\lambda\mu}\Big\{\frac{1}{n-2}(R-h^2+(2n-3)h_a^b h_b^a)+R-h^2-h_a^b h_b^a\Big\}\Big]\\
&= \frac{y}{n-2}\Big\{-2\frac{\partial}{\partial y}h_{\lambda\mu}-2(R_{\lambda\mu}+2h_\lambda^b h^b{}_\mu-h\,h_{\lambda\mu})\\
&\qquad\qquad+\frac{1}{n-2}g_{\lambda\mu}(R-h^2+h_a^b h_b^a)\Big\},
\end{aligned}$$

that is

$$(49)\qquad \begin{aligned}\frac{\partial}{\partial y}h_{\lambda\mu} &= \frac{n-2}{y}h_{\lambda\mu}-R_{\lambda\mu}-2h_\lambda^b h_{b\mu}+h\,h_{\lambda\mu}\\
&\qquad+\frac{1}{2(n-2)}g_{\lambda\mu}(R-h^2+h_a^b h_b^a).\end{aligned}$$

However (48) must follow from (49), hence by virtue of (39′) we obtain

$$\begin{aligned}\frac{\partial h}{\partial y} &= \frac{\partial}{\partial y}(g^{\lambda\mu}h_{\lambda\mu})=2h_a^b h_b^a+\frac{n-2}{y}h-R-2h_a^b h_b^a-h^2\\
&\qquad+\frac{n-1}{2(n-2)}(R-h^2+h_a^b h_b^a),\end{aligned}$$

that is

$$(48')\qquad \frac{\partial h}{\partial y}=\frac{n-2}{y}h-\frac{1}{2(n-2)}\Big\{(n-3)(R-h^2)-(n-1)h_a^b h_b^a\Big\}.$$

The last equation must be equivalent to (48), hence we get

$$(50)\qquad \frac{n-2}{y}h=\frac{1}{2}(R-h^2+h_a^b h_b^a).$$

Consequently we see that the first and second fundamental tensors $g_{\lambda\mu}(x,y)$ and $h_{\lambda\mu}(x,y)$ of $F_{n-1}(y)$ are solutions of the differential equations (39) and (49) where the Riemann tensor of $V_{n-1}(y)$ appears as known terms:

$$(39)\qquad \frac{\partial}{\partial y}g_{\lambda\mu}=-2h_{\lambda\mu},$$

$$(49)\qquad \frac{\partial}{\partial y}h_{\lambda\mu}=\frac{n-2}{y}h_{\lambda\mu}-R_{\lambda\mu}-2h_\lambda^b h_{b\mu}+h\,h_{\lambda\mu}+\frac{g_{\lambda\mu}}{2(n-2)}(R-h^2+h_a^b h_b^a),$$

provided that they satisfy the auxiliary conditions

$$h^{b}_{\lambda,\,b} - h_{,\lambda} = 0 \tag{47}$$

and

$$h - \frac{y}{2(n-2)}(R - h^2 + h^{b}_{a} h^{a}_{b}) = 0. \tag{50}$$

Conversely, if we consider a Riemannian space $V_n$ with the fundamental tensor $g_{ij}$ such that the components $g_{\lambda\mu}$'s are solutions of the last equations and $g_{\lambda n} = 0, g_{nn} = 1$, then in the space with normal conformal connexion associated with $V_n$ the hypersurface $F_{n-1}$ (0) will obviously be the image of the fixed hypersphere $S_{n-1}$ $(= A_n + y A_\infty)$.

§ 2

5. The result of the last section yields the following problem: is it possible to imbed any Riemannian space $V_{n-1}$ in a suitable Riemannian space $V_n$ as the image of $S_{n-1}$? Otherwise in what conditions is it possible? We now pass to study these problems.

As we knew in § 1, the Riemannian space $V_n$ in consideration is conformal with some Einstein space whose scalar curvature is $< 0$. On the other hand, J. E. Campbell proved that "any Riemannian space $V_{n-1}$ can always be imbedded as a hypersurface in some Einstein space". However, the latter is of scalar curvature 0, as this will be shown in a following section. Consequently this hypersurface will not be the image of $S_{n-1}$ because in this case $S_{n-1}$ becomes a point sphere.

By means of (39′), (49) reduces to

$$\frac{\partial}{\partial y} h^{\nu}_{\lambda} = \frac{n-2}{y} h^{\nu}_{\lambda} - R^{\nu}_{\lambda} + h\, h^{\nu}_{\lambda} + \frac{1}{2(n-2)} \delta^{\nu}_{\lambda}(R - h^2 + h^{b}_{a} h^{a}_{b}). \tag{49′}$$

From now on let us replace $n-1$ by $n$ in order to simplify our calculation and consider the problem of imbedding a Riemannian space $V_n$ in an Einstein space $A_{n+1}$ as the image of $S_n$. Then our system of equations will be

$$\frac{\partial}{\partial y} g_{\lambda\mu} = -2\, h_{\lambda\mu}, \tag{51}$$

$$\frac{\partial}{\partial y} h^{\nu}_{\lambda} = \frac{n-1}{y} h^{\nu}_{\lambda} - R^{\nu}_{\lambda} + h\, h^{\nu}_{\lambda} + \frac{1}{2(n-1)} \delta^{\nu}_{\lambda}(R - h^2 + h^{b}_{a} h^{a}_{b}), \tag{52}$$

$$h^{b}_{\lambda,\,b} - h_{,\lambda} = 0, \tag{53}$$

$$h - \frac{y}{2(n-1)}(R - h^2 + h^{b}_{a} h^{a}_{b}) = 0 \tag{54}$$

$$(\lambda, \mu, \cdots ; a, b, \cdots = 1, 2, \cdots, n) \ (n > 1).$$

In order to study conditions (53) and (54), let us consider the quantities of the left sides of (53) and (54) in which $g_{\lambda\mu}$ and $h_{\lambda\mu}$ are a set of solutions of (51) and (52). Then we have

$$D_\mu \frac{\partial}{\partial y} h^\nu_\lambda = \frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial y} h^\nu_\lambda\right) + \Gamma^\nu_{\alpha\mu} \frac{\partial}{\partial y} h^\alpha_\lambda - \Gamma^\alpha_{\lambda\mu} \frac{\partial}{\partial y} h^\nu_\alpha$$

$$= \frac{\partial}{\partial y} h^\nu_{\lambda,\,\mu} - h^\alpha_\lambda \frac{\partial}{\partial y} \Gamma^\nu_{\alpha\mu} + h^\nu_\alpha \frac{\partial}{\partial y} \Gamma^\alpha_{\lambda\mu},$$

hence we get from (52)

$$\frac{\partial}{\partial y} h^a_{\lambda,\,a} = h^a_\lambda \frac{\partial}{\partial y} \Gamma^b_{ab} - h^b_a \frac{\partial}{\partial y} \Gamma^a_{\lambda b} + \frac{n-1}{y} h^a_{\lambda,\,a} - R^a_{\lambda,\,a} \tag{55}$$

$$+ h_{,\,a} h^a_\lambda + h\, h^a_{\lambda,\,a} + \frac{1}{2(n-1)}(R - 2h\, h_{,\,\lambda} + 2\, h^b_a h^a_{b,\,\lambda}).$$

On the other hand we have by (39) and (39′)

$$\frac{\partial}{\partial y} \Gamma^\nu_{\lambda\mu} = \frac{1}{2} \frac{\partial}{\partial y}\left\{g^{\nu a}\left(\frac{\partial g_{\lambda a}}{\partial x^\mu} + \frac{\partial g_{a\mu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^a}\right)\right\}$$

$$= 2\, h^{\nu a} \Gamma_{\lambda a\mu} - g^{\nu a}\left(\frac{\partial h_{\lambda a}}{\partial x^\mu} + \frac{\partial h_{a\mu}}{\partial x^\lambda} - \frac{\partial h_{\lambda\mu}}{\partial x^a}\right)$$

$$= 2\, h^\nu_a \Gamma^a_{\lambda\mu} - g^{\nu a}\{h_{\lambda a,\,\mu} + h_{ba} \Gamma^b_{\lambda\mu} + h_{\lambda b} \Gamma^b_{a\mu}$$

$$+ h_{a\mu,\,\lambda} + h_{b\mu} \Gamma^b_{a\lambda} + h_{ab} \Gamma^b_{\mu\lambda}$$

$$- h_{\lambda\mu,\,a} - h_{b\mu} \Gamma^b_{\lambda a} - h_{\lambda b} \Gamma^b_{\mu a}\},$$

that is

$$\frac{\partial}{\partial y} \Gamma^\nu_{\lambda\mu} = g^{\nu a} h_{\lambda\mu,\,a} - h^\nu_{\lambda,\,\mu} - h^\nu_{\mu,\,\lambda} \tag{56}$$

and further we get

$$\frac{\partial}{\partial y} \Gamma^b_{\lambda b} = -h_{,\,\lambda}. \tag{57}$$

If we make use of (56) and (57), (55) takes the form

$$\frac{\partial}{\partial y} h^a_{\lambda,\,a} = -h^a_\lambda h_{,\,a} - h^b_a (g^{ac} h_{\lambda b,\,c} - h^a_{\lambda,\,b} - h^a_{b,\,\lambda}) + \frac{n-1}{y} h^a_{\lambda,\,a}$$

$$- R^a_{\lambda,\,a} + h_{,\,a} h^a_\lambda + h\, h^a_{\lambda,\,a} + \frac{1}{2(n-1)}(R_{,\,\lambda} - 2h\, h_{,\,\lambda} + 2h^b_a h^a_{b,\,\lambda}),$$

hence we get

$$\frac{\partial}{\partial y} h^a_{\lambda,\,a} = \frac{n-1}{y} h^a_{\lambda,\,a} - R^a_{\lambda,\,a} + h\, h^a_{\lambda,\,a} \tag{58}$$

$$+ \frac{1}{2(n-1)}(R_{,\,\lambda} - 2h\, h_{,\,\lambda} + 2h^b_a h^a_{b,\,\lambda}).$$

Now from (51) and (52) we get

$$\frac{\partial}{\partial y} h = \frac{n-1}{y} h - \frac{1}{2(n-1)}\left\{(n-2)(R - h^2) - n\, h^b_a h^a_b\right\}, \tag{59}$$

accordingly

$$\frac{\partial}{\partial y} h_{,\,\lambda} = \left(\frac{\partial h}{\partial y}\right)_{,\,\lambda} \tag{60}$$

$$= \frac{n-1}{y} h_{,\lambda} - \frac{1}{2(n-1)}\left\{(n-2)(R_{,\,\lambda} - 2h\, h_{,\,\lambda}) - 2n\, h^b_a h^a_{b,\,\lambda}\right\},$$

hence we get from (58) and (60)

$$\frac{\partial}{\partial y}(h^a_{\lambda,a} - h_{,a}) = \frac{n-1}{y}(h^a_{\lambda,a} - h_{,\lambda}) + R^a_{\lambda,a} - \frac{1}{2}R_{,\lambda} + h(h^a_{\lambda,a} - h_{,\lambda}).$$

As it is easily seen from the Bianchi's identity, $R^a_{\lambda,a} = \frac{1}{2}R_{,\lambda}$, we have the relation

$$\frac{\partial}{\partial y}(h^a_{\lambda,a} - h_{,\lambda}) = \left(\frac{n-1}{y} + h\right)\left(h^a_{\lambda,a} - h_{,\lambda}\right). \tag{61}$$

In the next place let us consider the quantity of the left hand side of (54). Making use of geodesic normal coordinate systems, we see, by virtue of (56) and (57), that

$$\begin{aligned}\frac{\partial}{\partial y}R &= 2h^{\lambda\mu}R_{\lambda\mu} + g^{\lambda\mu}\frac{\partial}{\partial y}\left(\frac{\partial \Gamma^a_{\lambda\mu}}{\partial x^a} - \frac{\partial \Gamma^a_{\lambda a}}{\partial x^\mu} + \Gamma^b_{\lambda\mu}\Gamma^a_{ba} - \Gamma^b_{\lambda a}\Gamma^a_{b\mu}\right)\\ &= 2h^\mu_\lambda R^\lambda_\mu + g^{\lambda\mu}(g^{ab}h_{\lambda\mu,b} - h^a_{\lambda,\mu} - h^a_{\mu,\lambda})_{,a}\\ &\quad - g^{\lambda\mu}(g^{ab}h_{\lambda a,b} - h^a_{\lambda,a} - h_{,\lambda})_{,\mu},\end{aligned}$$

that is

$$\frac{\partial}{\partial y}R = 2(h^b_a R^a_b + g^{ab}h_{,ab} - h^{ab}{}_{,ab}). \tag{62}$$

Hence, by means of (52), (59) and (61) we obtain

$$\begin{aligned}&\frac{\partial}{\partial y}\left\{h - \frac{y}{2(n-1)}(R - h^2 + h^b_a h^a_b)\right\}\\ &= \frac{n-1}{y}h - \frac{1}{2(n-1)}\left\{(n-2)(R-h^2) - n\,h^b_a h^a_b\right\}\\ &\quad - \frac{1}{2(n-1)}(R - h^2 - h^b_a h^a_b) - \frac{y}{n-1}(h^b_a R^a_b + g^{ab}h_{,ab} - h^{ab}{}_{,ab})\\ &\quad + \frac{y}{n-1}h\left[\frac{n-1}{y}h - \frac{1}{2(n-1)}\left\{(n-2)(R-h^2) - n\,h^b_a h^a_b\right\}\right]\\ &\quad - \frac{y}{n-1}h^\lambda_\nu\left\{\frac{n-1}{y}h^\nu_\lambda - R^\nu_\lambda + h\,h^\nu_\lambda + \frac{1}{2(n-1)}\delta^\nu_\lambda(R - h^2 + h^b_a h^a_b)\right\},\end{aligned}$$

that is, an analogous equation to (61)

$$\begin{aligned}&\frac{\partial}{\partial y}\left\{h - \frac{y}{2(n-1)}(R - h^2 + h^b_a h^a_b)\right\}\\ &= \left(\frac{n-1}{y} + h\right)\left\{h - \frac{y}{2(n-1)}(R - h^2 + h^b_a h^a_b)\right\}\\ &\quad + \frac{y}{n-1}g^{ab}(h^c_{b,c} - h_{,b})_{,a}.\end{aligned} \tag{63}$$

Thus we get the necessary equations (61) and (63), but these equations show that the differential equations of our problem do not belong to the familiar type in differential geometries in which solutions $g_{\lambda\mu}$ and $h_{\lambda\mu}$ are asked for under the initial conditions such that $h(x,0)=0$, and $h^{\alpha}_{\lambda,\alpha}(x,0)=0$. In the next section we shall change these apparently irregular equations into another ones which are easy to deal with.

**6.** First we get by virtue of (52) the relation $h^{\nu}_{\lambda}(0)=0$. Noticing that the solutions of the differential equation

$$\frac{\partial}{\partial y}\varphi=\frac{n-1}{y}\varphi+a(x),\quad (n>2)$$

are of the form $\varphi=y^{n-1}F(x)-\dfrac{y}{n-2}a(x)$, let us put

$$h^{\nu}_{\lambda}=y\left\{f^{\nu}_{\lambda}+\frac{1}{n-2}\left(R^{\nu}_{\lambda}-\frac{R}{2(n-1)}\delta^{\nu}_{\lambda}\right)\right\}. \tag{64}$$

Then (52) turns into the differential equation with respect to $f^{\nu}_{\lambda}$ i.e.

$$\frac{\partial}{\partial y}h^{\nu}_{\lambda}=f^{\nu}_{\lambda}+\frac{1}{n-2}\left(R^{\nu}_{\lambda}-\frac{R}{n-2}\delta^{\nu}_{\lambda}\right)+y\frac{\partial}{\partial y}f^{\nu}_{\lambda}+\frac{y}{n-2}\left(\frac{\partial}{\partial y}R^{\nu}_{\lambda}-\frac{1}{2(n-1)}\delta^{\nu}_{\lambda}\frac{\partial R}{\partial y}\right).$$

Let us now calculate $\dfrac{\partial}{\partial y}R^{\nu}_{\lambda}$ in the right hand side of the last equation. Making use of the equations which appear in the calculation of $\dfrac{\partial R}{\partial y}$ in (62), we get

$$\frac{\partial}{\partial y}R_{\lambda\mu}=g^{ab}h_{\lambda\mu,ba}-h^{\alpha}_{\lambda,\mu\alpha}-h^{\alpha}_{\mu,\lambda\alpha}+h_{,\lambda\mu}$$

or

$$\frac{\partial}{\partial y}R^{\nu}_{\lambda}=2R^{\alpha}_{\lambda}h^{\nu}_{\alpha}+\Delta_2h^{\nu}_{\lambda}-h^{\alpha}_{\lambda,}{}^{\nu}{}_{\alpha}-h^{\nu}_{\alpha,\lambda\alpha}+h_{,\lambda}{}^{\nu}, \tag{65}$$

where $\Delta_2$ means the second differential parameter of Beltrami. By (64) we have

$$h=y\left(f+\frac{R}{2(n-1)}\right),\quad f=f^{\alpha}_{\alpha}, \tag{66}$$

hence we get

$$\begin{aligned}\frac{\partial}{\partial y}h^{\nu}_{\lambda}=&\;y\frac{\partial}{\partial y}f^{\nu}_{\lambda}+f^{\nu}_{\lambda}+\frac{1}{n-2}\left(R^{\nu}_{\lambda}-\frac{R}{2(n-1)}\delta^{\nu}_{\lambda}\right)\\&+\frac{y^2}{n-2}\left[2\left\{R^{\alpha}_{\lambda}f^{\nu}_{\alpha}+\frac{1}{n-2}\left(R^{\alpha}_{\lambda}R^{\nu}_{\alpha}-\frac{R}{2(n-1)}\delta^{\nu}_{\lambda}\right)\right\}\right.\\&+\left\{\Delta_2f^{\nu}_{\lambda}+\frac{1}{n-2}\left(\Delta_2R^{\nu}_{\lambda}-\frac{1}{2(n-1)}\Delta_2R\,\delta^{\nu}_{\gamma}\right)\right\}\\&-\left\{f^{\alpha}_{\lambda,}{}^{\nu}{}_{\alpha}+\frac{1}{n-2}\left(R^{\alpha}_{\lambda,}{}^{\nu}{}_{\alpha}-\frac{1}{2(n-1)}R,{}^{\nu}{}_{\lambda}\right)\right\}\\&-\left\{f^{\nu}_{\alpha,\lambda}{}^{\alpha}+\frac{1}{n-2}\left(R^{\nu}_{\alpha,\lambda}{}^{\alpha}-\frac{1}{2(n-1)}R,{}_{\lambda}{}^{\nu}\right)\right\}\\&-\frac{y^2}{(n-1)(n-2)}\delta^{\nu}_{\lambda}\left[R^{b}_{a}f^{a}_{b}+\frac{1}{n-2}\left(R^{b}_{a}R^{a}_{b}-\frac{R^2}{2(n-1)}\right)+\Delta_2f\right.\end{aligned}$$

$$+\frac{1}{2(n-1)}\Delta_2 R - f^{ab}{}_{,ab} - \frac{1}{n-2}R^{ab}{}_{,ab} + \frac{\Delta_2 R}{2(n-1)(n-2)}\Bigg].$$

On the other hand, we have from the right hand side of (52)

$$\frac{\partial}{\partial y}h^\nu_\lambda = (n-1)\left\{f^\nu_\lambda + \frac{1}{n-2}\left(R^\nu_\lambda - \frac{R}{2(n-1)}\delta^\nu_\lambda\right)\right\} - R^\nu_\lambda +$$

$$+ y^2\left(f + \frac{R}{2(n-1)}\right)\left\{f^\nu_\lambda + \frac{1}{n-2}\left(R^\nu_\lambda - \frac{R}{2(n-1)}\delta^\nu_\lambda\right)\right\} + \frac{R}{2(n-1)}\delta^\nu_\lambda$$

$$- \frac{y^2}{2(n-1)}\delta^\nu_\lambda\left(f + \frac{R}{2(n-1)}\right)^2 + \frac{y^2}{2(n-1)}\delta^\nu_\lambda\Bigg[f^b_a f^a_b + \frac{2}{n-2}\Big(f^b_a R^a_b -$$

$$- \frac{Rf}{2(n-1)} + \frac{1}{(n-2)^2}\left\{R^b_a R^a_b - \frac{R^2}{n-1} + \frac{nR^2}{4(n-1)^2}\right\}\Bigg],$$

hence we obtain

$$\frac{\partial}{\partial y}f^\nu_\lambda = \frac{n-2}{y}f^\nu_\lambda - \frac{y}{n-2}\Bigg[\frac{1}{n-2}\Big\{2R^a_\lambda R^\nu_a - \frac{R}{n-1}R^\nu_\lambda + \Delta_2 R^\nu_\lambda -$$

$$- \frac{\Delta_2 R}{2(n-1)}\delta^\nu_\lambda - R^a_{\lambda,}{}^\nu{}_a + \frac{1}{2(n-1)}R,{}^\nu{}_\lambda - R^\nu_{a,\lambda}{}^a + \frac{1}{2}R,{}_\lambda{}^\nu\Big\}$$

$$+ 2R^a_\lambda f^\nu_a + \Delta_2 f^\nu_\lambda - f^a_{\lambda,}{}^\nu{}_a - f^\nu_{a,\lambda}{}^a + f,{}_\lambda{}^\nu\Bigg]$$

$$+ \frac{y}{(n-1)(n-2)}\delta^\nu_\lambda\Bigg[\frac{1}{n-2}\left\{R^b_a R^a_b - \frac{R^2}{2(n-1)} + \frac{1}{2}\Delta_2 R - R^{ab}{}_{,ab}\right\}$$

$$+ R^b_a f^a_b + \Delta_2 f - f^{ab}{}_{,ab}\Bigg]$$

$$+ y\Bigg[\left\{\frac{R}{2(n-1)(n-2)}\left(R^\nu_\lambda - \frac{R}{2(n-1)}\delta^\nu_\lambda\right) - \frac{R^2}{8(n-1)^3}\delta^\nu_\lambda\right.$$

$$\left. + \frac{R^b_a R^a_b \delta^\nu_\lambda}{2(n-1)(n-2)^3} - \frac{3n-4}{8(n-1)^3(n-2)^2}R^2\delta^\nu_\lambda\right\} + f f^\nu_\lambda + \frac{R}{2(n-1)}f^\nu_\lambda$$

$$+ \frac{f}{n-2}\left(R^\nu_\lambda - \frac{R}{2(n-1)}\delta^\nu_\lambda\right) - \frac{f^2}{2(n-1)}\delta^\nu_\lambda - \frac{fR}{2(n-1)^2}\delta^\nu_\lambda$$

$$+ \frac{1}{2(n-1)}\delta^\nu_\lambda\left\{f^b_a f^a_b + \frac{2}{n-2}\left(f^b_a R^a_b - \frac{Rf}{2(n-1)}\right)\right\}\Bigg].$$

If we make use of $\Delta_2 R - 2R^{ab}{}_{,ab} = 0$, the last equation reduces to

$$(67)\qquad \frac{\partial}{\partial y}f^\nu_\lambda = \frac{n-2}{y}f^\nu_\lambda - \frac{y}{(n-2)^2}\Bigg[2R^a_\lambda R^\nu_a - \frac{n}{2(n-1)}R R^\nu_\lambda + \Delta_2 R^\nu_\lambda +$$

$$+ \frac{1}{2(n-1)}R,{}^\nu{}_\lambda + \frac{1}{2}R,{}_\lambda{}^\nu - R^a_{\lambda,}{}^\nu{}_a - R^\nu_{a,\lambda}{}^a$$

$$- \frac{1}{2(n-1)}\delta^\nu_\lambda\left\{3R^b_a R^a_b - \frac{3n}{4(n-1)}R^2 + \Delta_2 R\right\}\Bigg]$$

$$+ y\Bigg[-\frac{2}{n-2}R^a_\lambda f^\nu_a + \frac{1}{2(n-1)}Rf^\nu_\lambda + \frac{1}{n-2}R^\nu_\lambda f + f f^\nu_\lambda$$

$$- \frac{1}{n-2}(\Delta_2 f^\nu_\lambda - f^a_{\lambda,}{}^\nu{}_a - f^\nu_{a,\lambda}{}^a + f,{}_\lambda{}^\nu)$$

$$+\frac{1}{n-1}\delta^\nu_\lambda\left\{\frac{1}{n-2}(2R^b_a f^a_b - Rf) + \frac{1}{2}(f^b_a f^a_b - f^2) + \right.$$
$$\left.\left. + \frac{1}{n-2}(\Delta_2 f - f^{ab}{}_{,ab})\right\}\right].$$

Thus we have an analogous equation to (52) with respect to $f^\nu_\lambda$. It is noted that the coefficient of $f^\nu_\lambda$ is now $\frac{n-2}{y}$ while that of $h^\nu_\lambda$ in (52) is $\frac{n-1}{y}$. Applying the same method successively this coefficient may be reduced to $\frac{1}{y}$. Let us discuss it generally in the next section.

7. By the last section we may set $h^\nu_\lambda(x,y)$ in the form

$$h^\nu_\lambda = \sum_{i=1}^{n-1} y^i \underset{(i)}{H^\nu_\lambda}, \quad (y^i = (y)^i) \tag{68}$$

where $\underset{(i)}{H^\nu_\lambda}(x:y)$ $(i = 1, 2, \cdots, n-2)$ are polynomials of $g_{ab}$, $R^b_a$ and their covariant derivatives.

Indeed, substituting (68) in (52), we get

$$\begin{aligned}\frac{\partial}{\partial y}h^\nu_\lambda &= y^{n-1}\frac{\partial}{\partial y}\underset{(n-1)}{H^\nu_\lambda} + \sum_{n=1}^{n-2} y^i\left\{(i+1)\underset{(i+1)}{H^\nu_\lambda} + \frac{\partial}{\partial y}\underset{(i)}{H^\nu_\lambda}\right\} + \underset{(1)}{H^\nu_\lambda} \\ &= (n-1)\sum_{i=0}^{n-2} y^i \underset{(i+1)}{H^\nu_\lambda} - R^\nu_\lambda + \frac{R}{2(n-1)}\delta^\nu_\lambda + \\ &+ \sum_{k=2}^{2n-2} y^k \sum_{i+j=k}\left\{\underset{(i)}{H}\underset{(j)}{H^\nu_\lambda} + \frac{1}{2(n-1)}\delta^\nu_\lambda(\underset{(i)}{H^b_a}\underset{(j)}{H^a_b} - \underset{(i)}{H}\underset{(j)}{H})\right\}.\end{aligned} \tag{69}$$

Hence our object will be to define $\underset{(i)}{H^\nu_\lambda}$ $(i = 1, 2, \cdots\cdots, n-2)$ suitably so that they satisfy (69) and the equations of $\underset{(n-1)}{H^\nu_\lambda}$ as unknown take forms as simple as possible. First let us put

$$\frac{\partial}{\partial y}\underset{(i)}{H^\nu_\lambda} = \sum_{j\geq 0} y^j \underset{(i,j)}{K^\nu_\lambda}. \tag{70}$$

Then the terms apparently constant with respect to $y$ on both sides of (69) cancel out when we define $\underset{(1)}{H^\nu_\lambda}$ by the relation

$$\underset{(1)}{H^\nu_\lambda} = (n-1)\underset{(1)}{H^\nu_\lambda} - R^\nu_\lambda + \frac{R}{2(n-1)}\delta^\nu_\lambda,$$

or

$$\underset{(1)}{H^\nu_\lambda} \equiv \frac{1}{n-2}\left(R^\nu_\lambda - \frac{R}{2(n-1)}\delta^\nu_\lambda\right). \tag{71}$$

From the last equation we gain at once

$$\underset{(1)}{H} \equiv \underset{(1)}{H^a_a} = \frac{R}{2(n-1)}. \tag{72}$$

Next, from the coefficients of $y$ we shall define $\underset{(2)}{H^\nu_\lambda}$ by

$$2\underset{(2)}{H}{}^{\nu}_{\lambda} + \underset{(1,0)}{K}{}^{\nu}_{\lambda} = (n-1)\underset{(2)}{H}{}^{\nu}_{\lambda},$$

that is,

$$\underset{(2)}{H}{}^{\nu}_{\lambda} = \frac{1}{n-3}\underset{(1,0)}{K}{}^{\nu}_{\lambda}.$$

In the last equation we assume that $\underset{(1,0)}{K}{}^{\nu}_{\lambda}$ has already been determined. In order to determine $\underset{(1,0)}{K}{}^{\nu}_{\lambda}$ we notice, by means of (71), (62) and (65), that

$$\frac{\partial}{\partial y}\underset{(1)}{H}{}^{\nu}_{\lambda} = \frac{1}{n-2}\Big\{2R^{a}_{\lambda}h^{\nu}_{a} + \Delta_2 h^{\nu}_{\lambda} - h^{a}_{\lambda},{}^{\nu}{}_{a} - h^{\nu}_{a},{}_{\lambda}{}^{a} + h,{}_{\lambda}{}^{\nu}$$
$$- \frac{1}{n-1}\delta^{\nu}_{\lambda}(h^{b}_{a}R^{a}_{b} + \Delta_2 h - h^{ab},{}_{ab})\Big\}.$$

Substituting (63) into the last equation and comparing with (70) $(i=1)$ we see that we may put

$$\underset{(1,0)}{K}{}^{\nu}_{\lambda} = 0.$$

Consequently we get

(73) $$\underset{(2)}{H}{}^{\nu}_{\lambda} = 0,$$

whence we get $\underset{(2,1)}{K}{}^{\nu}_{\lambda} = 0$ by virtue of (70) $(i=2)$,

Comparing coefficients of $y^2$ in both sides of (69) we see that we may put

$$3\underset{(3)}{H}{}^{\nu}_{\lambda} + \underset{(1,1)}{K}{}^{\nu}_{\lambda} = (n-1)\underset{(3)}{H}{}^{\nu}_{\lambda} + \underset{(1)}{H}\,\underset{(1)}{H}{}^{\nu}_{\lambda}$$
$$+ \frac{1}{2(n-1)}\delta^{\nu}_{\lambda}(\underset{(1)}{H}{}^{b}_{a}\underset{(1)}{H}{}^{a}_{b} - \underset{(1)}{H}\,\underset{(1)}{H}),$$

or

$$\underset{(3)}{H}{}^{\nu}_{\lambda} = \frac{1}{n-4}\Big\{\underset{(1,1)}{K}{}^{\nu}_{\lambda} - \underset{(1)}{H}\,\underset{(1)}{H}{}^{\nu}_{\lambda} + \frac{1}{2(n-1)}\delta^{\nu}_{\lambda}(\underset{(1)}{H}\,\underset{(1)}{H} - \underset{(1)}{H}{}^{b}_{a}\underset{(1)}{H}{}^{a}_{b})\Big\}.$$

In the last equation we assume that $\underset{(1,1)}{K}{}^{\nu}_{\lambda}$ has already been defined. However from the coefficients of $y$ in $\frac{\partial}{\partial y}\underset{(1)}{H}{}^{\nu}_{\lambda}$ we know that we may define

$$\underset{(1,1)}{K}{}^{\nu}_{\lambda} = \frac{1}{n-2}\Big\{2R^{a}_{\lambda}\underset{(1)}{H}{}^{\nu}_{a} + \Delta_2\underset{(1)}{H}{}^{\nu}_{\lambda} - \underset{(1)}{H}{}^{a}_{\lambda},{}^{\nu}{}_{a} - \underset{(1)}{H}{}^{\nu}_{a},{}_{\lambda}{}^{a} + \underset{(1)}{H},{}_{\lambda}{}^{\nu}$$
$$- \frac{1}{n-1}\delta^{\nu}_{\lambda}(\underset{(1)}{H}{}^{b}_{a}R^{a}_{b} + \Delta_2\underset{(1)}{H} - \underset{(1)}{H}{}^{ab},{}_{ab})\Big\}$$
$$= \frac{1}{(n-2)^2}\Big\{2\Big(R^{a}_{\lambda}R^{\nu}_{a} - \frac{1}{2(n-1)}R\,R^{\nu}_{\lambda}\Big) + \Delta_2 R - \frac{\Delta_2 R}{2(n-1)}\delta^{\nu}_{\lambda}$$
$$- R^{a}_{\lambda},{}^{\nu}{}_{a} - R^{\nu}_{a},{}_{\lambda}{}^{a} + \frac{1}{2(n-1)}(R,{}^{\nu}{}_{\lambda} + R,{}_{\lambda}{}^{\nu})\Big\}$$
$$+ \frac{1}{2(n-1)(n-2)}R,{}_{\lambda}{}^{\nu} - \frac{1}{(n-1)(n-2)}\delta^{\nu}_{\lambda}\Big\{\frac{1}{n-2}\Big(R^{b}_{a}R^{a}_{b} - \frac{R^2}{2(n-1)}\Big)$$
$$+ \frac{1}{2(n-1)}\Delta_2 R - \frac{1}{n-2}\Big(R^{ab},{}_{ab} - \frac{1}{2(n-1)}\Delta_2 R\Big)\Big\}$$

or

$$\underset{(1,1)}{K}{}^{\nu}_{\lambda} = \frac{1}{(n-2)^2}\Big\{2R^a_\lambda R^\nu_a - \frac{1}{n-1}RR^\nu_\lambda + \Delta_2 R^\nu_\lambda + \frac{1}{2(n-1)}R,{}_\lambda{}^\nu$$
$$+\frac{1}{2}R,{}_\lambda{}^\nu - R^a_\lambda,{}^\nu{}_a - R^\nu_a,{}_\lambda{}^a\Big\}$$
$$-\frac{1}{(n-1)(n-2)^2}\delta^\nu_\lambda\Big\{R^b_a R^a_b - \frac{1}{2(n-1)}R^2 - R^{ab},{}_{ab} + \Delta_2 R\Big\}.$$

By substitution we obtain $\underset{(3)}{H}{}^\nu_\lambda$ of the form

$$(74)\quad \underset{(3)}{H}{}^\nu_\lambda = \frac{1}{(n-2)^2(n-4)}\Big\{2R^a_\lambda R^\nu_a - \frac{n}{2(n-1)}R^2 + \Delta_2 R^\nu_\lambda + \frac{1}{2(n-1)}R,{}^\nu_\lambda$$
$$+\frac{1}{2}R,{}_\lambda{}^\nu - R^a_\lambda,{}^\nu{}_a - R^\nu_a,{}_\lambda{}^a\Big\}$$
$$-\frac{1}{2(n-1)(n-2)^2(n-4)}\delta^\nu_\lambda\Big\{3R^b_a R^a_b - \frac{3nR^2}{4(n-1)} + \Delta_2 R\Big\}.$$

In the following suppose inductively that we have obtained

$$\underset{(i)}{H}{}^\nu_\lambda = \underset{(i)}{H}{}^\nu_\lambda(g^{ab};R^b_a;R^b_{a,\rho_1}\cdots;R^b_{a,\rho_1\cdots\rho_{j-1}}),$$
$$\underset{(s,j-s)}{K}{}^\nu_\lambda = \underset{(s,j-s)}{K}{}^\nu_\lambda(g^{ab};R^b_a;R^b_{a,\rho_1};\cdots;R^b_{a,\rho_1\cdots\rho_j}),$$
$$(i=1,2,\cdots,p\leqq n-3;\ j=1,2,\cdots,p-1;\ s=1,2,\cdots,j),$$

which are polynomials of $g_{ab}$, $R^b_a$ and their covariant derivatives, so that the coefficients of $y^{i-1}$ in (69) are zero. Then we shall define $\underset{(p+1)}{H}{}^\nu_\lambda$, taking account of the cofficients of $y^p$ in (69), by

$$(p+1)\underset{(p+1)}{H}{}^\nu_\lambda + \sum_{s=1}^{p}\underset{(s,p-s)}{K}{}^\nu_\lambda = (n-1)\underset{(p+1)}{H}{}^\nu_\lambda + \sum_{i=1}^{p-1}\Big\{\underset{(i)}{H}\underset{(p-i)}{H}{}^\nu_\lambda +$$
$$+\frac{1}{2(n-1)}\delta^\nu_\lambda(\underset{(i)}{H}{}^b_a\underset{(p-i)}{H}{}^a_b - \underset{(i)}{H}\underset{(p-i)}{H})\Big\}$$

or

$$(75)\qquad \underset{(p+1)}{H}{}^\nu_\lambda = \frac{1}{n-p-2}\Big[\sum_{s=1}^{p}\underset{(s,p-s)}{K}{}^\nu_\lambda - \sum_{i=1}^{p-1}\Big\{\underset{(i)}{H}\underset{(p-i)}{H}{}^\nu_\lambda +$$
$$+\frac{1}{2(n-1)}\delta^\nu_\lambda(\underset{(i)}{H}{}^b_a\underset{(p-i)}{H}{}^a_b - \underset{(i)}{H}\underset{(p-i)}{H})\Big\}\Big].$$

In the last equation we assume that $\underset{(s,p-s)}{K}{}^\nu_\lambda$ has been defined, $\underset{(s)}{H}{}^\nu_\lambda$'s have already been defined by our assumption, since $1\leqq s\leqq p$. Now we get

$$\frac{\partial}{\partial y}\underset{(s)}{H}{}^\nu_\lambda = \sum_{a\leqq b}(\partial\underset{(s)}{H}{}^\nu_\lambda/\partial g^{ab})\,2h^{ab} + \sum_{\substack{a,b\\ k=0,1,,s-1}}(\partial\underset{(s)}{H}{}^\nu_\lambda/\partial R^a_{b,\rho_1\cdots\rho_k})\frac{\partial}{\partial y}R^a_{b,\rho_1\cdots\rho_k},$$

while for $k\geqq 1$

$$\frac{\partial}{\partial y}R^a_{b,\rho_1\cdots\rho_k} = \Big(\frac{\partial}{\partial y}R^a_{b,\rho_1\cdots\rho_{k-1}}\Big),{}_{\rho_k} + R^c_{b,\rho_1\cdots\rho_{k-1}}\frac{\partial}{\partial y}\Gamma^a_{c\rho_k}$$

$$- R^a_{c,\,\rho_1\cdots\rho_{k-1}}\frac{\partial}{\partial y}\Gamma^c_{b\rho_k} - \sum_{r=1}^{k-1} R^a_{b,\,\rho_1\cdots\rho_{r-1}c\,\rho_{r+1}\cdots\rho_{k-1}}\frac{\partial}{\partial y}\Gamma^c_{\rho_r\rho_k}.$$

On the other hand, we already knew in (56) and (65) that the following relations hold good:

$$\frac{\partial}{\partial y}\Gamma^\nu_{\lambda\mu} = h_{\lambda\mu,}{}^\nu - h^\nu_{\lambda,\,\mu} - h^\nu_{\mu,\,\lambda},$$

$$\frac{\partial}{\partial y}R^\nu_\lambda = 2R^a_\lambda h^\nu_a + \Delta_2 h^\nu_\lambda - h^a_{\lambda,}{}^\nu{}_a - h^\nu_{a,\,\lambda}{}^a + h,_\lambda{}^\nu.$$

Accordingly it follows inductively that $\frac{\partial}{\partial y}R^\nu_{\lambda,\,\rho_1\cdots\rho_k}$ are linear forms of $h^a_b, h^a_{b,\,\alpha_1}, \cdots, h^a_{b,\,\alpha_1\cdots\alpha_{k+2}}$ whose coefficients are polynomials of $R^a_b, R^a_{b,\,\alpha_1}, \cdots, R^a_{b,\,\alpha_1\cdots\alpha_k}$. Consequently, comparing the above equations with (70), we may define, by means of the known quantities, $\underset{(s,p-s)}{K}{}^\nu_\lambda$ as follows:

$$\underset{(s,p-s)}{K}{}^\nu_\lambda = \sum_{a\leqq b} 2\,(\partial \underset{(s)}{H}{}^\nu_\lambda/\partial g^{b})\,\underset{(p-s)}{H}{}^{ab} \tag{76}$$
$$+\sum_{\substack{a,b\\k=0,1,s-1}} (\partial \underset{(s)}{H}{}^\nu_\lambda/\partial R^a_{b,\,\rho_1\cdots\rho_k}) f^{b\,\rho_1\cdots\rho_k}_a,$$

where $f^{b\,\rho_1\cdots\rho_k}_a$ are linear forms of $\underset{(p-s)}{H}{}^c_e, \underset{(p-s)}{H}{}^c_{e,\,\alpha_1}, \cdots\cdots, \underset{(p-s)}{H}{}^c_{e,\,\alpha_1\cdots\alpha_{k+2}}$ whose coefficients are polynomials of $R^c_e, \cdots\cdots, R^c_{e,\,\alpha_1\cdots\alpha_k}$, accordingly $\underset{(s,p-s)}{K}{}^\nu_\lambda$ are polynomials of $g^{ab}, R^a_b, \cdots, R^a_{b,\,\rho_1\rho_2\cdots\rho_k}$. Hence we see that $\underset{(p+1)}{H}{}^\nu_\lambda$ can be defined by (75) without any ambiguity. Thus we can define successively $\underset{(1)}{H}{}^\nu_\lambda, \underset{(2)}{H}{}^\nu_\lambda, \cdots, \underset{(n-2)}{H}{}^\nu_\lambda$ so that the apparent coefficients of $y, y^2, \cdots, y^{n-3}$ in (69) are all zero.

We see that the apparent coefficients of $y^{n-2}$ on both sides of (69) are

$$(n-1)\underset{(n-1)}{H}{}^\nu_\lambda + \sum_{s=1}^{n-2}\underset{(s,n-2-s)}{K}{}^\nu_\lambda$$

and

$$(n-1)\underset{(n-1)}{H}{}^\nu_\lambda + \sum_{s=1}^{n-3}\left\{\underset{(s)}{H}\,\underset{(n-2-s)}{H}{}^\nu_\lambda + \frac{1}{2(n-1)}\delta^\nu_\lambda(H^b_a\underset{(s)}{\ }\underset{(n-2-s)}{H}{}^a_b - \underset{(s)}{H}\,\underset{(n-2-s)}{H})\right\},$$

then, let us define a tensor

$$L^\nu_\lambda = \sum_{s=1}^{n-3}\left\{-\underset{(s,n-2-s)}{K}{}^\nu_\lambda + \underset{(s)}{H}\,\underset{(n-2-s)}{H}{}^\nu_\lambda + \right. \tag{77}$$
$$\left. + \frac{1}{2(n-1)}\delta^\nu_\lambda(\underset{(s)}{H}{}^b_a\underset{(n-2-s)}{H}{}^a_b - \underset{(s)}{H}\,\underset{(n-2-s)}{H})\right\}$$

which depends on $R^a_b, \cdots, R^a_{b,\,\rho_1\cdots\rho_{n-2}}$. Although $\underset{(s,n-2-s)}{K}{}^\nu_\lambda$ is out of the above extent, it is evident that (76) is also applicable in this case.

From (76) and (72) we see that $\underset{(i,2)}{K}{}^\nu_\lambda = 0$ $(i \geqq 1)$. Moreover, there hold generally the following relations·

(78) $$\underset{(2i)}{H}{}^{\nu}_{\lambda}=0,\quad \underset{(2i,j)}{K}{}^{\nu}_{\lambda}=0,\quad \underset{(j,2i)}{K}{}^{\nu}_{\lambda}=0,$$

which are easily proved by induction by virtue of (76) and (75). Accordingly, it follows the relation $L^{\nu}_{\lambda}=0$ for $n$ = odd.

Making use of the auxiliary quantities $\underset{(i)}{H}{}^{\nu}_{\lambda}$ $(i=1,2,\cdots,n-2)$, let us rewrite (52). We get by virtue of (69)

(79) $$\frac{\partial}{\partial y}\underset{(n-1)}{H}{}^{\nu}_{\lambda}=\frac{1}{y}L^{\nu}_{\lambda}+\sum_{s=0}^{n-1}y^{s}\sum_{i+j=s+n-1}\Big\{-\underset{(i,j)}{K}{}^{\nu}_{\lambda}+\underset{(i)}{H}\underset{(j)}{H}{}^{\nu}_{\lambda}+ + \frac{1}{2(n-1)}\delta^{\nu}_{\lambda}(\underset{(i)}{H}{}^{b}_{a}\underset{(j)}{H}{}^{a}_{b}-\underset{(i)}{H}\underset{(j)}{H})\Big\},$$

where the quantities in { } are quadratic forms of $\underset{(n-1)}{H}{}^{\nu}_{\lambda}$ and its covariant derivatives.

8. In this section let us consider the condition (53). Although we get from (68) the relation

$$h^{\rho}_{\lambda,\rho}-h_{,\lambda}=\sum_{i=1}^{n-3}y^{i}(\underset{(i)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(i)}{H}{}_{,\lambda}),$$

we shall prove that factors of the right hand side $\underset{(i)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(i)}{H}{}_{,\lambda}=0$ $(i=1,2,\cdots,n-2)$.

First from (71) we see that

$$\underset{(1)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(1)}{H}{}_{,\lambda}=\frac{1}{n-2}\Big(R^{\rho}_{\lambda,\rho}-\frac{n}{2(n-1)}R_{,\lambda}\Big)-\frac{1}{2(n-1)}R_{,\lambda}$$
$$=\frac{1}{n-2}\Big(R^{\rho}_{\lambda,\rho}-\frac{1}{2}R_{,\lambda}\Big)=0.$$

Now we suppose that the relations

$$\underset{(i)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(1)}{H}{}_{,\lambda}=0\quad(i=3,4,\cdots,p;\,p<n-2)$$

hold good, and we shall prove $\underset{(p+1)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(p+1)}{H}{}_{,\lambda}=0.$ From (75) it follows

$$(n-p-2)\underset{(p+1)}{H}{}^{\rho}_{\lambda,\rho}=\sum_{s=1}^{p-1}\Big[-\underset{(s,p-s)}{K}{}^{\rho}_{\lambda,\rho}-\underset{(s)}{H}\underset{(p-s)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(s)}{H}{}_{,\rho}\underset{(p-s)}{H}{}^{\rho}_{\lambda}- -\frac{1}{n-1}(\underset{(s)}{H}{}^{b}_{a}\underset{(p-s)}{H}{}^{a}_{b,\lambda}-\underset{(s)}{H}\underset{(p-s)}{H}{}_{,\lambda})\Big]$$

and

$$(n-p-2)\underset{(p+1)}{H}=\sum_{s=1}^{p-1}\Big[\underset{(s,p-s)}{K}-\frac{1}{2(n-1)}\Big\{n\underset{(s)}{H}{}^{b}_{a}\underset{(p-s)}{H}{}^{a}_{b,\lambda}+(n-2)\underset{(s)}{H}\underset{(p-s)}{H}\Big\}\Big],$$

hence we get

$$(n-p-2)\underset{(p+1)}{H}{}_{,\lambda}=\sum_{s=1}^{p-1}\Big[\underset{(s,p-s)}{K}{}_{,\lambda}-\frac{1}{n-1}\Big\{n\underset{(s)}{H}{}^{b}_{a}\underset{(p-s)}{H}{}^{a}_{b,\lambda}+(n-2)\underset{(s)}{H}\underset{(p-s)}{H}{}_{,\lambda}\Big\}\Big].$$

Accordingly we obtain

$$(n-p-2)(\underset{(p+1)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(p+1)}{H}{}_{,\lambda})=\sum_{s=1}^{p-1}\Big[(\underset{(s,p-s)}{K}{}^{\rho}_{\lambda,\rho}-\underset{(s,p-s)}{K}{}_{,\lambda})-$$

$$-\{\underset{(s)}{H}\underset{(p-s)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(s)}{H}\underset{(p-s)}{H}{}_{,\lambda}-\underset{(s)}{H}{}^{b}_{a}\underset{(p-s)}{H}{}^{a}_{b,\lambda}+\underset{(s)}{H}{}_{,a}\underset{(p-s)}{H}{}^{a}_{\lambda}\}\Big],$$

or by our assumptions

$$(80)\qquad (n-p-2)(\underset{(p+1)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(p+1)}{H}{}_{,\lambda})=$$

$$\sum_{s=1}^{p-1}\Big[\underset{(s,p-s)}{K}{}^{\rho}_{\lambda,\rho}-\underset{(s,p-s)}{K}{}_{,}\;+\underset{(s)}{H}{}^{b}_{a}\underset{(p-s)}{H}{}^{a}_{b,\lambda}-\underset{(s)}{H}{}_{,a}\underset{(p-s)}{H}{}^{a}_{\lambda}\Big].$$

However, if we take account of the fact that the following relation holds good in general

$$\left(\frac{\partial}{\partial y}T^{\nu}_{\lambda}\right)_{,\mu}=\frac{\partial}{\partial y}T^{\nu}_{\lambda,\mu}-T^{\iota}_{\lambda}\frac{\partial}{\partial y}\Gamma^{\nu}_{\alpha\mu}+T^{\nu}_{a}\frac{\partial}{\partial y}\Gamma^{\iota}_{\lambda\mu},$$

we have

$$\left(\frac{\partial}{\partial y}\underset{(s)}{H}{}^{\rho}_{\lambda}\right)_{,\rho}-\left(\frac{\partial}{\partial y}H\right)_{,\lambda}=\frac{\partial}{\partial y}\left(\underset{(s)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(s)}{H}{}_{,\lambda}\right)-\underset{(s)}{H}{}^{a}_{\lambda}\frac{\partial}{\partial y}\Gamma^{\rho}_{a\rho}+\underset{(s)}{H}{}^{\rho}_{a}\frac{\partial}{\partial y}\Gamma^{a}_{\lambda\rho},$$

and further by virtue of (56), (57) and the assumptions, we get

$$\left(\frac{\partial}{\partial y}\underset{(s)}{H}{}^{\rho}_{\lambda}\right)_{,\rho}-\left(\frac{\partial}{\partial y}\underset{(s)}{H}\right)_{,\lambda}=\underset{(s)}{H}{}^{\rho}_{\lambda}h_{,\rho}+\underset{(s)}{H}{}^{\rho}_{a}(h_{\lambda\rho,}{}^{a}-h^{a}_{\lambda,\rho}-h^{a}_{\rho,\lambda}).$$

Thus, making use of (68) and (70), we get

$$\underset{(s,p-s)}{K}{}^{\rho}_{\lambda,\rho}-\underset{(s,p-s)}{K}{}_{,\lambda}=\underset{(s)}{H}{}^{a}_{\lambda}\underset{(p-s)}{H}{}_{,a}+\underset{(s)}{H}{}^{\rho}_{a}\Big(\underset{p-s)}{H}{}_{\lambda\rho,}{}^{a}-\underset{(p-s)}{H}{}^{a}_{\lambda,\rho}-\underset{(p-s)}{H}{}^{a}_{\rho,\lambda}\Big)$$

$$=\underset{(s)}{H}{}^{\rho}_{\lambda}\underset{(p-s)}{H}{}_{,\rho}-\underset{(s)}{H}{}^{b}_{a}\underset{(p-s)}{H}{}^{a}_{b,\lambda}.$$

Then substituting these into (80), we get

$$(n-p-2)\;(\underset{(p+1)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(p+1)}{H}{}_{,\lambda})=0.$$

but, as $n-p-2\neq 0$ we have

$$\underset{(p+1)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(p+1)}{H}{}_{,\lambda}=0.$$

Thus we have proved that

$$(81)\qquad \underset{(i)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(i)}{H}{}_{,\lambda}=0\quad(i=1,2,\cdots,n-2)$$

Consequently the condition (53) reduces to

$$(82)\qquad \underset{(n-1)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(n-1)}{H}{}_{,\lambda}=0.$$

In connection with the last equation we shall consider the condition (61), We see from (63) and (81), that the following relation holds good :

$$(n-1)\,y^{n-2}\left(\underset{(n-1)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(n-1)}{H}{}_{,\lambda}\right)+y^{n-1}\frac{\partial}{\partial y}\left(\underset{(n-1)}{H}{}^{\rho}_{\lambda,\rho}-\underset{(n-1)}{H}{}_{,\lambda}\right)$$

$$= \left(\frac{n-1}{y} + \sum_{i=1}^{n-1} y^i \underset{(i)}{H}\right) y^{n-1} \left( \underset{(n-1)}{H}{}^{\rho}_{\lambda,\rho} - \underset{(n-1)}{H}{}_{,\lambda}\right),$$

that is,

$$\text{(83)} \qquad \frac{\partial}{\partial y}\left( \underset{(n-1)}{H}{}^{\rho}_{\lambda,\rho} - \underset{(n-1)}{H}{}_{,\lambda}\right) = \sum_{i=1}^{n-1} y^i \underset{(i)}{H}\left( \underset{(n-1)}{H}{}^{\rho}_{\lambda,\rho} - \underset{(n-1)}{H}{}_{,\lambda}\right).$$

**9.** In this section we shall consider the condition (54). From (68) and (72) we get

$$h - \frac{y}{2(n-1)}(R - h^2 + h^b_a h^a_b)$$

$$= \sum_{i=2}^{n-1} y^i \underset{(i)}{H} - \frac{y}{2(n-1)} \sum_{i=2}^{2n-2} y^i \sum_{s=1}^{i-1}\left(\underset{(s)}{H}{}^b_a \underset{(i-s)}{H}{}^a_b - \underset{(s)}{H}\,\underset{(i-s)}{H}\right).$$

We shall prove that $\underset{(i)}{H}{}^\nu_\lambda$ satisfy the relations

$$\text{(84)} \qquad \underset{(i+1)}{H} = \frac{1}{2(n-1)}\sum_{s=1}^{i-1}\left(\underset{(s)}{H}{}^b_a \underset{(i-s)}{H}{}^a_b - \underset{(s)}{H}\,\underset{(i-s)}{H}\right) \quad (i = 1, 2, \cdots, n-3).$$

When $i$ is odd, owing to the fact $\underset{(2j)}{H}{}^\nu_\lambda = 0$, these are evidently true. If $i = 2$, it follows from (75) that

$$(n-4)\underset{(3)}{H} = \underset{(1,1)}{K} - \frac{1}{2(n-1)}\left\{n \underset{(1)}{H}{}^b_a \underset{(1)}{H}{}^a_b + (n-2)\underset{(1)}{H}\,\underset{(1)}{H}\right\}.$$

On the other hand, we get

$$\underset{(1,1)}{K} = \frac{1}{n-1}\left(\underset{(1)}{H}{}^b_a R^a_b + \Delta_2 \underset{(1)}{H} - \underset{(1)}{H}{}^{ab}{}_{,ab}\right),$$

and by (71) and (72)

$$\text{(85)} \qquad R^\nu_\lambda = (n-2)\underset{(1)}{H}{}^\nu_\lambda + \delta^\nu_\lambda H.$$

Hence, by means of (81), we get

$$\underset{(1,1)}{K} = \frac{1}{n-1}\underset{(1)}{H}{}^b_a R^a_b = \frac{n-2}{n-1}\underset{(1)}{H}{}^b_a \underset{(1)}{H}{}^a_b + \frac{1}{n-1}\underset{(1)}{H}\,\underset{(1)}{H}.$$

Substituting this in the above equation we see that

$$(n-4)\underset{(3)}{H} = \frac{n-4}{2(n-1)}\left(\underset{(1)}{H}{}^b_a \underset{(1)}{H}{}^a_b - \underset{(1)}{H}\,\underset{(1)}{H}\right)$$

viz.

$$\underset{(3)}{H} = \frac{1}{2(n-1)}\left(\underset{(1)}{H}{}^b_a \underset{(1)}{H}{}^a_b - \underset{(1)}{H}\,\underset{(1)}{H}\right).$$

Now let us suppose inductively that for $i = 2, 4, \cdots, 2p-2\ (2p < n-2)$ the relations (84) are true. Then we see from (75) that

$$\text{(86)} \quad (n-2p-2)\underset{(2p+1)}{H} = \sum_{s=1}^{2p-1}\left\{\underset{(s,2p-s)}{K} - \frac{n}{2(n-1)}\underset{(s)}{H}{}^b_a \underset{(2p-s)}{H}{}^a_b - \frac{n-2}{2(n-1)}\underset{(s)}{H}\,\underset{(2p-s)}{H}\right\}.$$

By virtue of the assumptions we get for $s \geq 3$

$$\frac{\partial}{\partial y}\underset{(s)}{H} = \frac{1}{2(n-1)}\frac{\partial}{\partial y}\sum_{t=1}^{s-2}\Big(\underset{(t)}{H^b_a}\;\underset{(s-t-1)}{H}{}^a_b - \underset{(t)}{H}\;\underset{(s-t-1)}{H}\Big)$$

$$= \frac{1}{n-1}\sum_{t=1}^{s-2}\Big\{\Big(\frac{\partial}{\partial y}\underset{(t)}{H^b_a}\Big)\underset{(s-t-1)}{H}{}^a_b - \Big(\frac{\partial}{\partial y}\underset{(t)}{H}\Big)\underset{(s-t-1)}{H}\Big\},$$

hence we have

$$\underset{(s,2p-s)}{K} = \frac{1}{n-1}\sum_{t=1}^{s-2}\Big\{\underset{(t,2p-s)}{K}{}^b_a\;\underset{(s-t-1)}{H}{}^a_b - \underset{(t,2p-s)}{K}\;\underset{(s-t-1)}{H}\Big\},$$

and for $s=1$, making use of (72), (62) and (81), we have

$$\underset{(1,2p-1)}{K} = \frac{1}{n-1}\Big\{(n-2)\underset{(2p-1)}{H}{}^b_a\underset{(1)}{H^a_b} + \underset{(2p-1)}{H}\;\underset{(1)}{H}\Big\}.$$

Accordingly we see that

$$\sum_{s=1}^{2p-1}\underset{(s,2p-s)}{K} = \frac{1}{n-1}\Big[\sum_{s=3}^{2p-1}\sum_{t=1}^{s-2}\Big\{\underset{(t,2p-s)}{K}{}^b_a\;\underset{(s-t-1)}{H}{}^a_b - \underset{(t,2p-s)}{K}\;\underset{(s-t-1)}{H}\Big\}$$

$$+ (n-2)\underset{(2p-1)}{H}{}^b_a\underset{(1)}{H^a_b} + \underset{(2p-1)}{H}\;\underset{(1)}{H}\Big]$$

$$= \frac{1}{n-1}\Big[\sum_{l=1}^{2p-3}\;\sum_{s=l+2}^{2p-1}\Big\{\underset{(s-l-1,2p-3)}{K}{}^b_a\;\underset{(l)}{H^a_b} - \underset{(s-l-1,2p-3)}{K}\;\underset{(l)}{H}\Big\}$$

$$+ (n-2)\underset{(2p-1)}{H}{}^b_a\underset{(1)}{H^a_b} + \underset{(2p-1)}{H}\;\underset{(1)}{H}\Big].$$

But, we have by (75)

$$\sum_{s=1}^{q-1}\underset{(s,q-s)}{K}{}^\mu_\lambda = (n-q-2)\underset{(q+1)}{H}{}^\mu_\lambda$$

$$+ \sum_{s=1}^{q-1}\Big\{\underset{(s)}{H}\;\underset{(q-s)}{H}{}^\mu_\lambda + \frac{1}{2(n-1)}\,\delta^\mu_\lambda\Big(\underset{(s)}{H^b_a}\;\underset{(q-s)}{H}{}^a_b - \underset{(s)}{H}\;\underset{(q-s)}{H}\Big)\Big\},$$

$$\sum_{s=1}^{q-1}\underset{(s,q-s)}{K} = (n-q-2)\underset{(q+1)}{H} + \frac{1}{2(n-1)}\sum_{s=1}^{q-1}\Big\{n\underset{(s)}{H^b_a}\;\underset{(q-s)}{H}{}^a_b + (n-2)\underset{(s)}{H}\;\underset{(q-s)}{H}\Big\}.$$

Substituting these in the above equations we get

$$\sum_{s=1}^{2p-1}\underset{(s,2p-s)}{K} = \frac{1}{n-1}\sum_{l=1}^{2p-3}\Big[(n-2p+l-1)\underset{(2p-l)}{H}{}^b_a\underset{(l)}{H^a_b} + \sum_{s=1}^{2p-l-1}\Big\{\underset{(s)}{H}\;\underset{(2p-l-1-s)}{H}{}^b_a\;\underset{(l)}{H^a_b} +$$

$$+ \frac{1}{2(n-1)}\underset{(l)}{H}\;\underset{(s)}{H^a_b}\;\underset{(2p-l-1-s)}{H}{}^a_b - \frac{1}{2(n-1)}\underset{(l)}{H}\;\underset{(s)}{H}\;\underset{(2p-l-1-s)}{H}$$

$$- (n-2p+l-1)\underset{(2p-l)}{H}\;\underset{(l)}{H} - \frac{n}{2(n-1)}\underset{(l)}{H}\;\underset{(s)}{H^b_a}\;\underset{(2p-l-1-s)}{H}{}^a_b$$

$$- \frac{n-2}{2(n-1)}\underset{(l)}{H}\;\underset{(s)}{H}\;\underset{(2p-l-1-s)}{H}\Big\}\Big] + \frac{1}{n-1}\Big\{(n-2)\underset{(2p-1)}{H}{}^b_a\underset{(l)}{H^a_b} + \underset{(2p-1)}{H}\;\underset{(1)}{H}\Big\}$$

$$= \frac{1}{n-1}\sum_{l=1}^{2p-3}\Big[(n-2p+l-1)\Big\{\underset{(2p-l)}{H}{}^a_b\underset{(l)}{H^a_b} - \underset{(2p-l)}{H}\;\underset{(l)}{H}\Big\} -$$

$$
-\frac{1}{2}\underset{(l)}{H}\sum_{s}\Big\{\underset{(s)}{H}{}_a^b\underset{(2p-l-1-s)}{H}{}_b^a+\underset{(s)}{H}\underset{(2p-l-1-s)}{H}\Big\}+\sum_{s}\underset{(s)}{H}\underset{(2p-l-1-s)}{H}\underset{(l)}{H}{}_b^a\Big]
$$

$$
+\frac{1}{n-1}\Big\{(n-2)\underset{(2p-1)}{H}{}_a^b\underset{(1)}{H}{}_b^a+\underset{(2p-1)}{H}\underset{(1)}{H}\Big\}
$$

$$
=\frac{1}{n-1}\sum_{l=1}^{2p-3}\Big[(n-2p+l-1)\Big\{\underset{(2p-l)}{H}{}_a^b\underset{(l)}{H}{}_b^a-\underset{(2p-l)}{H}\underset{(l)}{H}\Big\}
$$

$$
+\frac{1}{2}\underset{(l)}{H}\sum_{s=1}^{2p-l-2}\Big\{\underset{(2p-l-1-s)}{H}{}_a^b\underset{(s)}{H}{}_b^a-\underset{(2p-l-1-s)}{H}\underset{(s)}{H}\Big\}\Big]
$$

$$
+\frac{1}{n-1}\Big\{(n-2)\underset{(2p-1)}{H}{}_a^b\underset{(1)}{H}{}_b^a+\underset{(2p-1)}{H}\underset{(1)}{H}\Big\}.
$$

By induction, replacing the quantities in the second term in [ ], we see

$$
=\frac{1}{n-1}\sum_{l=1}^{2p-3}(n-2p+l-1)\Big\{\underset{(2p-l)}{H}{}_a^b\underset{(l)}{H}{}_b^a-\underset{(2p-l)}{H}\underset{(l)}{H}\Big\}+
$$

$$
+(n-1)\sum_{l=1}^{2p-3}\underset{(l)}{H}\underset{(2p-l)}{H}+(n-2)\underset{(2p-1)}{H}{}_a^b\underset{(1)}{H}{}_b^a+\underset{(2p-1)}{H}\underset{(1)}{H}\Big].
$$

Making use of this, we have

$$
\sum_{l=1}^{2p-1}\Big[\underset{(l,2p-l)}{K}-\frac{1}{n-1}\Big\{(n-p-1)\underset{(2p-l)}{H}{}_a^b\underset{(l)}{H}{}_b^a+p\underset{(2p-l)}{H}\underset{(l)}{H}\Big\}\Big]
$$

$$
=\frac{1}{n-1}\Big[\sum_{l=1}^{2p-1}(l-p)\Big\{\underset{(2p-l)}{H}{}_a^b\underset{(l)}{H}{}_b^a-\underset{(2p-l)}{H}\underset{(l)}{H}\Big\}-
$$

$$
-(n-2)\Big\{\underset{(2p-1)}{H}{}_a^b\underset{(1)}{H}{}_b^a-\underset{(2p-1)}{H}\underset{(1)}{H}\Big\}-(n-1)\underset{(2p-1)}{H}\underset{(1)}{H}
$$

$$
+(n-2)\underset{(2p-1)}{H}{}_a^b\underset{(1)}{H}{}_b^a+\underset{(2p-1)}{H}\underset{(1)}{H}\Big]=0,
$$

that is,

$$
\sum_{s=1}^{2p-1}\underset{(s,2p-s)}{K}=\frac{1}{n-1}\sum_{s=1}^{2p-1}\Big\{(n-p-1)\underset{(2p-s)}{H}{}_a^b\underset{(s)}{H}{}_b^a+p\underset{(2p-s)}{H}\underset{(s)}{H}\Big\}. \tag{87}
$$

Substituting this in (86) we see that

$$
(n-2p-2)\underset{(2p+1)}{H}=\frac{1}{2(n-1)}\sum_{s=1}^{2p-1}\Big[2(n-p-1)\underset{(2p-s)}{H}{}_a^b\underset{(s)}{H}{}_b^a+2p\underset{(2p-s)}{H}\underset{(s)}{H}-
$$

$$
-n\underset{(2p-s)}{H}{}_a^b\underset{(s)}{H}{}_b^a-(n-2)\underset{(2p-s)}{H}\underset{(s)}{H}\Big]
$$

$$
=\frac{n-2p-2}{2(n-1)}\sum_{s=1}^{2p-1}\Big\{\underset{(2p-s)}{H}{}_a^b\underset{(s)}{H}{}_b^a-\underset{(2p-s)}{H}\underset{(s)}{H}\Big\},
$$

hence noticing that $n-2p-2\neq 0$ by hypothesis, we get

$$\underset{(2p+1)}{H} = \frac{1}{2(n-1)} \sum_{s=1}^{2p-1} \left\{ \underset{(2p-s)}{H}{}_a^b \underset{(s)}{H}{}_b^a - \underset{(2p-s)}{H} \underset{(s)}{H} \right\},$$

thus (84) is generally true.

Accordingly we see that

$$h - \frac{y}{2(n-1)} (R - h^2 + h_a^b h_b^a)$$

$$= y^{n-1} \left[ \underset{(n-1)}{H} - \frac{1}{2(n-1)} \sum_{j=0}^{n} y^j \sum_{s=1}^{n+j-3} \left\{ \underset{(s)}{H}{}_a^b \underset{(n+j-2-s)}{H}{}_b^a - \underset{(s)}{H} \underset{(n+j-2-s)}{H} \right\}.\right.$$

Thus the condition (54) reduces to

$$(88) \qquad F \equiv \underset{(n-1)}{H} - \frac{1}{2(n-1)} \sum_{j=0}^{n} y^j \sum_{s=3}^{n+j-1} \left\{ \underset{(s-2)}{H}{}_a^b \underset{(n+j-s)}{H}{}_b^a - \underset{(s-2)}{H} \underset{(n+j-s)}{H} \right\} = 0,$$

and the equation (63) may be replaced by an analogous equation as (83):

$$(89) \qquad \frac{\partial}{\partial y} F = \left( \sum_{i=1}^{n-1} y^i \underset{(i)}{H} \right) F + \frac{y}{n-1} g^{ab} \left( \underset{(n-1)}{H}{}_{b,\,c}^{c} - \underset{(n-1)}{H}{}_{,\,b} \right)_{,\,a}.$$

**10.** In the above sections 5–9 we have seen, for $n > 2$, by virtue of (79), (82) and (88), that the system of equations (51)–(54) are replaced by

$$(90) \qquad \frac{\partial}{\partial y} g_{\lambda\mu} = -2 \left\{ \sum_{i=1}^{n-2} y^i \underset{(i)}{H}_{\lambda\mu} + y^{n-1} \underset{(n-1)}{H}_{\lambda\mu} \right\},$$

$$(79) \qquad \frac{\partial}{\partial y} \underset{(n-1)}{H}{}_\lambda^\nu = \frac{1}{y} L_\lambda^\nu + \sum_{s=0}^{n-1} y^s \sum_{i+j=s+n-1} \left\{ - \underset{(i,j)}{K}{}_\lambda^\nu + \underset{(i)}{H} \underset{(j)}{H}{}_\lambda^\nu \right.$$

$$\left. + \frac{1}{2(n-1)} \delta_\lambda^\nu \left( \underset{(i)}{H}{}_a^b \underset{(j)}{H}{}_b^a - \underset{(i)}{H} \underset{(j)}{H} \right) \right\},$$

$$(82) \qquad \underset{(n-1)}{H}{}_{\lambda,\,\rho}^{\rho} - \underset{(n-1)}{H}{}_{,\,\lambda} = 0,$$

$$(88) \qquad F \equiv \underset{(n-1)}{H} - \frac{1}{2(n-1)} \sum_{j=0}^{n} y^j \sum_{s=3}^{n+j-1} \left\{ \underset{(s-2)}{H}{}_a^b \underset{(n+j-s)}{H}{}_b^a - \underset{(s-2)}{H} \underset{(n+j-s)}{H} \right\},$$

and the general solutions $g_{\lambda\mu}(x, y)$, $H_{\lambda\mu}(x, y)$ of (90) and (79) satisfy the equations

$$(83) \qquad \frac{\partial}{\partial y} \left( \underset{(n-1)}{H}{}_{\lambda,\,\rho}^{\rho} - \underset{(n-1)}{H}{}_{,\,\lambda} \right) = \left( \sum_{i=1}^{n-1} y^i \underset{(i)}{H} \right) \left( \underset{(n-1)}{H}{}_{\lambda,\,\rho}^{\rho} - \underset{(n-1)}{H}{}_{,\,\lambda} \right),$$

$$(89) \qquad \frac{\partial}{\partial y} F = \left( \sum_{i=1}^{n-1} y^i \underset{(i)}{H} \right) F + \frac{y}{n-1} g^{ab} \left( \underset{(n-1)}{H}{}_{b,\,c}^{c} - \underset{(n-1)}{H}{}_{,\,b} \right)_{,\,a}.$$

As we have already seen in the section 7,

$$L_\lambda^\nu \equiv 0 \qquad \text{for } n = 2m+1 \ (m \geq 2),$$

we can solve (90) and (79) under the initial conditi ns

$$[g_{\lambda\mu}(x, y)]_{y=0} = g_{\lambda\mu}(x),$$

$$[\underset{(n-1)}{H}{}_\lambda^\nu (x, y)]_{y=0} = \underset{(n-1)}{H}{}_\lambda^\nu (x).$$

Accordingly, if we can determine the initial functions $\underset{(n-1)}{H}{}^{\nu}_{\lambda}(x)$ so that they satisfy the conditions

$$\underset{(n-1)}{H}{}^{\rho}_{\lambda,\rho} - \underset{(n-1)}{H}{}_{,\lambda} = 0 \tag{91}$$

and

$$\underset{(n-1)}{H} - \frac{1}{2(n-1)}\sum_{s=3}^{n-1}\left\{\underset{(s-2)}{H}{}^{b}_{a}\underset{(n-s)}{H}{}^{a}_{b} - \underset{(s-2)}{H}\;\underset{(n-s)}{H}\right\} = 0 \tag{92}$$

then the solutions $g_{\lambda\mu}(x,y)$, $\underset{(n-1)}{H}{}^{\nu}_{\lambda}(x,y)$ will satisfy (82) and (88).

However, as $n = 2m+1$, (92) becomes

$$\underset{(n-1)}{H} = 0, \tag{92$'$}$$

hence it will be sufficient to set $\underset{(n-1)}{H}{}^{\nu}_{\lambda}(x) = 0$. Thus we obtain a result that any Riemannian space $V_n$ of $n=2m+1$ $(m \geqq 2)$ dimensions can be imbedded in some Riemannian space $V_{n+1}$ conformal with an Einstein space as a hypersurface which is the image of $S_n$ invariant under the group of holonomy of the conformal connexion associated with $V_{n+1}$.

In the case of $n=3$ the equations (92), etc. lose their meaning, but from (67) it is evident that $L^{\nu}_{\lambda} = 0$. Accordingly the above result remains true.

But in the case of $n = 1$, we must consider it in a different way from sections $1-2$.

**11.** Next let us consider the case $n = 2m$ $(m \geqq 1)$. Owing to the form of (79) it is necessary for the tensor $L^{\nu}_{\lambda}$ $(R^{\cdot}_{b}\,;\, R^{\cdot}_{b},{}_{\rho_1}\,;\, \cdots\cdots\,;\, R^{\cdot}_{b},{}_{\rho_1\cdots\cdots\rho_{n-2}})$

$$L^{\nu}_{\lambda}(x\,;0) = 0. \tag{93}$$

Accordingly for the space which satisfy this condition, if we can give $H^{\nu}_{\lambda}(x,0) = H^{\nu}_{\lambda}(x)$ such that (91) and (92) are satisfied, then the solutions of (90) and (79) obeying the initial conditions (91) and (92) will satisfy our system of equations. But there exists always such $\underset{(n-1)}{H}{}^{\nu}_{\lambda}$ for any $V_n$. Thus we obtain a result that any Riemannian space $V_n$ of $n=2m$ dimensions for which $L^{\nu}_{\lambda} = 0$ can be imbedded in some Riemannian space $V_{n+1}$ conformal with an Einstein space as a hypersurface which is the image of $S_n$.

Lastly let us consider the case $n = 2$. Since the Gaussian curvature is $K = R_{1212}/g$, it follows, as is well known, that

$$R_{\lambda\mu} = Kg_{\lambda\mu} \text{ or } R^{\nu}_{\lambda} = K\,\delta^{\nu}_{\lambda}$$

and $R = 2K$, hence we get

$$R^{\nu}_{\lambda} - \frac{1}{2}R\,\delta^{\nu}_{\lambda} = 0.$$

Accordingly our system of equations is replaced by

$$\frac{\partial}{\partial y}\,g_{\lambda\mu} = -2h_{\lambda\mu}, \tag{51}$$

(52′) $$\frac{\partial}{\partial y} h^\nu_\lambda = \frac{1}{y} h^\nu_\lambda + h\, h^\nu_\lambda - \delta^\nu_\lambda |h^b_a|,$$

(53) $$h^\rho_{\lambda,\rho} - h_{,\lambda} = 0,$$

(54′) $$h + h|h^b_a| - y\,K = 0,\ \ h = h^a_a.$$

General solutions of (51) and (52′) will sutisfy

(61′) $$\frac{\partial}{\partial y}(h^\rho_{\lambda,\rho} - h_{,\lambda}) = \Big(\frac{1}{y} + h\Big)(h^\rho_{\lambda,\rho} - h_{,\lambda}),$$

(63′) $$\frac{\partial}{\partial y}\{h + y|h^a_b| - yK\} = \Big(\frac{1}{y} + h\Big)\{h + y|h^a_b| - yK\} + y\,g^{ab}\,(h^c_{b,c} - h_{,b})_{,a}.$$

If we substitute $h^\nu_\lambda$ by

(94) $$h^\nu_\lambda = y f^\nu_\lambda,$$

the above system of equations will reduce to

(51″) $$\frac{\partial}{\partial y} g_{\lambda\mu} = -2y f_{\lambda\mu},$$

(52″) $$\frac{\partial}{\partial y} f^\nu_\lambda = y\{f f^\nu_\lambda - \delta^\nu_\lambda |f^b_a|\},$$

(53″) $$f^\rho_{\lambda,\rho} - f_{,\lambda} = 0,$$

(54″) $$f + y^2|f^b_a| - K = 0$$

and

(61″) $$\frac{\partial}{\partial y}(f^\rho_{\lambda,\rho} - f_{,\lambda}) = yf(f^\rho_{\lambda,\rho} - f_{,\lambda}),$$

(63″) $$\frac{\partial}{\partial y}\{f + y^2|f^b_a| - K\} = yf\{f + y^2|f^b_a| - K\} + y\,g^{ab}(f^c_{b,c} - f_{,b})_{,a}.$$

Hence if we can give a tensor $f^\nu_\lambda$ so that $f \equiv f^a_a = K(x)$ and $f^\rho_{\lambda,\rho} - f_{,\lambda} = 0$ for a given fundamental tensor $g_{\lambda\mu}(x)$, we shall obtain our solution of (51″)-(54″) when we solve (51′) and (52″) under the initial conditions $g_{\lambda\mu}(x,0) = g_{\lambda\mu}(x)$, $f^\nu_\lambda(x,0) = f^\nu_\lambda(x)$. But it is obvious that we can find such $f^\nu_\lambda$ as stated above.

Thus we have proved the following theorem.

**Theorem.** *For $n = 2m+1$ $(m \geq 1)$ and 2 any Riemannian space $V_n$, and for $n = 2m$ $(m \geq 2)$ any $V_n$ satisfying the condition $L^\nu_\lambda = 0$ can be imbedded in a Riemannian space $V_{n+1}$ conformal with some Einstein space as a hypersurface which is the image of a hypersphere invariant under the group of holonomy of the space with normal conformal connection associated with this $V_{n+1}$.*

Further the above proof shows that, as $h_{\lambda\mu}(x,0) = 0$, this hypersurface is a minimal surface. For $n = 2$, $V_3$ is conformally flat as it is evident from the result of E. Cartan.

**12.** In this section we shall show that, similarly as in above sections, we can treat the problem of imbedding any Riemannian space $V_n$ in some Einstein space $A_{n+1}$. According to Campbell's work, using our notations, this problem reduces to the one to solve the following equations

(51*) $$\frac{\partial}{\partial y} g_{\lambda\mu} = -2h_{\lambda\mu}\phi,$$

(52*) $$\frac{\partial}{\partial y} h^\nu_\lambda = \phi\,(h\,h^\nu_\lambda - R^\nu_\lambda) + \phi^\nu_\lambda, \quad \phi^\nu_\lambda = g^{\nu\rho}\phi,_{\lambda\rho}$$

under additional conditions

(53) $$h^\rho_{\lambda},_\rho - h,_\lambda = 0,$$

(54*) $$R - h^2 + h^b_a h^a_b = 0.$$

If $g_{\lambda\mu}$ and $h^\nu_\lambda$ satisfy only (51*) and (52*), then for the quantities of the left hand sides of (53) and (54*), we get

(61*) $$\frac{\partial}{\partial y}(h^\rho_{\lambda},_\rho - h,_\lambda) = \phi,_\lambda(R - h^2 + h^b_a h^a_b) + \phi\,h\,(h^\rho_{\lambda},_\rho - h,_\lambda) + \frac{1}{2}\phi\,(R - h^2 + h^b_a h^a_b),_\lambda$$

and

(63*) $$\frac{\partial}{\partial y}(R - h^2 + h^b_a h^a_b) = 2\,\phi\,h\,(R - h^2 + h^b_a h^b_a) - 4g^{ab}\phi,_b(h^\rho_{a},_\rho - h,_a) - 2\,\phi\,g^{ab}(h^\rho_{b},_\rho - h,_b),_a.$$

Hence it is sufficient that there exists at $y = 0$ a tensor $h^\nu_\lambda$ satisfying (53) and (54*). But this is always possible, hence there will exist our solution $g_{\lambda\mu}(x, y)$ for any $g_{\lambda\mu}(x)$. Thus we obtain a $V_{n+1}$ with a line element

$$ds^2 = g_{\lambda\mu}(x, y)\,dx^\lambda dx^\mu + \phi\,(x, y)\,dy\,dy,$$

where $\phi\,(x, y)$ is an arbitrary function of $x^\lambda$ and $y$, and it will be easily shown that this space is an Einstein space with vanishing scalar curvature.

## § 3.

In the following section we shall consider spaces with normal conformal connexions whose groups of holonomy fix a hypersphere $S'_{n-1}$ and another real one $S''_{n-1}$, and investigate properties of the hypersurface of image of $S''_{n-1}$. S. Sasaki[4)] and K. Yano[5)] already studied such spaces by another way.

**13.** According to the section 1, if a space with conformal connexion has the group of holonomy which fixes a hypersphere $S'_{n-1}$, the connexion is represented with respect to normal frames by a system of Pfaffians $\omega^0_0$, $\omega^i$, $\omega^0_i$, $\omega^i_j$ satisfying the conditions

4), 5) Cf. loc. cit, 1).

(95) $$\begin{cases} \omega_0^0 = 0, \\ \omega_i^0 = \varepsilon\,\omega^i \quad (i = 1, 2, \cdots\cdots, n) \end{cases}$$

and $S'_{n-1}$ is represented by

(96) $$\varepsilon A_0 + A_\infty$$

where $\varepsilon$ is 0, 1 or $-1$, according as $S'_{n-1}$ is point, real or imaginary.

Moreover if the connexion is normal, that is, if

(17) $$\omega_i^0 = -\frac{1}{n-2} R_{ik}\omega^k + \frac{R}{2(n-1)(n-2)}\omega^i,$$

the Riemannian space $V_n$ defined by the line element $ds^2 = \omega^i\omega^i$ is, as it was shown in section 1, an Einstein space such that

(96) $$R_{ij} = -2\varepsilon(n-1)\delta_{ij}, \quad R = -2\varepsilon n(n-1).$$

Using only such frames, let us consider another fixed real hypersphere $S''_{n-1}$ represented by

(4) $$X = x^0 A_0 + x^i A_i + x^\infty A_\infty,$$

then (5) must be integrable. As we consider only a neighborhood of the image $F''_{n-1}$ of this hypersphere, we may consider that $x^n \neq 0$ since $S''_{n-1}$ is real. Accordingly, as (22), if we put

$$y^a = \frac{x^a}{x^n}, \quad y^0 = \frac{x^0}{x^n}, \quad y^\infty = \frac{x^\infty}{x^n} \quad (a = 1, 2, \cdots\cdot, n-1)$$

(5) turns into (23). If we further choose frames such that $y^a = 0$, we get

(25) $$\begin{cases} dy^0 + \omega_n^0 - y^0(y^0\omega^n - y^\infty\omega_n^0) = 0, \\ dy^\infty - \omega^n - y^\infty(y^0\omega^n - y^\infty\omega_n^0) = 0, \\ \omega_n^a + y^0\omega^a - y^\infty\omega_a^0 = 0. \end{cases}$$

Substituting (95) in these, we have

(97) $$\begin{cases} dy^0 + \{\varepsilon(1 + y^0 y^\infty) - y^0 y^0\}\omega^n = 0, \\ dy^\infty - \{1 + y^0 y^\infty - \varepsilon y^\infty y^\infty\}\omega^n = 0, \\ \omega_n^a + (y^0 - \varepsilon y^\infty)\omega^a = 0. \end{cases}$$

From the last equations we get also

$$(1 + y^0 y^\infty - \varepsilon y^\infty y^\infty)\,dy^0 + \{\varepsilon(1 + y^0 y^\infty) - y^0 y^0\}\,dy^\infty = 0,$$

or

$$(1 + 2y^0 y^\infty)\,d(y^0 + \varepsilon y^\infty) - (y^0 + \varepsilon y^\infty)\,d(y^0 y^\infty) = 0,$$

viz.

$$y^0 + \varepsilon y^\infty = c\sqrt{1 + 2y^0 y^\infty}, \quad c = \text{constant},$$

or

(98) $$y^0 = (c^2 - \varepsilon)y^\infty + c\sqrt{(c^2 - 2\varepsilon)y^\infty y^\infty + 1}\,.$$

Using the relation

$$1 + y^0 y^\infty - \varepsilon y^\infty y^\infty = 1 + (c^2 - 2\varepsilon)y^\infty y^\infty + c y^\infty\sqrt{1 + (c^2 - 2\varepsilon)y^\infty y^\infty}$$

which follows by substitution of (98) in (97), we have

(99) $$\omega^n = \frac{dy^\infty}{1+(c^2-2\varepsilon)y^\infty y^\infty + cy^\infty\sqrt{1+(c^2-2\varepsilon)y^\infty y^\infty}},$$

(100) $$\begin{aligned}\omega^n_a &= -(y^0-\varepsilon y^\infty)\,\omega^a\\ &= -\{(c^2-2\varepsilon)y^\infty + c\sqrt{1+(c^2-2\varepsilon)y^\infty y^\infty}\}\,\omega^a.\end{aligned}$$

Putting $y^\infty = y$, we have

$$\begin{aligned}(\omega^a)' &= [\omega^b\omega^a_b] + [\omega^n\omega^a_n]\\ &= [\omega^b\omega^a_b] - \frac{(c^2-2\varepsilon)y + c\sqrt{1+(c^2-2\varepsilon)yy}}{1+(c^2-2\varepsilon)yy+cy\sqrt{1+(c^2-2\varepsilon)yy}}[dy\,\omega^a],\end{aligned}$$

hence we can define a function $f(y)$ such that

(101) $$(f\omega^a)' = [f\omega^b, \omega^a_b].$$

Comparing this with the last equation we get

$$\begin{aligned}d\log f &= \frac{(c^2-2\varepsilon)y + c\sqrt{1+(c^2-2\varepsilon)y^2}}{1+(c^2-2\varepsilon)y^2+cy\sqrt{1+(c^2-2\varepsilon)y^2}}\,dy\\ &= d\log\{\sqrt{1+(c^2-2\varepsilon)y^2}+cy\}\end{aligned}$$

or

(102) $$f(y) = \sqrt{1+(c^2-2\varepsilon)y^2} + cy.$$

Next we denote by $x^1, x^2, \cdots, x^{n-1}$, the integrals of the following system of Pfaff equations

(103) $$\omega^1 = \omega^2 = \cdots\cdots = \omega^{n-1} = 0,$$

then it is evident from (99) that we may take $(x^a, y)$ as a coordinate system in $V_n$. If we consider in (101) only variations with respect to $y$, we obtain

$$-\delta(f\omega^a(d)) = \omega^a_b(\delta)f\omega^b(d),$$

accordingly we get

$$\delta(f^2\omega^a\omega^a) = -f^2\omega^a_b(\delta)\,\omega^a(d)\,\omega^b(d) = 0\,,$$

and hence we have

$$f^2\omega^a\omega^a = \gamma_{ab}(x^1,\cdots,x^{n-1})\,dx^a dx^b.$$

Thus we see that we can choose a coordinate system such that the line-element takes the form

(104) $$ds^2 = \frac{1}{\{\sqrt{1+(c^2-2\varepsilon)y^2}+cy\}^2}\left\{\gamma_{ab}(x)\,dx^a dx^b + \frac{dy\,dy}{1+(c^2-2\varepsilon)y^2}\right\}.$$

The image of $S''_{n-1}$ is given by $y = 0$, because the condition that $A_0$ lies on this hypersphere $y^0A_0 + A_n + y^\infty A_\infty$ is equivalent to $y^\infty \equiv y = 0$. Denoting the fundamental tensor of $V_n$ by $g_{ij}(x,y)$, we have

(105) $$g_{ab}(x,y) = \frac{\gamma_{ab}(x)}{\{\sqrt{1+(c^2-2\varepsilon)y^2}+cy\}^2},\quad g_{an} = 0,$$

$$g_{nn} = \frac{1}{\{1+(c^2-2\varepsilon)y^2\}\{\sqrt{1+(c^2-2\varepsilon)y^2}+cy\}^2}.$$

**14.** Let us consider $F^{v'}_{n-1}$ as an $(n-1)$-dimensional Riemannian space $V_{n-1}$ and designate its quantities by $*$. In order to simplify the calculation we transform the coordinates $(x^a, y)$ into $(x^a, x^n)$ so that

$$x^n = \int_0^y \frac{dy}{\sqrt{1+(c^2-2\varepsilon)y^2}} \tag{106}$$

and put

$$\frac{1}{\sqrt{1+(c^2-2\varepsilon)y^2}+cy} = \varphi(x^n). \tag{107}$$

Then it follows

$$ds^2 = \varphi^2 \{\gamma_{ab}\, dx^a\, dx^b + dx^n\, dx^n\}, \tag{104'}$$

hence we have

$$g_{ab} = \varphi^2 \gamma_{ab}, \quad g_{an} = 0, \quad g_{nn} = \varphi^2.$$

Accordingly we can find that

$$\Gamma_{acb} = \varphi^2 \Gamma^*_{acb}, \; \Gamma_{anb} = -g_{ab}\frac{d}{dx^n}\log\varphi = -\Gamma_{nab},$$

$$\Gamma_{nnc} = \Gamma_{nan} = 0, \quad \Gamma_{nnn} = g_{nn}\frac{d}{dx^n}\log\varphi,$$

or

$$\Gamma^c_{ab} = \Gamma^{*c}_{ab}, \; \Gamma^n_{ab} = -\gamma_{ab}\frac{d}{dx^n}\log\varphi, \; \Gamma^a_{nb} = \delta^a_b\frac{d}{dx^n}\log\varphi,$$

$$\Gamma^a_{nn} = \Gamma^n_{nb} = 0, \quad \Gamma^n_{nn} = \frac{d}{dx^n}\log\varphi,$$

where $\Gamma^i_{jk}$ and $\Gamma^{*a}_{bc}$ are the Christoffel's symbols with respect to $g_{ij}$ and $\varphi^2\gamma_{ab}$.

Thus we obtain, in terms of natural frames,

$$\omega^c_a = \omega^{*c}_a + \delta^c_a\, d\sigma,$$

$$\omega^n_a = -\sigma'\gamma_{ab}\, dx^b, \quad \omega^a_n = \sigma'\, dx^a, \quad \omega^n_n = d\sigma,$$

$$\sigma(x^n) = \log\varphi(x^n).$$

On account of the relations

$$\Omega^b_a = -(\omega^b_a)' + [\omega^i_a\, \omega^b_i] = -(\omega^{*b}_a)' + [\omega^{*c}_a + \delta^c_a\, d\sigma, \omega^{*b}_c + \delta^b_c\, d\sigma] -$$
$$-(\sigma')^2\gamma_{ac}[dx^c\, dx^b] = \Omega^{*b}_a - (\sigma')^2\gamma_{ac}[dx^c\, dx^b],$$

$$\Omega^n_a = -(\omega^n_a)' + [\omega^i_a\, \omega^n_i] = [d(\sigma'\gamma_{ab})\, dx^b] - \sigma'[\omega^{*c}_a\gamma_{cb}\, dx^b]$$
$$= \sigma''\gamma_{ab}[dx^n\, dx^b] + \sigma'\frac{\partial}{\partial x^c}\gamma_{ab}[dx^c\, dx^b] - \sigma'[\omega^*_{ab}\, dx^b],$$

$$\Omega^n_n = -(\omega^n_n)' + [\omega^a_n\, \omega^n_a] = 0,$$

we get the following relations between the components of the Riemann curvature tensors:

$$\left\{\begin{aligned} R_a{}^b{}_{cd} &= R^*{}_a{}^b{}_{cd} - (\sigma')^2\{\gamma_{ac}\delta^b_d - \gamma_{ad}\delta^b_c\}, \\ R_a{}^n{}_{cn} &= -\sigma''\gamma_{ac}, \\ R_a{}^b{}_{cn} &= R_a{}^n{}_{cd} = R_n{}^n{}_{ij} = 0. \end{aligned}\right. \tag{108}$$

Accordingly we get for the Ricci tensor

$$(109)\qquad \begin{cases} R_{ac} = R_{a\ cb}^{\ b} + R_{a\ cn}^{\ n} = R^*_{ac} - \{(n-2)(\sigma')^2 + \sigma''\}\gamma_{ac}, \\ R_{an} = 0,\quad R_{nn} = R_{n\ na}^{\ a} = -(n-1)\sigma''. \end{cases}$$

From (96) and (104′) we have

$$(110)\qquad \sigma'' = 2\varepsilon\varphi^2$$

and

$$R_{ac} = R^*_{ac} - \{(n-2)(\sigma')^2 + \sigma''\}\gamma_{ac} = -2\varepsilon(n-1)\varphi^2\gamma_{ac},$$

hence we get

$$(111)\qquad R^*_{ac} = \{(n-2)(\sigma')^2 + \sigma'' - 2\varepsilon(n-1)\varphi^2\}\gamma_{ac}.$$

Making use of (110), we have

$$(n-2)(\sigma')^2 + \sigma'' - 2\varepsilon(n-1)\varphi^2 = (n-2)\{(\sigma')^2 - 2\varepsilon\varphi^2\},$$

moreover from (106) and (107) we get

$$\frac{d\sigma}{dx^n} = \left(\frac{d}{dy}\log\varphi\right)\frac{dy}{dx^n} = -\left(\frac{d}{dy}\log f\right)\sqrt{1+(c^2-2\varepsilon)y^2}$$

$$= -\frac{(c^2-2\varepsilon)y + c\sqrt{1+(c^2-2\varepsilon)y^2}}{\sqrt{1+(c^2-2\varepsilon)y^2}+cy},$$

therefore we obtain

$$(\sigma')^2 - 2\varepsilon\varphi^2 = \frac{\{(c^2-2\varepsilon)y + c\sqrt{1+(c^2-2\varepsilon)y^2}\}^2 - 2\varepsilon}{\{\sqrt{1+(c^2-2\varepsilon)y^2}+cy\}^2}$$

$$= c^2 - 2\varepsilon.$$

Accordingly (111) reduces to

$$(111')\qquad R^*_{ac} = (n-2)(c^2-2\varepsilon)\gamma_{ac},$$

whence we get

$$(112)\qquad R^* = (n-1)(n-2)(c^2-2\varepsilon).$$

Thus we see that the hypersurfaces $F_{n-1}(y)$ defined by $y = \text{const.}$ are totally umbilical and Einstein spaces and that the orthogonal trajectories of these $F_{n-1}(y)$ are conformal circles. For, as $x^a = \text{const.}$ $(a = 1, 2, \cdots, n-1)$ yields

$$\omega^a = 0,\quad \omega_a^n = 0,\quad \omega_n^0 = \varepsilon\,\omega^n$$

and

$$dA_0 = \omega^n A_n,\quad d^2A = \varepsilon\,\omega^n\omega^n A_0 + d\omega^n A_n - \omega^n\omega^n A_\infty,$$

the developement of any orthogonal trajectory, is the intersections of the hyperspheres $A_1, A_2, \cdots, A_{n-1}$.

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