NOTES ON FOURIER ANALYSIS (XXVI): SOME NEGATIVE EXAMPLES IN THE THEOREY OF FOURIER SERIES*)

By

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Introduction. W. Randels[1] has proved the following Theorem.

THEOREM A. *There is a function* $f(t) \in L^2$ *such that*

1°. $\int_0^t |\varphi_x(u)|\,du \neq o(t)$, $\varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$,

2°. *the series*

$$\sum_{n=2}^{\infty} (a_n \cos nx + b_n \sin nx)/\sqrt{\log n} \tag{0.1}$$

converges, where

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{0.2}$$

R. E. A. C. Paley[2] has proved

THEOREM B. *There is an integrable function* $f(t)$ *such that*

1°. $\int_0^t \varphi_x(u)\,du \neq o(t)$,

2°. *the Fourier series* (0.2) *of* $f(t)$ *converges at* $t = x$.

As a generalization of Theorem A we prove that

THEOREM 1. *There is a bounded function* $f(t)$ *such that*

1°. $\int_0^t |\varphi_x(u)|\,du \neq 0$,

2°. *the Fourier series* (0.2) *of* $f(t)$ *converges at* $t = x$.

We prove also the following theorem containing Theorem A and B. That is,

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1) W. Randels, Bull. Am. Math. Soc., 46 (1940).

2) R. E. A. C. Paley, Proc. Cambridge Phil. Soc., 26 (1930).

THEOREM 2. *Let* $\infty > p \geqq 1$. *Then there is a function* $f(t) \varepsilon L^p$ *such that*

1°. $\int_0^t \varphi_x(u)\,du \neq o(t)$,

2°. *the Fourier series* (0.2) *of* $f(t)$ *converges at* $t = x$.

It is known that[3]

THEOREM C. *If*

1°. $a_n = O(1/n^\delta)$, $b^n = O(1/n^\delta)$ $(n = 1, 2, \cdots)$

where $\delta > 0$ *and* a_n, b_n *are Fourier coefficients of* $f(t)$,

2°. $s_n(x) - f(x) = o(1/\log n)$,

where s_n *denotes the partial sum of the Fourier series* (1.2) *of* $f(t)$, *then*

$$\int_0^t \varphi_x(u)\,du = o(t).$$

In this theorem the condition 2° is the best possible, that is, o cannot be replaced by O. In fact we prove

THEOREM 3. *There is an integrable function* $f(t) \varepsilon L^p$ *such that*

1°. $a_n = O(1/n^\delta)$, $b_n = O(1/n^\delta)$ $(n = 1, 2, \cdots)$

where $\delta > 0$ *and* a_n, b_n *are Fourier coefficients of* $f(t)$,

2°. $s_n(x) - f(x) \neq o(1/\log n)$, $s_n(x) - f(x) = O(1/\log n)$,

3°. $\int_0^t \varphi_x(u)\,du \neq o(t)$.

This is a generalization of Theorem 2.

On the other hand L.R. Bosanquet[4] and R.E.A.C. Paley proved that

THEOREM D. *Let* $\alpha \geqq 0$. *If the Fourier series* (0.2) *of* $f(t)$ *is* (C, α)*-summable to* $f(x)$ *at* $t = x$, *then*

$$\Phi_{\alpha+1+\varepsilon}(t) = o(1) \qquad (t \to 0), \tag{0.3}$$

where $\Phi_\beta(t)$ *is the* β*-th mean of* $\varphi_x(u)$, *that is*,

$$\Phi_\beta(t) \equiv \frac{1}{t_\beta} \int_0^t (t-u)^{\beta-1} \varphi_x(u)\,du.$$

Conversely, if

$$\Phi_\alpha(t) = o(1) \qquad (t \to 0),$$

3) G.H. Hardy and J.E. Littlewood, Annali di Pisa, 3 (1932).

4) L.S. Bosanquet, Proc. London Math. Soc., 31 (1930).

then the Fourier series of $f(t)$ at $t=x$ is $(C, \alpha+\varepsilon)$-summable.

It is said that the Theorem is not true for $\varepsilon=0$. But Paley has proved the case $\alpha=0$ and Hahn[5] has proved the case $\alpha=1$ of the converse part. Bosanquent[6] states that Wiener's general Tauberian Theorem implies that the converse part of the theorem is not true for $\varepsilon=0$ and any $\alpha \geqq 0$. Therefore there is no concrete example for general α. We prove the following theorems.

THEOREM 4. *Let $1 \leqq p<\infty$ $\alpha \geqq 0$. Then there is an integrable function $f \varepsilon L^p$ such that*

1°. $\int_0^t |\Phi_\alpha(u)|\, du \neq o(t)$.

2°. *The Fourier series* (0.2) *of $f(t)$ is summable (C, α) at $t=x$.*

The case $\alpha=0$ is contained in Theorem 1.

THEOREM 5. *Let $1 \leqq p<\infty$ and $\alpha \geqq 0$. Then there is $f(t) \varepsilon L^p$ such that*

1°: $\Phi_\alpha(u)=o(t)$,

2°. *the Fourier series* (0.2) *of $f(t)$ is not summable (C, α) at $t=x$.*

THEOREM 6. *Let $\alpha \geqq 0$ and $1 \leqq p<\infty$. There is an integrable function $f(t) \varepsilon L^p$ such that*

1°. $\sigma_n^\alpha(x)-f(x) \neq o(1/\log n)$, $\sigma_n^\alpha(x)-f(x)=O(1/\log n)$,

where $\sigma_n^\alpha(t)$ denotes the α-th Cesàro mean of the Fourier series (0.2) *of $f(t)$*,

2°. $\Phi_{\alpha+1}(t) \neq o(1)$.

The case $\alpha=0$ is contained in Theorem 3. This containes Theorem 4 as a special case. In spite of this we prove Theorem 4, for its proof is simpler than that of Theorem 6, and suggests the method of proof of Theorem 6.

§1. Before going to the proof of theorems we explain the type of examples used. We take a sequence of disjoint intervals

$$\Delta_k \equiv\left(\frac{\pi}{n_k}, \frac{\pi}{n_k}+\frac{\pi}{m_k}\right) \qquad (k=1,2, \cdots) \tag{1.1}$$

and define an even periodic function $f(t)$ such that

$$f(t)=c_k \sin M_k t \qquad (t \varepsilon \Delta_k) \tag{1.2}$$

for $k=1,2, \cdots$ and $f(t) \equiv 0$ in $(0, \pi)-\bigcup \Delta_k$. Here (n_k), (m_k) and (M_k) are increasing sequences of integers and (c_k) is a sequence of positive numbers.

5) H. Hahn, Jahrbte. Deutschen Math. Ver., 25 (1916).

6) L.S. Bosanquet, Proc. London Math. Soc., 37 (1934).

They are suitably determined in each problem and (1.1) is sometimes replaced by

$$\Delta_k \equiv \left(\frac{\pi}{n_k}, \frac{\pi}{m_k} \right) \quad (k = 1, 2, \cdots),$$

and further (1.2) may be changed to

$$f(t) \equiv c_k \cos M_k t \qquad (t \in \Delta_k),$$

or

$$f(t) \equiv c_k t \sin M_k t \qquad (t \in \Delta_k),$$

and so on.

This is a function-analogy of the Fejér example defined by series in a sense. Many problems which solved by the Fejér's example, are also proved by this type of examples, and we can go more in some cases.

As an illustration we will prove some classical theorems by our example.

THEOREM D. *There is a continuous function $f(t)$ with Fourier series divergent at a point x.*

PROOF. Whithout loss of generality we can suppose that $x = 0$. Let $s_n(x)$ be the n-th partial sum of Fourier series of $f(t)$, $f(t)$ being defined by (1.2). Then

$$\begin{aligned} (1.3) \qquad s_{M_k}(0) &= \frac{2}{\pi} \int_0^\pi f(t) \frac{\sin M_k t}{t} dt + o(1) \\ &= \frac{2}{\pi} \sum_{i=1}^{\infty} c_i \int_{\Delta_i} \frac{\sin M_i t \sin M_k t}{t} dt + o(1) \\ &= \frac{2}{\pi} \left[\sum_{i=1}^{k-1} c_i \int_{\Delta_i} + c_k \int_{\Delta_k} + \sum_{i=k+1}^{\infty} c_i \int_{\Delta_i} \right] + o(1) \\ &= \frac{2}{\pi} \Big[I_1 + I_2 + I_3 \Big] + o(1), \end{aligned}$$

say. First we have

$$\begin{aligned} (1.4) \qquad I_2 &= c_k \int_{\Delta_k} \frac{\sin^2 M_k t}{t} dt = \frac{c_k}{2} \int_{\Delta_k} \frac{dt}{t} + \frac{c_k}{2} \int_{\Delta_k} \frac{\cos 2M_k t}{t} dt \\ &= \frac{c_k}{2} \log \left(1 + \frac{n_k}{m_k} \right) - \frac{c_k}{2} \int_{2\pi M_k n_k}^{2(\pi n_k + \pi/m_k) M_k} \frac{\cos t}{t} dt \\ &= \frac{c_k}{2} \log \left(1 + \frac{n_k}{m_k} \right) + O\left(\frac{c_k n_k}{M_k} \right), \end{aligned}$$

concerning I_1,

$$(1.5)\qquad I_1=\sum_{i=1}^{k-1} c_i \int_{\Delta_i} \frac{\sin M_i t \sin M_k t}{t}\,dt$$

$$=\sum_{i=1}^{k-1} \frac{c_i}{2}\left[\int_{\pi/n_i}^{\pi/n_i+\pi/m_i} \frac{\cos(M_k-M_i)t}{t}\,dt\right.$$

$$\left.-\int_{\pi/n_i}^{\pi/n_i+\pi/m_i} \frac{\cos(M_k+M_i)t}{t}\,dt\right]=\sum_{i=1}^{k-1} O\left(\frac{c_i n_i}{M_k-M_i}\right).$$

Similarly we have

$$(1.6)\qquad I_3=\sum_{i=1}^{k+1} O\left(\frac{c_i n_i}{M_i-M_k}\right).$$

If we take $c_k\equiv 1/\sqrt{k}$, then $f(t)$ is continuous at $t=0$. When

$$(1.7)\qquad n_k \mid M_k,\quad m_k \mid M_k \qquad (k=1,2,\cdots),$$

$f(t)$ is continuous everywhere. Let us take

$$n_k\equiv 2^k\cdot 2^{k^2},\quad m_k\equiv 2^{k^2},\quad M_k\equiv 2^{(k+1)^2}.$$

Then $n_k/m_k=2^k$ and $c_k n_k/M_k\to 0$. Thus by (1.4) $I_2\to 0$ as $k\to\infty$. As easily may be seen $I_1+I_3\to 0$ by (1.5) and (1.6), and also the intervals (1.1) are disjoint mutually. Thus the theorem is proved.

THEOREM E. *There is a continuous function $f(t)$ such that the Fourier series of $f(t)$ diverges at $t=0$ and the continuity modulus $\omega(\delta)$ of $f(t)$ satisfies*

$$(1.8)\qquad \omega(\delta)=O\left(1\Big/\log\frac{1}{\delta}\right).$$

PROOF. In stead of (1.2), we take

$$f(t)\equiv(-1)^k c_k \sin M_k t \qquad (t\in\Delta_k)$$

for $(k=1,2,\cdots)$, and put

$$c_k\equiv 1/k^2,\quad n_k\equiv 2^{2k^2},\quad m_k\equiv 2^{k^2},\quad M_k\equiv 2^{4k^2}.$$

Then, using the notation in the proof of Theorem D,

$$I_2=(-1)^k\frac{c_k}{2}\log\left(1+\frac{n_k}{m_k}\right)+o(1)$$

$$=(-1)^k\frac{1}{2k^2}\log 2^{k^2}+o(1)=(-1)^k\log 2+o(1)$$

and $c_k \log n_k = O(1)$. Thus s_{M_k} does not converge and (1.7) is satisfied.

THEOREM F. *There is a continuous function $f(t)$ such that its Fourier series converges everywhere but does not converge uniformly at a point.*

PROOF. Let us take the function $f(t)$ defined by (1.2). We will take $n_k = m_k$. Then if we denote by I_2' the term in $s_{M_k}(x_k)$ corresponding to I_2, putting $x_k \equiv 2\pi/n_k - \pi/n_k^2$,

$$\begin{aligned} I_2' &= c_k \int_{\Delta_k} \sin M_k t \; \frac{\sin M_k (x-t)}{x-t} \; dt \\ &= \frac{c_k}{2} \int_{\Delta_k} \{-\cos M_k x + \cos (M_k x - 2M_k t)\} \frac{dt}{x-t} \\ &= -J_1 + J_2, \end{aligned}$$

say, where integrals are taken in the Cauchy sense. First,

$$J_1 = \frac{1}{2} c_k \cos M_k x \int_{\Delta_k} \frac{dt}{x-t} = \frac{1}{2} c_k \cos M_k x \log (n_k - 2).$$

If we suppose $n_k^2 | M_k$, then

$$\cos (M_k x - 2M_k t) = + \cos 2M_k (x - t),$$

and then $J_2 = O(c_k/M_k)$. Let us take

$$c_k \equiv 1/k, \quad m_k \equiv n_k \equiv 2^{k^2}, \quad M_k \equiv 2^{3k^2}.$$

Then I_2' does not converge. As easily may be seen from the proof of Theorem D, the Fourier series of $f(t)$ converges everywhere, but does not converge uniformly at $t = 0$.

We are now easy to construct a continuous function which satisfies the condition in Theorem F and (1.8).

§ 2. PROOF OF THEOREM 1. Let us consider a sequence of intervals

$$(2.1) \qquad \Delta_n \equiv \left(\frac{1}{n}, \frac{1}{n} + \frac{1}{n^2}\right) \qquad (n = 2, 3, \cdots)$$

which are mutually disjoint. Let us define the even periodic function $f(t)$ by

$$(2.2) \qquad f(t) \equiv \sin M_n t \quad (t \in \Delta_n)$$

for $n = 2, 3, \cdots$ and $f(t) \equiv 0$ in $(0, \pi) - \bigcup \Delta_n$, where (M_n) is an increasing sequence of integers determined later.

For the proof of Theorem 1 we can suppose $x = 0$ without any loss of generality. Since $f(0) = 0$, $f(t) = \phi_0(t)/2$. Evidently $f(t)$ is bounded and

$$\int_0^t |f(u)|\,du \geqq \frac{1}{\pi} \sum_{n=[1/t]+1}^{\infty} \frac{1}{n^2} > \frac{t}{2\pi}.$$

Thus the condition 1 of the theorem is satisfied. Let us prove that the condition 2 is also satisfied. For this purpose it is sufficient to prove that

$$I \equiv \int_0^\pi f(t)\,\frac{\sin mt}{t}\,dt = o(1) \qquad (m \to \infty). \tag{2.3}$$

Substituting (2.2) we get

$$\begin{aligned} I &= \sum_{n=2}^{\infty} \int_{\Delta_n} \sin M_n t\,\frac{\sin mt}{t}\,dt \\ &= \sum_{n=2}^{\infty} \int_{1/n}^{1/n+1/n^2} \sin M_n t\,\frac{\sin mt}{t}\,dt \\ &= \sum_{n=2}^{\infty} \left[\int_{1/n}^{1/n+1/n^2} \cos (M_n - m)t\,\frac{dt}{t} - \int_{1/n}^{1/n+1/n^2} \cos (M_n + m)t\,\frac{dt}{t} \right]. \end{aligned}$$

For any m there is a μ such as $M_\mu \leqq m < M_{\mu+1}$. Thus $\mu \to \infty$ as $m \to \infty$. Let us devide I into two parts.

$$I = \sum_{n=2}^{\infty} = \sum_{n=2}^{\mu} + \sum_{n=\mu+1}^{\infty} \equiv I_1 + I_2, \tag{2.4}$$

say. The term $n = \mu$ in I_1 is

$$\begin{aligned} &\int_{1/\mu}^{1/\mu+1/\mu^2} \cos (m - M_\mu)t\,\frac{dt}{t} - \int_{1/\mu}^{1/\mu+1/\mu^2} \cos (m + M_\mu)t\,\frac{dt}{t} \\ &= \int_{(m-M_\mu)1/\mu}^{(m-M_\mu)(1/\mu+1/\mu^2)} \frac{\cos t}{t}\,dt - \int_{(m+M_\mu)1/\mu}^{(m+M_\mu)(1/\mu+1/\mu^2)} \frac{\cos t}{t}\,dt \\ &= O\left(\log\left(1 + \frac{1}{\mu}\right)\right) = o(1) \end{aligned}$$

as $m \to \infty$. The term $n = \mu + 1$ in I_2 is similarly $o(1)$ as $m \to \infty$.

In order to estimate the remaining terms of I_1, we suppose $n < \mu$. If (M_n) is convex, then

$$m - M_n > M_\mu - M_n \geqq (\mu - n)\,(M_{n+1} - M_n). \tag{2.5}$$

If we take $M_n \equiv n^3$, then $M_{n+1} - M_n \geqq n^2$, whence

$$(m - M_n)/n \geqq (\mu - n)\,n.$$

by (2.5). Hence the n-th term of I_1 is

$$\int_{(m-M_n)/n}^{(m-M_n)(1/n+1/n^2)} \frac{\cos t}{t}\,dt - \int_{(m+M_n)/n}^{(m+M_n)(1/n+1/n^2)} \frac{\cos t}{t}\,dt$$

$$= O\left(\frac{n}{m-M_n}\right) = O\left(\frac{1}{(\mu-n)\,n}\right).$$

Thus

$$I_1 = \sum_{n=2}^{\mu-1} + o(1) = O\left(\sum_{n=2}^{\mu-1} \frac{1}{(\mu-n)\,n}\right) + o(1) \tag{2.6}$$

$$= O\left(\frac{\log \mu}{\mu}\right) + o(1) = o(1).$$

Similarly we have

$$I_2 = o(1) + \sum_{n=\mu+1}^{\infty} = o(1) + O\left(\sum_{n=\mu+1}^{\infty} \frac{n}{M_n - m}\right) \tag{2.7}$$

$$= o(1) + O\left(\sum_{n=\mu+1}^{\infty} \frac{1}{(n-\mu)\,n}\right) = o(1).$$

By (2.4), (2.6) and (2.7) we get $I = o(1)$. Thus we get (2.3), which is the required.

§ 3. Proof of Theorem 2. Let (m_k) and (n_k) be increasing sequences of integers, which will be determined later and let us take a sequence of intervals

$$\Delta_k \equiv \left(\frac{\pi}{n_k}, \frac{\pi}{n_k} + \frac{\pi}{m_k}\right) \qquad (k = 1, 2, \cdots) \tag{3.1}$$

which are taken disjoint mutually. Let us define an even function such that

$$f(t) = c_k\left[t \cos M_k t + \frac{1}{M_k} \sin M_k t\right] \qquad (t \in \Delta_k) \tag{3.2}$$

for $k = 1, 2, \cdots$ and $f(t) \equiv 0$ in $(0, \pi) - \bigcup \Delta_k$, where (c_k) is a sequence of positive numbers and (M_k) is an increasing sequence of positive integers, which will be determined later. We shall first suppose that

$$n_k \mid M_k, \quad m_k \mid M_k \qquad (k = 1, 2, \cdots). \tag{3.3}$$

Then

$$I_k \equiv \int_{\Delta_k} f(t)\,dt = \left[\frac{c_k}{M_k}\, t \sin M_k t\right]_{t=\pi/n_k}^{t=\pi/n_k + \pi/m_k} = 0. \tag{3.4}$$

By

$$\int_0^\pi |f(t)|^p\,dt \leqq \sum_{k=1}^\infty \int_{\Delta_k} |f(t)|^p\,dt \leqq \sum_{k=1}^\infty \frac{c_k^p}{n_k^p} \cdot \frac{1}{m_k},$$

in order that $f(t)$ belongs to L^p it is sufficient that

$$\sum_{k=1}^\infty \frac{c_k^p}{n_k^p} \frac{1}{m_k} < \infty. \tag{3.5}$$

By (3.4)

$$f_1(t) \equiv \int_0^t f(u)\,du = \frac{c_k}{M_k} t \sin M_k t$$

for $t \in \Delta_k$, and $f(t) = 0$ outside $\bigcup \Delta_k$. If we take

$$c_k = M_k \qquad (k = 1, 2, \cdots), \tag{3.6}$$

then the condition 1^0 is satisfied. In this case (3.5) becomes

$$\sum_{k=1}^\infty \frac{M_k^p}{n_k^p} \frac{1}{m_k} < \infty. \tag{3.7}$$

Let us now consider the Fourier series (1.2) of $f(t)$ and by s_n we denote the n-th partial sum of (0.2) at $t = 0$. The condition 2^0 is satisfied when

$$I \equiv \int_0^\pi f(t) \frac{\sin nt}{t}\,dt = o(1). \tag{3.8}$$

We will begin by the case $n = M_k$. Dividing I into three parts, we put

$$I = \sum_{i=1}^\infty \int_{\Delta_i} = \sum_{i=1}^\infty I_i = \sum_{i=1}^{k-1} I_i + I_k + \sum_{i=k+1}^\infty I_i \equiv J_1 + J_2 + J_3,$$

say. We have by (3.6),

$$\begin{aligned} J_2 &= \int_{\Delta_k} \sin^2 M_k t \frac{dt}{t} + M_k \int_{\Delta_k} \sin M_k t \cos M_k t\,dt \\ &= \frac{1}{2} \int_{\Delta_k} (1 - \cos 2M_k t) + \frac{M_k}{2} \int_{\Delta_k} \sin 2M_k t\,dt \\ &= \frac{1}{2} \log\left(1 + \frac{n_k}{m_k}\right) + O\left(\frac{n_k}{M_k}\right). \end{aligned} \tag{3.9}$$

If we take

$$n_k/m_k \to 0 \quad (k \to \infty), \tag{3.10}$$

then $J_2 = o(1)$.

$$J_1 = \sum_{i=1}^{k-1} \left\{ \int_{\Delta_i} \sin M_i t \sin M_k t \frac{dt}{t} + M_i \int_{\Delta_i} \cos M_i t \sin M_k t\,dt \right. \tag{3.11}$$

$$= O\left(\sum_{i=1}^{k-1} \frac{n_i}{M_k - M_i}\right) + O\left(\sum_{i=1}^{k-1} \frac{M_i}{M_k - M_i}\right) = O\left(\sum_{i=1}^{k-1} \frac{M_i}{M_k - M_i}\right).$$

Similarly

$$J_2 = O\left(\sum_{i=k+1}^{\infty} \frac{n_i}{M_i - M_k} + M_k \sum_{i=k+1}^{\infty} \frac{M_i}{m_i\, n_i}\right). \tag{3.12}$$

Let us take

$$c_k \equiv M_k \equiv k^3 2^{k^2}, \quad m_k \equiv k^2 2^{k^2}, \quad n_k \equiv 2^{k^2}. \tag{3.13}$$

Then the conditions (3.3), (3.6), (3.7) and (3.10) are satisfied, and the sequence of intervals (3.1) is disjoint. By (3.11) and (3.12), $J_1 = o(1)$, $J_2 = o(1)$.

Thus the condition 2^0 is satisfied for $n = M_k$ $(k = 1, 2, \dots)$. We have to show it for all n. For this purpose we take k such as $M_k < n < M_{k+1}$, and put

$$I = \sum_{i=1}^{\infty} I_i = \sum_{i=1}^{k-1} I_i + (I_k + I_{k+1}) + \sum_{i=k+2}^{\infty} I \equiv J_1' + J_2' + J_3',$$

J_2' can be estimated similarly as J_2, except that the second term of (3.9) does not vanish but is sufficiently small. J_1' and J_3' are similarly estimated as J_1 and J_3, respectively. Thus Theorem 2 is completely proved.

§ 4. Proof of Theorem 3. Let us take the sequence of intervals (3.1) and define $f(t)$ by (3.2). If the condition (3.3), (3.6) and (3.7) are satisfied, then (3.4) holds, $f \in L^p$ and the condition 3^0 is satisfied. The condition 2^0 is satisfied when

$$I \equiv \int_0^{\pi} f(t) \frac{\sin nt}{t}\, dt = O\left(\frac{1}{\log n}\right). \tag{4.1}$$

Putting $n \equiv M_k$ and $I \equiv J_1 + J_2 + J_3$ as in § 3, we get, by (3.9),

$$J_2 = \frac{n_k}{2m_k}(1 + o(1)) \tag{4.2}$$

when (3.10) is satisfied. If we take

$$\frac{n_k}{m_k} \log M_k \to a \neq 0, \tag{4.3}$$

then $J_2 = O(1/\log M_k)$.

Let us put

(4.4) $$c_k \equiv M_k \equiv k^3 \cdot {}^{k^2}, \qquad m_k \equiv k^2 \cdot 2^{k^2}, \qquad n_k \equiv 2^{k^2}.$$

Then the condition (3.3), (3.6), (3.7), (3.10) and (4.3) are satisfied and $J_1 + J_3 = O(1/\log M_k)$. Thus we get (4.1). For general n we get also (4.1) as in § 3.

Concerning the condition 1^0,

$$a_{M_k} = \frac{2}{\pi} \int_0^\pi J(t) \cos M_k t \, dt$$

$$= \frac{2}{\pi} \sum_{i=1}^\infty M_i \int_{\Delta i} \left[t \cos M_k t + \frac{1}{M_k} \sin M_k t \right] \cos M_k t \, dt$$

$$= \frac{1}{\pi} \frac{M_k}{n_k m_k} (1 + o(1)) = O(M_k^{-\delta})$$

for $0 < \delta < 1$. For general n, a_n becomes also $O(1/n^\delta)$. Thus the theorem is completely proved.

§ 5. Proof of Theorem 4. The case $\alpha = 1$. Let us consider the sequence of intervals (3.1) and let $f(t)$ be an even function such that

(5.1) $$f(t) \equiv \frac{M_k}{n_k} \cos M_k t \qquad (t \in \Delta_k)$$

for $k = 1, 2, \cdots$ and $f(t) \equiv 0$ in $(0, \pi) - \bigcup \Delta_k$. If we suppose (3.3), then

(5.2) $$\int_{\Delta_k} f(t) \, dt = \left[\frac{\sin M_k t}{n_k} \right]_{t=\pi n_k}^{t=\pi n_k + \pi/m_k} = 0.$$

By

$$\int_0^\pi |f(t)|^p \le \sum_{k=1}^\infty \left(\frac{M_k}{n_k} \right)^p \int_{\Delta k} dt = \pi \sum_{k=1}^\infty M_\varphi^k / n^p m_k,$$

In order that $f \in L^p$, it is sufficient to take

(5.3) $$\sum_{k=1}^\infty M_k^p / n_k^p m_k < \infty.$$

For $t \in \Delta_k$,

(5.4) $$\int_0^t f(u) \, du = \int_{\pi n_k}^t f(u) \, du = \frac{\sin M_k t}{n_k}$$

by (5.1) and (5.2). If we suppose (3.10), then, by (5.2) and (5.4), $\Phi(t) \equiv \frac{1}{t} \int_0^t f(u) \, du = (1 + o(1)) \sin M_k t$ $(t \in \Delta_k)$ for $k = 1, 2, \cdots$ and $\Phi(t) = 0$ in $(0, \pi) - \bigcup \Delta_k$. And then, for $t \in \Delta_k$,

$$\int_0^t |\Phi(u)|\,du \geqq \frac{1}{\pi}\sum_{j=k+1}^{\infty}\int_{\Delta_j} du \geqq \sum_{j=k+1}^{\infty}\frac{1}{m_j}. \tag{5.5}$$

If we suppose

$$\sum_{j=k+1}^{\infty}\frac{1}{m_j} > \frac{1}{2n_k}, \tag{5.6}$$

then

$$\int_0^t |\Phi(u)|\,du \geqq \frac{t}{2}, \tag{5.7'}$$

which implies the condition 1°.

Let us consider the Fourier series (1.2) of $f(t)$, and by σ_n denote its $(C, 1)$ mean at $t = 0$. Then

$$\begin{aligned}
\sigma_n - f(0) = \sigma_n &= \frac{1}{\pi(n+1)}\int_0^\pi f(t)\,\frac{\sin^2 (n+1)t/2}{\sin^2 t/2}\,dt \\
&= \frac{4}{\pi n}\int_0^\pi f(t)\,\frac{\sin^2 nt/2}{t^2}\,dt + o(1) \\
&= \frac{2}{\pi}\int_0^\pi \Phi(t)\,\frac{\sin nt}{t}\,dt + \frac{8}{\pi n}\int_0^\pi \Phi(t)\frac{\sin^2 nt/2}{t^2}\,dt + o(1) \\
&\equiv \frac{2}{\pi} I + J + o(1),
\end{aligned} \tag{5.8}$$

say. By the definition

$$\int_0^t \Phi(u)\,du = o(t), \quad \int_0^t |\Phi(u)|\,du = O(t),$$

which implies $J = o(1)$ by a well known theorem. We have also

$$\begin{aligned}
I &= \int_0^\pi \Phi(t)\,\frac{\sin nt}{t}\,dt \\
&= \sum_{k=1}^{\infty}\int_{\Delta_k}(1 + o(1))\,\sin M_k t\,\frac{\sin nt}{t}\,dt \\
&= \sum_{k=1}^{\infty}\left[\int_{\Delta_k}(1 + o(1))\,\frac{\cos (M_k - n)t}{t}\,dt - \int_{\Delta_k}(1 + o(1))\,\frac{\cos (M_k + n)t}{t}\,dt\right].
\end{aligned}$$

For any n there is a μ such as $M_\mu < n \leqq M_{\mu+1}$. Let

$$I = \sum_{k=1}^{\infty} = \sum_{k=1}^{\mu} + \sum_{k=\mu+1}^{\infty} \equiv I_1 + I_2.$$

As I'_2 in §3, the last term in I_1 and the first term of I_2 are

$$O\left(\log\left(1 + n_k/m_k\right)\right) = O\left(n_k/m_k\right) = o(1)$$

by (3.10). If (M_k) is taken as convex and

$$M_{k+1} - M_k > kn_k \qquad (k=1,2,\ldots), \tag{5.9}$$

then, for $k < \mu$,

$$(n - M_k)/n > (M_\mu - M_k)/n_k > (\mu - k)(M_{k+1} - M_k)/n_k > (\mu - k)k.$$

Thus

$$I = \sum_{k=1}^{\mu-1} + o(1) = O\left(\sum_{k=1}^{\mu-1} \frac{n_k}{n - M_k}\right) + o(1)$$

$$= O\left(\sum_{k=1}^{\mu-1} \frac{1}{(\mu - k)k}\right) + o(1) = o(1).$$

Let a be an integer $> 2p + 1$ and put

$$M_k \equiv k^{a+2}, \quad m_k \equiv k^{a+1}, \quad n_k \equiv k^a. \tag{5.10}$$

Then the conditions (3.3), (5.3), (3.10), (5.7) and (5.9) are satisfied. Moreover

$$I_2 = o(1) + \sum_{k=\mu+1}^{\infty} O\left(\frac{n_k}{M_k - M_\mu}\right) = o(1).$$

Thus $I = o(1)$, where $\sigma_n = o(1)$ by (5.8). Thus Theorem 4 is proved for $\alpha = 1$.

§ 6. Proof of Theorem 4. The general case. We may restrict to the case $0 < \alpha < 1$ only. For, the cases $\alpha = 0$ and $\alpha = 1$ were proved and the general case $\alpha > 1$ is obtained by the combination of two methods in § 5 and § 6.

We need a lemma[4].

Lemma. *If* $0 < \alpha < 1$ *and*

$$\int_0^t \varphi_x(u)\, du = o(t^{1-\alpha}) \qquad (t \to 0), \tag{6.1}$$

then the Fourier series (1.2) *of* $f(t)$ *is summable* (C,a) *at* $t = x$.

For the proof of Theorem 4, we may suppose that $x = 0$. Taking a sequence of disjoint intervals (3.1) and put

$$f(t) \equiv \frac{M_k^\alpha}{n_k^\alpha} \cos M_k t \qquad (t \in \Delta_k) \tag{6.3}$$

for $k = 1, 2, \ldots$ and $f(t) \equiv 0$ in $(0,\pi) - \bigcup \Delta_k$, $f(t) \equiv f(-t)$ in $(-\pi, 0)$. If we suppose (3.3), then (5.2) holds. If

4) Cf. Izumi and Sunouchi, Notes on Fourier Analysis (XXXI): Theorems concerning Cesàro summability.

$$\sum_{k=1}^{\infty} \frac{M_k^{\alpha p}}{n_k^{\alpha p}\, m_k} < \infty, \tag{6.4}$$

then $f \in L^p$ for $p > 1$. By $f_\alpha(u)$ we denote the α-th integral of $f(u)$, then

$$f_\alpha(t) = \sum_{j=k+1}^{\infty} \int_{\Delta_j} f(u)(t-u)^{\alpha-1}\,du + \int_{\pi n_k}^{t} f(u)\,(t-u)^{\alpha-1}\,du$$

$$= \sum_{j=k+1}^{\infty} \frac{M_j^\alpha}{n_j^\alpha} \int_{\Delta_j} \cos M_j t\,(t-u)^{\alpha-1}\,du + \frac{M_k^\alpha}{n_k^\alpha} \int_{\pi n_k}^{t} \cos M_k t\,(t-u)^{\alpha-1}\,du$$

for t in Δ_k. If we put

$$\Psi_\alpha(t) = \frac{M_k^\alpha}{t^\alpha n_k^\alpha} \int_{\pi n_k}^{t} \cos M_k u\,(t-u)^{\alpha-1}\,du \qquad (t \in \Delta_k)$$

for $k = 1, 2, \ldots$ and $\Psi_\alpha(t) \equiv 0$ in $(0, \pi) - \bigcup \Delta_k$, and put

$$\Theta_\alpha(t) \equiv \sum_{j=k+1}^{\infty} \frac{M_j^\alpha}{t^\alpha n_j^\alpha} \int_{\Delta_j} \cos M_j u\,(t-u)^{\alpha-1}\,du$$

in the interval $\Delta_k^1 \equiv \left(\frac{\pi}{n_{k+1}} + \frac{\pi}{m_{k+1}}, \frac{\pi}{n_k} + \frac{\pi}{m_k}\right)$ $(k = 1, 2, \ldots)$, then we have

$$\Phi_\alpha(t) = \Psi_\alpha(t) + \Theta_\alpha(t).$$

Since

$$\left|\int_{\Delta_i} \cos M_i u\,(t-u)^{\alpha-1}\,du\right| \leqq \frac{1}{M_i}\left(t - \frac{\pi}{n_i}\right)^{\alpha-1},$$

for $i \geqq k+1$, we have

$$|\Theta_\alpha(t)| \leqq \frac{1}{t} \sum_{i=k+1}^{\infty} \frac{1}{n_i^\alpha M_i^{1-\alpha}},$$

$$\int_0^t |\Theta_\alpha(u)|\,du \leqq 2 \sum_{j=k+1}^{\infty} \log \frac{n_{j+1}}{n_j} \sum_{i=j+1}^{\infty} \frac{1}{n_i^\alpha M_i^{1-\alpha}}.$$

Hence, if

$$\sum_{j=k+1}^{\infty} \log \frac{n_{j+1}}{n_j} \sum_{i=j+1}^{\infty} \frac{1}{n_i^\alpha M_j^{1-\alpha}} = o\left(\frac{1}{n_k}\right), \tag{6.5}$$

then we have

$$\int_0^t |\Theta_\alpha(u)|\,du = o(t). \tag{6.6}$$

On the other hand, for $t \in \Delta_k$,

$$\begin{aligned}\Psi_\alpha(t) &= \frac{1}{t^\alpha}\int_{\pi/n_k}^{t} \frac{M_k^\alpha}{n_k} \cos M_k u \,(t-u)^{\alpha-1}\,du \\ &= \frac{M_k^\alpha}{t^\alpha n_k}\int_{\pi/n_k}^{t} \cos M_k u\,(t-u)^{\alpha-1}\,du \\ &= \frac{M_k^\alpha}{n_k}\int_{\pi/tn_k}^{1} \cos M_k t u\,(1-u)^{\alpha-1}\,du.\end{aligned}$$

As easily may be seen by elementary estimation, we have

$$\int_{\Delta_k} |\Psi_\alpha(t)|\,dt \geqq \frac{\text{const}}{m_k}.$$

Thus, if the condition (5.7) is satisfied, we get

$$\int_0^t |\Psi_\alpha(u)|\,du \neq o(t).$$

whence, by (6.6),

$$\int_0^t |\Phi_\alpha(u)|\,du \neq o(t).$$

This is nothing but the condition 1°.

Concerning the condition 2°, it is sufficient to prove (6.1).[5] We have

$$\int_0^t f(u)\,du = \int_{\pi/n_k}^t f(u)\,du = \frac{M_k^\alpha}{n_k^\alpha}\int_{\pi/n_k}^t \cos M_k u\,du = O\left(\frac{1}{n_k^\alpha M_k^{1-\alpha}}\right)$$

for t in Δ_k, which is $o(n_k^{1/\alpha})$ when

$$n_k^{\frac{1+\alpha}{\alpha}} / M_k = o(1). \tag{6.7}$$

Let us take

$$M_k \equiv k^b, \quad m_k \equiv k^{a+1}, \quad n_k \equiv k^a, \tag{6.8}$$

a and b being positive integers. (3.3) and (3.10) are evident when $b > a+1$. (6.4) and (6.7) are satisfied when

$$a(\alpha+1) > \alpha b. \tag{6.9}$$

(6.5) is satisfied for sufficiently large p.

Hence the theorem is proved.

For the general α we remark that, for integral α we use α times the integration by parts to the α-th Cesàro mean σ_n^α of the Fourier series and use

5) Cordition 2° can, of course, be proved directly, without use of Lemma. Direct estimation leads also to (6.7).

the method of §5, and for non-integral α we use $[\alpha]$ times integration by parts to σ_n^α and the above method. In these case $\Phi_\alpha(t)$ does not vanish in $(0,\pi)-\bigcup\Delta_k$ for $\alpha \geqq 2$. Estimation of terms rising from such part is easy.[6]

§7. Proof of Theorem 5. We can prove the theorem modifying the example of theorem 4. We will now show the method of modification for the case $\alpha=1$. In the example of Theorem 4 $n_k/m_n \to 0$, but in this case $m_k/n_k \to 0$, that is, the length of Δ_k is taken longer in this case. Therefore we denote the sequence of intervals by

$$\Delta_k = \left(\frac{\pi}{n_k}, \frac{\pi}{m_k}\right) \qquad (k=1,2,\cdots). \tag{7.1}$$

We define an even function by

$$f(u) \equiv \frac{M_k}{m_k} c_k \cos M_k u \qquad (u \varepsilon \Delta_k) \tag{7.2}$$

for $k=1,2,\cdots$ and $f(u)\equiv 0$ in $(0,\pi)-\bigcup\Delta_k$. When (3.3) is satisfied, (5.2) holds. When

$$\sum_{k=1}^{\infty} \frac{N_k^p}{m_k^{p+1}} < \infty, \tag{7.3}$$

$f \varepsilon L^p$. Moreover

$$\Phi(t) = \frac{1}{t}\int_0^t f(u)\,du = c_k(1+o(1))\sin M_k t \qquad (t \varepsilon \Delta_k)$$

for $k=1,2,\cdots$ and $\Phi(t)=0$ in $(0,\pi)-\bigcup\Delta_k$. Thus $\Phi(t)=o(1)$ as $t\to 0$, when

$$c_k \to 0 \qquad (k\to\infty). \tag{7.4}$$

Hence the condition 1° is satisfied.

In order to verify the condition 2°, we consider (5.8). Evidently $J \rightleftarrows o(1)$. Hence it is sufficient to show that $I\to\infty$. Now

$$\frac{\pi}{2} I = \int_0^\pi \Phi(t)\frac{\sin M_\mu t}{t}\,dt$$
$$= \sum_{k=1}^{\infty} c_k \int_{\Delta_k} (1+o(1))\sin M_k t \frac{\sin M_\mu t}{t}\,dt$$

for $n=M_\mu\ (\mu=1,2,\cdots)$. Here we can omit the term $o(1)$ by its structure. Hence

6) Cf. §8. In the case $\alpha \geqq 1$, insted of (6.1) use $\int_0^t \varphi_x(u)\,du = O(t^{2-1/\alpha})$. See Izumi and Sunouchi; loc. cit.

$$I' \equiv \sum_{k=1}^{\infty} c_k \int_{\Delta_k} \frac{\sin M_k t \sin M_\mu t}{t}\, dt$$
$$= \sum_{k=1}^{\mu-1} + c_\mu \int_{\Delta_\mu} \frac{\sin^2 M_\mu t}{t}\, dt + \sum_{k=\mu+1}^{\infty}$$
$$\equiv I_1' + I_2' + I_3',$$

say, where

$$I_2' = c_\mu \int_{\pi/n_\mu}^{\pi/m_\mu} \frac{\sin^2 M_\mu t}{t}\, dt$$
$$= \frac{c_\mu}{2} \int_{\pi/n_\mu}^{\pi/m_\mu} \frac{dt}{t} - \frac{c_\mu}{2} \int_{\pi/n_\mu}^{\pi/m_\mu} \frac{\cos 2M_\mu t}{t}\, dt$$
$$= \frac{c_\mu}{2} \log \frac{n_\mu}{m_\mu} + o\left(\frac{n_\mu}{M_\mu}\right),$$

which tends to ∞ when c_μ tends to zero sufficiently showly and

(7.5) $$n_k / m_k \to \infty, \quad n_k / M_k \to 0 \qquad (k \to \infty).$$

Now, concerning I_1,

$$I_1' = \sum_{k=1}^{\mu-1} \frac{c_k}{2} \left[\int_{\pi/n_k}^{\pi/m_k} \frac{\cos (M_\mu - M_k) t}{t}\, dt - \int_{\pi/n_k}^{\pi/m_k} \frac{\cos (M_\mu + M_k) t}{t}\, dt \right]$$
$$= \sum_{k=1}^{\mu-1} \frac{c_k}{2} \left[\int_{\pi(M_\mu - M_k)/n_k}^{\pi(M_\mu - M_k)/m_k} \frac{\cos t}{t}\, dt - \int_{\pi(M_\mu + M_k)/n_k}^{\pi(M_\mu + M_k)/m_k} \frac{\cos t}{t}\, dt \right],$$

and so

$$|I_1'| \leqq 2 \sum_{k=1}^{\mu-1} \frac{n_k}{M_\mu - M_k}.$$

Let (M_k) be a convex sequence and

(7.6) $$M_k \equiv k^2 n_k \qquad (k = 1, 2, \cdots),$$

then $M_\mu - M_k \geqq (\mu - k) k n_k$ and then $I_1' = o(1)$. Similarly $I_3' = o(1)$. Thus we get $I \to \infty$.

If we take

$$M_k \equiv k^2 2^k, \quad m_k \equiv k 2^k, \quad n_k \equiv 2^k, \quad c_k \equiv 1/\log\log k,$$

then the conditions (3.3), (7.3), (7.4), (7.5) and (7.6) are satisfied. And (M_k) is convex and (Δ_k) is a system of disjoint intervals.

§ 8. Proof of Theorem 6. The case $\alpha = 0$ is proved in Theorem 3

Let us first consider the case $\alpha = 1$. Taking the sequence of intervals (3.1), we define an function $f(t)$ by

(8.1) $$f(t) \equiv \frac{c_k}{M_k} \sin M_k t + 3c_k t \cos M_k t - c_k M t^2 \sin M t$$

in Δ_k ($k = 1, 2, \cdots$) and $f(t) \equiv 0$ in $(0, \pi) - \bigcup \Delta_k$. If we put

$$y(t) \equiv \frac{c_k}{M_k} t \sin M_k t.$$

then $f(t) = (ty'(t))'$ in Δ_k, dash denoting differentiation with respect to t. We have

$$\frac{1}{\pi} \int_{\Delta_k} |f(t)| \, dt \leqq \frac{c_k}{m_k M_k} + \frac{3c_k}{n_k m_k} + \frac{c_k M_k}{n_k^2 m_k}.$$

If we suppose (3.3), then

$$\int_0^\pi |f(t)| \, dt = \sum_{k=1}^\infty \int_{\Delta k} |f(t)| \, dt < \infty$$

when

(8.2) $$\sum \frac{c_k}{n_k m_k} < \infty, \quad \sum \frac{c_k M_k}{n_k^2 m_k} < \infty.$$

Further, if

(8.3) $$M_k / m_k, \quad M_k / n_k \quad \text{are even,}$$

then

$$\int_{I_k} f(t) \, dt = \left[\frac{c_k}{M_k} t \sin M t + c_k t^2 \cos M_k t \right]_{t=\frac{\pi}{n_k}}^{t=\frac{\pi}{n_k}+\frac{\pi}{m_k}}$$

$$= c_k \frac{\pi}{m_k} \left(\frac{2\pi}{n_k} + \frac{\pi}{m_k} \right).$$

Since

$$\int_{\pi/n_k}^t f(u) \, du = \left[\frac{c_k}{M_k} u \sin M_k u + c \, u^2 \cos M_k u \right]_{u=\pi/n_k}^{u=t}$$

$$= \frac{c_k}{M_k} t \sin M_k t + c_k t^2 \cos M_k t - c_k \left(\frac{\pi}{n_k} \right)^2,$$

we have

$$f_1(t) = \int_0^t f(u) \, du$$

$$= \frac{c_k}{M_k} t \sin M_k t + c_k t^2 \cos M_k t$$

$$+ \sum_{i=k+1}^{\infty} c_k \frac{\pi}{m_k}\left(\frac{2\pi}{n_k} + \frac{\pi}{m_k}\right) - c\left(\frac{\pi}{n_k}\right)^2,$$

for $t \in \Delta_k$, and

$$f_1(t) = \sum_{i=k+1}^{\infty} c_k \frac{\pi}{m_i}\left(\frac{2\pi}{n_i} + \frac{\pi}{m_i}\right) - c_i\left(\frac{\pi}{n_i}\right)^2$$

for $\frac{\pi}{n_{k+1}} + \frac{\pi}{m_{k+1}} < t < \frac{\pi}{n_k}$. Let us put

$$g_1(t) \equiv \frac{c_k}{M_k} t \sin M_k t + c_k t^2 \cos M_k t \qquad (t \in \Delta_k)$$

for $k = 1, 2, \cdots$ and $g_1(t) \equiv 0$ in $(0, \pi) - \bigcup \Delta_k$;

$$h_1(t) \equiv \sum_{i=k+1}^{\infty} c_i \frac{\pi}{m_i}\left(\frac{\pi}{n_i} + \frac{\pi}{m_i}\right)$$

in the interval $\Delta_k' \equiv \left(\frac{\pi}{n_{k+1}} + \frac{\pi}{m_{k+1}}, \frac{\pi}{n_k} + \frac{\pi}{m_k}\right)$ $(k = 1, 2, \cdots)$;

$$k_1(t) \equiv - c_k (\pi/n_k) \qquad (t \in \Delta_k)$$

for $k = 1, 2, \cdots$ and $k_1(t) \equiv 0$ in $(0, \pi) - \bigcup \Delta_k$. Then we have

$$f_1(t) = g_1(t) + h_1(t) + k_1(t)$$

for all t in $(0, \pi)$, and

$$\text{(8.4)} \qquad \Phi(t) = \frac{f_1(t)}{t} = \frac{g_1(t)}{t} + \frac{h_1(t)}{t} + \frac{k_1(t)}{t}$$

$$\equiv \psi_1(t) + \chi_1(t) + \theta_1(t),$$

say.

Let us consider the Fourier series of $f(t)$ and σ_n be its Cesàro mean of order 1. In order to get $\sigma_n = O(1/\log n)$, it is sufficient to prove that

$$\sigma_n' \equiv \frac{1}{n} \int_0^{\pi} f(t) \frac{\sin^2 nt/2}{t^2} dt = O(1/\log n).$$

Now

$$\sigma_n' = \int_0^{\pi} \Phi(t) \frac{\sin nt}{t} dt - \frac{2}{n} \int_0^{\pi} \Phi(t) \frac{\sin^2 nt}{t^2} dt \equiv J_1 + J_2,$$

say. Substituting (8.4) into J_1,

$$J=\int_0^\pi \psi_1(t)\frac{\sin nt}{t}dt+\int_0^\pi \chi_1(t)\frac{\sin nt}{t}dt+\int_0^\pi \theta_1(t)\frac{\sin nt}{t}dt\equiv K_1+K_2+K_3,$$

say. Similarly we put $J_2\equiv K_1'+K_2'+K_3'$.

$$K_1=\int_0^\pi \psi_1(t)\frac{\sin nt}{t}dt=\sum_{i=0}^\infty\int_{\Delta_i}\frac{g_1(t)}{t}\frac{\sin nt}{t}dt$$

$$=\sum_{i=0}^\infty\int_{\Delta_i}\left\{\frac{c_i}{M_i}\sin M_it+c_it\cos M_it\right\}\frac{\sin nt}{t}dt.$$

Putting first $n=M_k$, and dividing K_1 into three parts, we put

$$K_1=\sum_{i=0}^\infty\int_{\Delta_i}=\sum_{i=0}^{k-1}\int_{\Delta_i}+\int_{\Delta_k}+\sum_{i=k+1}^\infty\int_{\Delta_i}\equiv L_1+L_2+L_3,$$

where

$$L_2=\frac{c_k}{M_k}\int_{\Delta_k}\frac{\sin^2 M_kt}{t}dt+c_k\int_{\Delta_k}\cos M_kt\sin M_kt\,dt$$

$$=\frac{c_k}{2M_k}\int_{\Delta_k}\left(\frac{1}{t}-\frac{\cos 2M_kt}{t}\right)dt$$

$$=\frac{c_k}{2M_k}\log\left(1+\frac{n_k}{m_k}\right)-\frac{c_k}{2M_k}\int_{\frac{2\pi M_k}{n_k}}^{2\left(\frac{\pi}{n_k}+\frac{\pi}{m_k}\right)M_k}\frac{\cos t}{t}dt.$$

If the condition (3.10) is supposed,

$$L_2=\frac{c_k n_k}{2M_km_k}(1+o(1))+O\left(c_kn_k/M_k^2\right).$$

In order that $L_2=O(1/\log M_k)$, it is sufficient that

$$\frac{c_k n_k}{M_k m_k}\log M_k\to 0. \tag{8.5}$$

Similarly estimating as (3.11) and (3.12),

$$L_1=O\left(\sum_{i=1}^{k-1}\frac{M_i}{M_k-M_i}\right), \tag{8.6}$$

$$L_2=O\left(\sum_{i=k+1}^{\infty}\frac{n_i}{M_i-M_k}+M_k\sum_{i=k+1}^{\infty}\frac{M_i}{m_i n_i}\right). \tag{8.7}$$

Before going to the estimation of K_2, K_3 and J_2, we will consider the condition 2°. We put

$$f_2(t) \equiv \int_0^t \Phi(t)\,dt = \int_0^t \psi_1(u)\,du + \int_0^t \chi_1(u)\,du + \int_0^t \theta_1(u)\,du$$
$$\equiv g_2(t) + h_2(t) + k_2(t).$$

Since

$$g_2(t) = \frac{c_k}{M_k}\, t \sin M_k t,$$

we have

$$\psi_2(t) \equiv g_2(t)/t = O(1), \text{ and } \neq o(1),$$

when

(8.8) $$c_k = M_k \qquad (k = 1, 2, \cdots).$$

Then (8.5) becomes

(8.9) $$\frac{n_k}{m_k} \log M_k \to a \neq 0.$$

Let us take

(8.10) $$c_k \equiv M_k \equiv (2k)^3 2^{k^2}, \quad m \equiv (2k)^2 2^{k^2}, \quad n_k \equiv k^2.$$

Then the conditions (8.2), (8.3), (8.8), and (8.9) are satisfied. By (8.6) and (8.7),

$$L_1 + L_2 = O(1/\log M_k).$$

Thus we get $K_1 = O(1/\log M_k)$. Similarly $K_1' = O(1\ \log M_k)$.

Concering K_2,

$$K_2 = \int_0^\pi \chi_1(t) \frac{\sin M_k t}{t}\, dt$$
$$= \sum_{i=1}^\infty \int_{\Delta_i} \frac{\sin M_k t}{t^2}\, dt \sum_{j=i+1}^\infty c_j \frac{\pi}{m_j}\left(\frac{2\pi}{n_j} + \frac{\pi}{m_j}\right)$$
$$= \sum_{i=1}^{k-1} + \sum_{i=k}^\infty \equiv K_2' + K_2'',$$

say. We have

$$K_2' = O\left(\sum_{i=1}^{k-1} \frac{n_{i+1} M_{i+1}}{M_k\, m_{i+1}}\right) = O\left(\frac{1}{\log M_k}\right),$$

$$K_2'' = O\left(\sum_{i=k}^\infty \frac{M_k \cdot i M_{i+1}}{m_{i+1}\, n_{i+1}}\right) = O\left(\frac{1}{\log M_k}\right).$$

Thus we have $K_2 = O(1/\log M_k)$. K_3 is also of order $O(1/\log M_k)$. For general n we can estimate similarly as in the last part of §3. Thus we have proved

Theorem 6 for $\alpha=1$. For the case $\alpha=2$, it is sufficient to use

$$f(t)=(t(ty'(t))')',$$

and so on. For fractional α such as $1<\alpha<2$, we have

$$f(t)=(t^{\alpha-1}(ty'(t))')^{(\alpha-1)}$$

where $z^{(\beta)}$ denotes a sort of the β-th derivative $(0<\beta<1)$ such that

$$(t^n)^{(\alpha-1)}=nt^{n-(\alpha-1)},\ (\sin nt)^{(\alpha-1)}=n^{\alpha-1}\sin nt,\ (\cos nt)^{(\alpha)}=n^{\alpha-1}\cos nt$$

and

$$(zw)^{(\beta)}=z^{(\beta)}w+zw^{(\beta)}.$$

For general α, it is easy to write the form of function. Estimation is quite similar.

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