EXISTENCE OF A POTENTIAL FUNCTION WITH A PRESCRIBED SINGULARITY ON ANY RIEMANN SURFACE

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Let F be a closed or an open Riemann surface spread over the z-plane, then by the Dirichlet principle, it is proved[1] that there exists a potential function $u(z, z_0)$, which has a polar singularity at z_0 and a potential function $u(z; \zeta_1, \zeta_2)$, which has logarithmic singularities at ζ_1, ζ_2. In this paper, I shall prove this theorem and a more general Osgood's theorem simply by means of the modified Green's function defined in § 1.

1. Modified Green's function

1. Let F be an open Riemann surface, which contains $z = 0$. We approximate F by a sequence of compact Riemann surfaces $F_0 \subset F_1 \subset \cdots \subset F_n \to F$, where F_0 contains $z = 0$ and the boundary Γ_n of F_n consists of a finite number of analytic Jordan curves. Let $g_n(z, 0)$ be the Green's function of F_n with $z = 0$ as its pole and let at $z = 0$,

$$g_n(z, 0) = \log 1/|z| + \gamma_n(0) + \varepsilon_n(z) \qquad (\varepsilon_n(0) = 0), \tag{1}$$

where $\gamma_n(0)$ is the Robin's constant.

Let

$$M_n = \operatorname*{Max}_{\Gamma_0} g_n(z, 0), \tag{2}$$

then Heins[2] proved that

$$|M_n - g_n(z, 0)| \leqq K(\Delta) \quad (n = 1, 2, \cdots) \tag{3}$$

in any compact domain Δ, which lies outside F_0, where $K(\Delta)$ is a constant, which depends on Δ only.

Hence on Γ_1,

$$|M_n - g_n(z, 0) - \log 1/|z|| \leqq K(= \text{const.}) \qquad (n = 1, 2, \cdots), \tag{4}$$

so that at $z = 0$,

$$|M_n - \gamma_n(0)| \leqq K \qquad (n = 1, 2, \cdots). \tag{5}$$

From (3), (4), (5), we have easily the following theorem.

THEOREM 1. $\quad |g_n(z, 0) - \gamma_n(0)| \leqq K(\Delta) \qquad (n = 1, 2, \cdots)$

1) H. WEYL., Die Idee der Riemannschen Fläche, Berlin (1923). HURWITZ-COURANT: Funktionentheorie, Berlin (1929).

2) HEINS, The conformal mapping of simply connected Riemann surfaces, Annals of Math., **50** (1949).

in any comcapt domain Δ, *which does not contain* $z=0$ *and for any* z *in* F_0
$|g_n(z, 0)-\log 1/|z|-\gamma_n(0)| \leqq K|z| \qquad (n=1, 2, \cdots) \qquad (K=\text{const.})$.

Hence we can find a partial sequence n_ν, such that

$$\lim_\nu (g_{n_\nu}(z, 0)-\gamma_{n_\nu}(0)) = g(z, 0) \tag{6}$$

converges uniformly in any compact domain, which does not contain $z=0$. $g(z, 0)$ is harmonic on F, except at $z=0$, where $g(z, 0)-\log 1/|z|$ is harmonic and vanishes.

2. For a closed surface, the following theorem holds.

THEOREM 2. *Let* F *be a closed Riemann surface and* z_0 $(\neq 0)$ *be a point of* F *and* $F_n = F-\Delta_n$, *where* Δ_n: $|z-z_0| \leqq r_n$ $(r_1 > r_2 > \cdots > r_n \to 0)$. *Let* $g_n(z, 0)$ *be the Green's function of* F_n, *then*

$$\lim_n (g_n(z, 0)-\gamma_n(0)) = g(z, 0)$$

and

$$\lim_n \left(\gamma_n(0)-\log \frac{1}{r_n}\right)$$

exist and $g(z, 0)+\log 1/|z-z_0|$ *is harmonic at* $z=z_0$.

PROOF. Let

$$\lim_\nu (g_{n_\nu}(z, 0)-\gamma_{n_\nu}(0)) = g(z, 0) \tag{1}$$

and in $0 < |z-z_0| \leqq r_1$, $(z-z_0 = re^{i\theta})$,

$$g(z, 0) = \log r + a_0 + \sum_{k=1}^\infty (a_k r^k + a_{-k} r^{-k}) \cos k\theta$$
$$+ \sum_{k=1}^\infty (b_k r^k + b_{-k} r^{-k}) \sin k\theta, \tag{2}$$

the coefficient of $\log r$ is 1, since $\displaystyle\int_0^{2\pi} \frac{\partial g}{\partial r} r d\theta = 2\pi$. Let in $r_n \leqq |z-z_0| \leqq r_1$,

$$g_n(z, 0)-\gamma_n(0) = \log r + a_0^{(n)} + \sum_{k=1}^\infty (a_k^{(n)} r^k + a_{-k}^{(n)} r^{-k}) \cos k\theta$$
$$+ \sum_{k=1}^\infty (b_k^{(n)} r^k + b_{-k}^{(n)} r^{-k}) \sin k\theta. \tag{3}$$

Then by (1),

$$\lim_\nu a_0^{(n\nu)} = a_0,\ \lim_\nu a_k^{(n\nu)} = a_k,\ \lim_\nu a_{-k}^{(n\nu)} = a_{-k},\ \lim_\nu b_k^{(n\nu)} = b_k,\ \lim_\nu b_{-k}^{(n\nu)} = b_{-k}. \tag{4}$$

Since $g_n(z, 0) = 0$ on $|z-z_0| = r_n$,

$$\log r_n + a_0^{(n)} = -\gamma_n(0),\quad a_k^{(n)} r_n^k + a_{-k}^{(n)} r_n^{-k} = 0,\quad b_k^{(n)} r_n^k + b_{-k}^{(n)} r_n^{-k} = 0. \tag{5}$$

Since by (4), $a_k^{(n\nu)}$, $b_k^{(n\nu)}$ $(\nu = 1, 2, \cdots)$ are bounded and $r_{n_\nu} \to 0$, we have from (5),

$$\lim_\nu a_{-k}^{(n\nu)} = a_{-k} = 0, \qquad \lim_\nu b_{-k}^{(n\nu)} = b_{-k} = 0,$$

so that

$$g(z,\ 0) = \log r + a_0 + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta)\, r^k. \tag{6}$$

Hence $g(z, 0) + \log 1/|z - z_0|$ is harmonic at z_0 and

$$\lim_{\nu}(\gamma_{n_\nu}(0) - \log 1/r_{n_\nu}) = -a_0. \tag{7}$$

Next we shall prove that $\lim\limits_{n} (g_n(z,\ 0) - \gamma_n(0))$ exists. For, suppose that the limit does not exist, then we can find two partial sequences n_ν, m_ν, such that

$$\lim_{\nu} (g_{n_\nu}(z,\ 0) - \gamma_{n_\nu}(0)) = g_1(z,\ 0),\ \lim_{\nu} (g_{m_\nu}(z, 0) - \gamma_{m_\nu}(0)) = g_2(z,\ 0),$$
$$g_1(z, 0) \not\equiv g_2(z, 0).$$

Since by (6), $g_1(z, 0) - g_2(z, 0)$ is harmonic on F, it is a constant and since it vanishes at $z = 0$, $g_1(z, 0) \equiv g_2(z, 0)$, which contradicts the hypothesis. Hence $\lim\limits_{n} (g_n(z, 0) - \gamma_n(0)$ and so $\lim\limits_{n} (\gamma_n(0) - \log 1/r_n)$ exists.

3. The following theorem plays an important rôle in this paper.

THEOREM 3. *Let F be an open Riemann surface, which contains $z = 0$ and $F_n \to F$ be its exhaustion, where F_0 contains $z = 0$ and Γ_n be the boundary of F_n. Let $g_n(z, \zeta)$ be the Green's function of F_n with ζ as its pole and $\gamma_n(\zeta)$ be its Robin's constant.*

Let a disc $\Delta_0 : |\zeta - \zeta_0| \leqq \rho_0$ be contained in F_n $(n \geqq n_0)$ and Δ be a compact domain, which lies outside Δ_0 and is contained in F_n. Then

(i) $|\gamma_n(\zeta) - \gamma_n(\zeta_0)| \leqq K(\Delta_0)|\zeta - \zeta_0|,\ \zeta \in \Delta_0$ $\qquad (n \geqq n_0)$,

(ii) $|g_n(z, \zeta) - g_n(z, \zeta_0)| \leqq K(\Delta_0, \Delta)|\zeta - \zeta_0|, \zeta \in \Delta_0,\ z \in \Delta$ $\qquad (n \geqq n_0)$,

where $K(\Delta_0)$, $K(\Delta_0, \Delta)$ are constants, which depend on Δ_0 or Δ_0, Δ only.

PROOF. Let $\rho_0 < \rho_1 < \rho_2$,

$$C_1 : |\zeta - \zeta_0| = \rho_1,\ \ \Delta_1 : |\zeta - \zeta_0| \leqq \rho_1,\ \ C_2 : |\zeta - \zeta_0| = \rho_2,\ \ \Delta_2 : |\zeta - \zeta_0| \leqq \rho_2, \tag{1}$$

such that Δ lies outside C_1. Let $g(z,\ \zeta)$ $(\zeta \in \Delta_0)$ be the Green's function of Δ_2, such that

$$g(z,\ \zeta) = \log \left| \frac{\rho_2^2 - \overline{(\zeta - \zeta_0)}\,(z - \zeta_0)}{\rho_2(z - \zeta)} \right|. \tag{2}$$

We put for $\zeta \in \Delta_0$,

$$M_n(\zeta) = \operatorname*{Max}_{C_1} \frac{g_n(z,\ \zeta) - g_n(z,\ \zeta_0)}{|\zeta - \zeta_0|}\ (\zeta \in \Delta_0)\ (n \geqq n_0),$$
$$M(\zeta) = \operatorname*{Max}_{C_1} \frac{g(z,\ \zeta) - g(z,\ \zeta_0)}{|\zeta - \zeta_0|} \tag{3}$$

and

$$u_n(z) = M_n(\zeta) - \frac{g_n(z,\ \zeta) - g_n(z,\ \zeta_0)}{|\zeta - \zeta_0|} \qquad (n \geqq n_0), \tag{4}$$
$$u(z) = M(\zeta) - \frac{g(z,\ \zeta) - g(z,\ \zeta_0)}{|\zeta - \zeta_0|}.$$

We may assume that $M_n(\zeta) \geqq 0$, since, otherwise, we interchange $g_n(z,\ \zeta)$, $g_n(z,\ \zeta_0)$ and $g(z,\ \zeta)$, $g(z,\ \zeta_0)$. Since $u_n(z) = M_n(\zeta) \geqq 0$ on Γ_n and $\geqq 0$ on C_1, by the maximum principle,

$$u_n(z) \geqq 0 \qquad \text{in } F_n - \Delta_1. \tag{5}$$

Since $u(z) - u_n(z)$ is harmonic in C_2 and at a point z_0 on C_1, $u_n(z_0) = 0$, $u(z_0) \geqq 0$, $u(z_0) - u_n(z_0) \geqq 0$, by the maximum principle,

$$\operatorname*{Max}_{C_2} \, (u(z) - u_n(z)) \geqq 0.$$

Since $u(z) = M(\zeta)$ on C_2,

$$\operatorname*{Min}_{C_2} \, u_n(z) \leqq M(\zeta). \tag{6}$$

From (2), for any $\zeta \in \Delta_0$, $|M(\zeta)| \leqq K(\Delta_0)$, where $K(\Delta_0)$ depends on Δ_0 only, so that

$$\operatorname*{Min}_{C_2} \, u_n(z) \leqq K(\Delta_0). \tag{7}$$

In the following, we denote constants, which depend on Δ_0 or Δ_0, Δ only by the same letter $K(\Delta_0)$, $K(\Delta_0, \Delta)$.

Since $u_n(z) \geqq 0$ in $F_n - \Delta_1$, we have by (7)

$$|u_n(z)| = \left| M_n(\zeta) - \frac{g_n(z, \zeta) - g_n(z, \zeta_0)}{|\zeta - \zeta_0|} \right| \leqq K(\Delta_0, \Delta), \quad \zeta \in \Delta_0, \quad z \in \Delta \quad (n \geqq n_0). \tag{8}$$

Hence for z on C_2,

$$\left| M_n(\zeta) - \frac{1}{|\zeta - \zeta_0|}\left(g_n(z, \zeta) - \log\frac{1}{|z - \zeta|} - g_n(z,\zeta_0) + \log\frac{1}{|z - \zeta_0|}\right)\right| \leqq K(\Delta_0). \tag{9}$$

Since the left-hand side of (9) is harmonic in C_2, (9) holds in C_2, so that at $z = \zeta_0$, $z = \zeta$, we have

$$\left| M_n(\zeta) - \frac{1}{|\zeta - \zeta_0|}\left(g_n(\zeta_0, \zeta) - \log\frac{1}{|\zeta - \zeta_0|} - \gamma_n(\zeta_0)\right)\right| \leqq K(\Delta_0),$$

$$\left| M_n(\zeta) - \frac{1}{|\zeta - \zeta_0|}\left(\gamma_n(\zeta) - g_n(\zeta, \zeta_0) + \log\frac{1}{|\zeta - \zeta_0|}\right)\right| \leqq K(\Delta_0).$$

Hence adding and subtracting each other, we have

$$\left| M_n(\zeta) - \frac{\gamma_n(\zeta) - \gamma_n(\zeta_0)}{|\zeta - \zeta_0|} \right| \leqq K(\Delta_0), \tag{10}$$

$$\left| \frac{\gamma_n(\zeta) - \gamma_n(\zeta_0)}{2} - \left(g_n(\zeta, \zeta_0) - \log\frac{1}{|\zeta - \zeta_0|} - \gamma_n(\zeta_0)\right)\right| \leqq K(\Delta_0)|\zeta - \zeta_0|. \tag{11}$$

Since by Theorem 1,

$$\left| g_n(\zeta, \zeta_0) - \log\frac{1}{|\zeta - \zeta_0|} - \gamma_n(\zeta_0)\right| \leqq K(\Delta_0)|\zeta - \zeta_0| \qquad (n \geqq n_0),$$

we have

$$|\gamma_n(\zeta) - \gamma_n(\zeta_0)| \leqq K(\Delta_0)|\zeta - \zeta_0| \qquad (n \geqq n_0), \tag{12}$$

so that from (10), $|M_n(\zeta)| \leqq K(\Delta_0)$, hence from (8),

$$|g_n(z, \zeta) - g_n(z, \zeta_0)| \leqq K(\Delta_0, \Delta)|\zeta - \zeta_0|, \quad \zeta \in \Delta_0, \quad z \in \Delta \ (n \geqq n_0). \tag{13}$$

Hence our theorem is proved.

4. Let Δ_0, Δ be two compact domains on F, which have no common points and $\Delta_0 \subset F_n$, $\Delta \subset F_n$ $(n \geqq n_0)$.

By Theorem 3 and Borel's covering theorem, we can prove easily that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \Delta_0, \Delta)$, which depends on ε, Δ_0, Δ only, such that for any $z \in \Delta$, $|g_n(z, \zeta_1) - g_n(z, \zeta_2)| < \varepsilon$, if $|\zeta_1 - \zeta_2| < \delta$, $\zeta_1 \in \Delta_0$, $\zeta_2 \in \Delta_0$ $(n \geqq n_0)$ and for any $\zeta \in \Delta_0$,

$|g_n(z_1, \zeta) - g_n(z_2, \zeta)| < \varepsilon$, if $|z_1 - z_2| < \delta$, $z_1 \in \Delta$, $z_2 \in \Delta$ $(n \geqq n_0)$,

so that

$|g_n(z_1, \zeta_1) - g_n(z_2, \zeta_2)| < 2\varepsilon$, if $|z_1 - z_2| < \delta$, $|\zeta_1 - \zeta_2| < \delta$ $(n \geqq n_0)$.

Hence

$$\varphi_n(z, \zeta) = g_n(z, \zeta) - \gamma_n(0) \qquad (n \geqq n_0) \tag{1}$$

is equi-continuous for $z \in \Delta$, $\zeta \in \Delta_0$. Since by Theorem 1, $g_n(z, \zeta_0) - \gamma_n(\zeta_0)$ is uniformly bounded in Δ, $\gamma_n(\zeta_0) - \gamma_n(0)$ and $g_n(z, \zeta) - g_n(z, \zeta_0)$ are bounded by (i), $\varphi_n(z, \zeta)$ $(n = 1, 2, \cdots)$ is uniformly bounded for $z \in \Delta$, $\zeta \in \Delta_0$. Hence by Arzelà's theorem, we can find a partial sequence n_ν, such that

$$\lim_\nu \varphi_{n_\nu}(z, \zeta) = \lim_\nu (g_{n_\nu}(z, \zeta) - \gamma_{n_\nu}(0)) \tag{2}$$

uniformly in $z \in \Delta, \zeta \in \Delta_0$. From this, we can prove easily that we can find a partial sequence, which we denote n_ν, such that for any fixed ζ on F, (2) converges uniformly in z in any compact domain, which does not contain ζ and for any fixed z on F, (2) converges uniformly in ζ in any compact domain, which does not contain z. Hence if we put

$$\lim_\nu (g_{n_\nu}(z, \zeta) - \gamma_{n_\nu}(0)) = g(z, \zeta), \tag{3}$$

then for any fixed ζ, $g(z, \zeta)$ is a harmonic function of z at z $(\neq \zeta)$ and for any fixed z, $g(z, \zeta)$ is a harmonic function of ζ at ζ $(\neq z)$.

Let U: $|z - \zeta_0| \leqq \rho$, V: $|\zeta - \zeta_0| \leqq \rho$ be a neighbourhood of ζ_0 and we put for $z \in U$, $\zeta \in V$,

$$g_n(z, \zeta) = \log \frac{1}{|z - \zeta|} + \gamma_n(0) + \psi_n(z, \zeta), \tag{4}$$

then $\psi_n(z, \zeta)$ is a harmonic function with respect to each variable in $z \in U$, $\zeta \in V$. Since on $|z - \zeta_0| = \rho$, $|\zeta - \zeta_0| = \rho_0$ $(> \rho)$,

$$\lim_\nu \psi_{n_\nu}(z, \zeta) = \psi(z, \zeta) = g(z, \zeta) - \log \frac{1}{|z - \zeta|} \tag{5}$$

uniformly, (5) converges uniformly in $z \in U$, $\zeta \in V$, so that $\psi(z, \zeta)$ is a harmonic function of z in U for a fixed $\zeta \in V$ and is a harmonic function of ζ in V for a fixed $z \in U$. Since $g_{n_\nu}(z, \zeta) - \gamma_{n_\nu}(0) = g_{n_\nu}(\zeta, z) - \gamma_{n_\nu}(0)$, we have $g(z, \zeta) = g(\zeta, z)$. Hence we have proved the following theorem.

THEOREM 4. *Let F be an open Riemann surface and $F_n \to F$ be its exhaustion, where F_0 contains $z = 0$ and $g_n(z, \zeta)$ be the Green's function of F_n with ζ as its pole. Then there exists a partial sequence n_ν, such that*

$$\lim_\nu (g_{n_\nu}(z, \zeta) - \gamma_{n_\nu}(0)) = g(z, \zeta) \qquad (g(z, \zeta) = g(\zeta, z))$$

(for a fixed ζ) converges uniformly in z in any compact domain, which does not contain ζ and (for a fixed z) converges uniformly in ζ in any compact domain, which does not contain z. Hence for a fixed ζ, $g(z, \zeta)$ is a harmonic function

of z at z $(\neq \zeta)$ and for a fixed z, $g(z, \zeta)$ is a harmonic function of ζ at ζ $(\neq z)$. Let U: $|z-\zeta_0| \leqq \rho$, V: $|\zeta-\zeta_0| \leqq \rho$ be a neighbourhood of ζ_0 and for $z \in U$, $\zeta \in V$, let

$$g(z, \zeta) = \log \frac{1}{|z-\zeta|} + \psi(z, \zeta),$$

then for a fixed $\zeta \in V$, $\psi(z, \zeta)$ is a harmonic function of z in U and for a fixed $z \in U$, $\psi(z, \zeta)$ is a harmonic function of ζ in V.

We shall call $g(z, \zeta)$ the modified Green's function of F with ζ as its pole. In the following, $g(z, \zeta)$ denotes always the modified Green's function.

2. Potential functions with two logarithmic singularities

We shall prove

THEOREM 5. *Let F be an open Riemann surface and ζ_1, ζ_2 be two inner points and put*

$$g(z; \zeta_1, \zeta_2) = g(z, \zeta_1) - g(z, \zeta_2).$$

Then $g(z; \zeta_1, \zeta_2)$ is harmonic on F, except at ζ_1, ζ_2, where

$$g(z; \zeta_1, \zeta_2) - \log \frac{1}{|z-\zeta_1|} \text{ is harmonic at } \zeta_1,$$

$$g(z; \zeta_1, \zeta_2) + \log \frac{1}{|z-\zeta_2|} \text{ is harmonic at } \zeta_2.$$

Let Γ be an analytic Jordan curve, which contains ζ_1, ζ_2 in its inside and $\Gamma(F)$ be the part of F, which is contained in Γ. Then the Dirichlet integral of $g = g(z; \zeta_1, \zeta_2)$ in $F - \Gamma(F)$ is finite, such that

$$D_{F-\Gamma(F)}[g] \leqq \int_\Gamma g \frac{\partial g}{\partial \nu} ds,$$

where ν is the inner normal of Γ and ds is its arc element.

PROOF. Since the first part is evident, we shall prove the second part. Let $F_n \to F$ be the exhaustion of F, where F_0 contains the inside of Γ and Γ_n be the boundary of F_n. We put

$$g_n = g_n(z; \zeta_1, \zeta_2) = (g_n(z, \zeta_1) - \gamma_n(0)) - (g_n(z, \zeta_2) - \gamma_n(0)), \tag{1}$$

then

$$\lim_\nu g_{n_\nu}(z; \zeta_1, \zeta_2) = g(z, \zeta_1) - g(z, \zeta_2) = g(z; \zeta_1, \zeta_2). \tag{2}$$

Since $g_n = 0$ on Γ_n, we have

$$D_{F_n-\Gamma(F)}[g_n] = \int_\Gamma g_n \frac{\partial g_n}{\partial \nu} ds,$$

so that

$$D_{F_m-\Gamma(F)}[g_n] \leqq \int_\Gamma g_n \frac{\partial g_n}{\partial \nu} ds \qquad (m < n).$$

Hence if we make $n = n_\nu \to \infty$ and then $m \to \infty$, we have

$$D_{F-\Gamma(F)}[g] \leqq \int_{\Gamma} g \frac{\partial g}{\partial \nu} ds. \qquad \text{q.e.d.}$$

REMARK. If F is a closed surface, we take off a point z_0 ($\neq \zeta_1, \neq \zeta_2$) from F and put $F' = F - (z_0)$ and for the open surface F', we construct $g(z;\ \zeta_1, \zeta_2)$, then since the Dirichlet integral of $g(z;\zeta_1,\zeta_2)$ in the neighbourhood of z_0 is finite, $g(z;\ \zeta_1,\zeta_2)$ is harmonic at z_0. Hence there exists a potential function on F, which has logarithmic singularities at ζ_1, ζ_2.

In the following Theorem 6 and 7, we assume that F is open and though we do not repeat the same remark, if F is closed we make the same modification to establish the existence of a potential function with the prescribed singularity.

THEOREM 6. *Let F be an open Riemann surface and ζ_1, ζ_2 be two inner points. We connect ζ_1, ζ_2 by an analytic Jordan arc C and put*

$$h(z;\zeta_1,\zeta_2) = \int_C \frac{\partial g(z,\zeta)}{\partial \nu} ds_\zeta = \int_{\zeta_1}^{\zeta_2} \frac{\partial g(z,\zeta)}{\partial \nu} ds_\zeta,$$

where ν is the normal of C at ζ, which is obtained from the direction of ds by a rotation of an angle $-\pi/2$ then $h(z;\zeta_1,\zeta_2)$ is harmonic on F, but is many valued, such that

$$h(z;\zeta_1,\zeta_2) - \arg(z-\zeta_1) \text{ is harmonic at } \zeta_1,$$
$$h(z;\zeta_1,\zeta_2) + \arg(z-\zeta_2) \text{ is harmonic at } \zeta_2.$$

Let Γ be an analytic Jordan curve, which contains ζ_1, ζ_2 in its inside, then the Dirichlet integral of $h = h(z;\zeta_1,\zeta_2)$ in $F-\Gamma(F)$ is finite, such that

$$D_{F-\Gamma(F)}[h] \leqq \int_{\Gamma} h \frac{\partial h}{\partial \nu} ds.$$

PROOF. For a fixed $z \in F - C$, let $h(z, \zeta)$ be the conjugate harmonic function of $g(z,\zeta)$, then

$$h(z;\zeta_1,\zeta_2) = \int_{\zeta_1}^{\zeta_2} \frac{\partial g(z,\zeta)}{\partial \nu} ds_\zeta = \int_{\zeta_1}^{\zeta_2} dh(z,\zeta) = h(z,\zeta_2) - h(z,\zeta_1). \qquad (1)$$

Since by Theorem 4, $\partial g(z,\zeta)/\partial\nu$ is a harmonic function of z, $h(z;\zeta_1,\zeta_2)$ is a harmonic function of z, except at ζ_1,ζ_2. Let $U: |z-\zeta_1| \leqq \rho$, $V: |\zeta - \zeta_1| \leqq \rho$ be a neighbourhood of ζ_1 and for $z \in U$, $\zeta \in V$, put

$$g(z,\zeta) = \log \frac{1}{|z-\zeta|} + \psi(z,\zeta),$$

then by Theorem 4, $\psi(z,\zeta)$ is a harmonic function in each variable. Let ζ_0 be the first point of intersection of C with $|\zeta - \zeta_1| = \rho$, when we proceed from ζ_1 to ζ_2 on C and let C_1 be the part of C, which is bounded by ζ_1, and ζ_0 and $C_2 = C - C_1$, then

$$h(z\,;\zeta_1,\zeta_2)=\int_C \frac{\partial g(z,\zeta)}{\partial\nu}\,ds_\zeta=\int_{C_1}\frac{\partial g(z,\zeta)}{\partial\nu}\,ds_\zeta+\int_{C_2}\frac{g(z,\zeta)}{\partial\nu}\,ds_\zeta$$

$$=\arg(z-\zeta_1)-\arg(z-\zeta_0)+\int_{C_1}\frac{\partial\psi(z,\zeta)}{\partial\nu}\,ds_\zeta+\int_{C_2}\frac{\partial g(z,\zeta)}{\partial\nu}\,ds_\zeta. \tag{2}$$

Since the three terms other than $\arg(z-\zeta_1)$ on the right hand side of (2) are harmonic at ζ_1, $h(z\,;\zeta_1,\zeta_2)-\arg(z-\zeta_1)$ is harmonic at ζ_1. Similarly $h(z\,;\zeta_1,\zeta_2)+\arg(z-\zeta_2)$ is harmonic at ζ_2. Now we divide C into N arcs of equal length Δs and ξ_k $(k=0,1,\cdots,N)$ $(\xi_0=\zeta_1,\ \xi_N=\zeta_2)$ be the point of division and $e^{i\theta_k}$ be a unit vector at ξ_k, which is orthogonal to C and put

$$u_n^N=u_n^N(z,\delta)=\sum_{k=1}^{N}\frac{g_n(z,\xi_k+\delta e^{i\theta_k})-g_n(z,\xi_k)}{\delta}\,\Delta s\quad(\delta>0), \tag{3}$$

then

$$\lim_{\nu\to\infty}\lim_{N\to\infty}\lim_{\delta\to\infty}u_{n_\nu}^N=\int_C\frac{\partial g(z,\zeta)}{\partial\nu}\,ds_\zeta=h(z\,;\zeta_1,\zeta_2). \tag{4}$$

Since $u_n^N=0$ on Γ_n, we have

$$D_{F_m-\Gamma(F)}[u_n^N]\leqq\int_\Gamma u_n^N\frac{\partial u_n^N}{\partial\nu}\,ds\qquad(m<n),$$

so that if we make succesively $\delta\to 0, N\to\infty,\ n=n_\nu\to\infty, m\to\infty$, we have

$$D_{F-\Gamma(F)}[h]\leqq\int_\Gamma h\,\frac{\partial h}{\partial\nu}\,ds,\qquad\text{q. e. d.}$$

3. Osgood's theorem

THEOREM 7. *Let F be an open Riemann surface and a schlicht disc $F_1:|z|\leqq R_1$ be contained in F and let $F_0:|z|\leqq R_0$ $(0\leqq R_0<R_1)$. Let $f(z)=\sum_{k=1}^{\infty}c_k/z^k$ be regular for $|z|>R_0$ and*

$$U(z)=\Re(f(z))=\sum_{k=1}^{\infty}\frac{a_k\cos k\theta+b_k\sin k\theta}{r^k}\qquad(z=re^{i\theta}).$$

Then

$$u(z)=\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left(a_k\frac{\partial^k g(z,0)}{\partial\xi^k}+b_k\frac{\partial^k g(z,0)}{\partial\xi^{k-1}\partial\eta}\right)^{3)}\qquad(\zeta=\xi+i\eta=0)$$

converges uniformly in any compact domain, which lies outside F_0, hence $u(z)$ is harmonic in $F-F_0$ and $u(z)$ can be continued harmonically in F_1, such that $u(z)-U(z)=V(z)$ is harmonic in F_1. Let Γ be an analytic Jordan curve, which contains F_0, then the Dirichlet integral of $u(z)$ in $F-\Gamma(F)$ is finite, such that

3) $$\frac{\partial^k g(z,0)}{\partial\xi^k}=\left[\frac{\partial^k g(z,\zeta)}{\partial\xi^k}\right]_{\zeta=0},\quad\frac{\partial^k g(z,0)}{\partial\xi^{k-1}\partial\eta}=\left[\frac{\partial^k g(z,\zeta)}{\partial\xi^{k-1}\partial\eta}\right]_{\zeta=0}\quad(\zeta=\xi+i\eta).$$

$$D_{F-\Gamma(F)}[u] \leqq \int_\Gamma u \frac{\partial u}{\partial \nu} ds.$$

That such a potential function exists (except the finiteness of the Dirichlet integral) was proved by Osgood.[4]

PROOF. Let for $z \in F_1$, $\zeta \in F_1$,

$$g(z,\zeta) = \log \frac{1}{|z-\zeta|} + \psi(z,\zeta),$$

then $\psi(z,\zeta)$ is harmonic in $z \in F_1$, $\zeta \in F_1$.

Since

$$\left[\frac{\partial^k}{\partial \xi^k} \log \frac{1}{|z-\zeta|}\right]_{\zeta=0} = \frac{(k-1)!\cos k\theta}{r^k},$$

$$\left[\frac{\partial^k}{\partial \xi^{k-1}\partial\eta} \log \frac{1}{|z-\zeta|}\right]_{\zeta=0} = \frac{(k-1)!\sin k\theta}{r^k} \quad (\zeta = \xi + i\eta),$$

if we put

$$u(z) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!}\left(a_k \frac{\partial^k g(z,0)}{\partial \xi^k} + b_k \frac{\partial^k g(z,0)}{\partial \xi^{k-1}\partial\eta}\right), \tag{1}$$

$$V(z) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!}\left(a_k \frac{\partial^k \psi(z,0)}{\partial \xi^k} + b_k \frac{\partial^k \psi(z,0)}{\partial \xi^{k-1}\partial\eta}\right), \tag{2}$$

then

$$u(z) = U(z) + V(z). \tag{3}$$

We shall prove that $V(z)$ converges in F_1 uniformly.

Let $|\psi(z,\zeta)| \leqq K$ in $|z| \leqq R_1$, $|\zeta| \leqq R_1$, then for $|z| \leqq R_1$,

$$\left|\frac{\partial^k \psi(z,0)}{\partial \xi^k}\right| \leqq \frac{k!M}{R_1^k}, \quad \left|\frac{\partial^k \psi(z,0)}{\partial \xi^{k-1}\partial\eta}\right| \leqq \frac{k!M}{R_1^k} \tag{4}$$

where M is a constant. Since $\sum_{k=1}^{\infty} k(|a_k| + |b_k|)/R_1^k < \infty$, $V(z)$ converges uniformly in F_1, so that $u(z) - U(z) = V(z)$ is harmonic in F_1. Let Δ be a compact domain in $F - F_0$, then Δ lies outside a certain disc F': $|\zeta| \leqq R$ $(R_0 < R < R_1)$ and

$$|g(z,\zeta)| \leqq K(\Delta), \quad z \in \Delta, \ \zeta \in F'. \tag{5}$$

Hence for $z \in \Delta$,

$$\left|\frac{\partial^k g(z,0)}{\partial \xi^k}\right| \leqq \frac{k!M}{R^k}, \quad \left|\frac{\partial^k g(z,0)}{\partial \xi^{k-1}\partial\eta}\right| \leqq \frac{k!M}{R^k},$$

hence $u(z)$ converges uniformly in Δ, so that is harmonic in Δ, hence $u(z)$ is harmonic in $F - F_0$. Let Γ be an analytic Jordan curve, which contains F_0 in its inside. Let $F_n \to F$ be the exhaustion of F and $g_n(z,\zeta) = g_n(z,\xi,\eta)$ $(\zeta = \xi + i\eta)$ be the Green's function of F_n. We put for $\delta > 0$,

4) W. F. OSGOOD, Lehrbuch d. Funktionentheorie II_2, Leipzig u. Berlin (1932).

$$\Delta^k g_n = \Delta^k g_n(z, 0, 0) = g_n(z, k\delta, 0) - \binom{k}{1} g_n(z, (k-1)\delta, 0)$$
$$+ \binom{k}{2} g_n(z, (k-2)\delta, 0) - \cdots \pm g_n(z, 0, 0), \tag{6}$$

$$\Delta_1^k g_n = \Delta^{k-1}(g_n(z, 0, \delta) - g_n(z, 0, 0)),$$

then

$$\lim_{\delta\to 0} \frac{\Delta^k g_n}{\delta^k} = \frac{\partial^k g_n(z, 0)}{\partial \xi^k}, \quad \lim_{\delta\to 0} \frac{\Delta_1^k g_n}{\delta^k} = \frac{\partial^k g_n(z, 0)}{\partial \xi^{k-1}\partial \eta}. \tag{7}$$

We put

$$u_n^N = u_n^N(z, \delta) = \sum_{k=1}^{N} \frac{1}{(k-1)!}\left(a_k \frac{\Delta^k g_n}{\delta^k} + b_k \frac{\Delta_1^k g_n}{\delta^k}\right). \tag{8}$$

Since $u_n^N = 0$ on Γ_n, we have

$$D_{F_m - \Gamma(F)}[u_n^N] \leqq \int_\Gamma u_n^N \frac{\partial u_n^N}{\partial \nu}\, ds \qquad (m < n).$$

If we make successively $\delta \to 0$, $n = n_\nu \to \infty$, $N \to \infty$, $m \to \infty$, we have

$$D_{F-\Gamma(F)}[u] \leqq \int_\Gamma u \frac{\partial u}{\partial \nu}\, ds, \qquad \text{q.e.d.}$$

4. Potential functions with polar singularities

1. Let F be an open Riemann surface. If we put

$$u(z) = \frac{\partial^k g(z, \zeta)}{\partial \xi^k}, \quad v(z) = -\frac{\partial^k g(z, \zeta)}{\partial \xi^{k-1}\partial \eta} \qquad (\zeta = \xi + i\eta),$$

then by Theorem 7, $u(z)$ and $v(z)$ have singularities

$$(k-1)!\Re \frac{1}{(z-\zeta)^k} = \frac{(k-1)!\cos k\theta}{r^k},$$
$$(k-1)!\Im \frac{1}{(z-\zeta)^k} = \frac{-(k-1)!\sin k\theta}{r^k}, \quad (z - \zeta = re^i)$$

at $z = \zeta$. Let $\tau_\zeta^k(z)$, $\tau_\zeta'^k(z)$ be the analytic functions of z, whose real parts are $u(z)$, $v(z)$ respectively, such that

$$\tau_\zeta^k(z) = \frac{\partial^k g(z, \zeta)}{\partial \xi^k} + i(\quad), \quad \tau_\zeta'^k(z) = -\frac{\partial^k g(z, \zeta)}{\partial \xi^{k-1}\partial \eta} + i(\quad). \tag{1}$$

Let α be an analytic Jordan curve on F, which is not homotop null, then by Theorem 6,

$$\omega_\alpha(z) = \frac{1}{2\pi}\int_\alpha \frac{\partial g(z, \zeta)}{\partial \nu} ds_\zeta \tag{2}$$

is harmonic on F, but is many valued, such that

$$\int_{\alpha'} d\omega_\alpha(z) = 1, \tag{3}$$

where α' is an analytic Jordan curve, which connects a point of α on one shore to the corresponding point on the opposite shore, the direction of ν being so chosen, that if we rotate it by an angle $\pi/2$, it coincides with the direction of ds_ζ.

Let $w_\alpha(z)$ be the analytic function, whose real part is $\omega_\alpha(z)$, such that

$$w_\alpha(z) = \frac{1}{2\pi}\int_\alpha \frac{\partial g(z,\zeta)}{\partial\nu}\, ds_\zeta + i(\quad), \qquad \Re\int_{\alpha'} dw_\alpha(z) = 1. \tag{4}$$

For a fixed z, let $\psi_z(\zeta)$ be the analytic function of ζ, whsoe real part is $g(z,\zeta)$, such that

$$\psi_z(\zeta) = g(z,\zeta) + i(\quad). \tag{5}$$

Let z_0 be a fixed point of F and we connect z_0 to a point z by an analytic Jordan arc C and let

$$h_z(\zeta) = \int_{z_0}^{z} \frac{\partial g(t,\zeta)}{\partial\nu}\, ds_t \qquad (\zeta \neq z_0, \neq z), \tag{6}$$

where we integrate on C and ν is the normal of C at t, such that if we rotate it by an angle $\pi/2$, it coincides with the direction of ds_t.

Let $\psi_z'(\zeta)$ be the analytic function of ζ, whose real part is $h_z(\zeta)$, such that

$$\psi_z'(\zeta) = \int_{z_0}^{z} \frac{\partial g(t,\zeta)}{\partial\nu}\, ds_t + i(\quad). \tag{7}$$

Then we can prove easily the following relations.[5)]

THEOREM 8.

$$\left.\begin{aligned}
&\Re\left(\frac{d^l\tau_\zeta^k(z)}{dz^l}\right) = \Re\left(\frac{d^k\tau_z^l(\zeta)}{d\zeta^k}\right), \quad \Im\left(\frac{d^l\tau_\zeta^k(z)}{dz^l}\right) = \Re\left(\frac{d^k\tau_z'^l(\zeta)}{d\zeta^k}\right),\\
&\Re\left(\frac{d^l\tau_\zeta'^k(z)}{dz^l}\right) = \Im\left(\frac{d^k\tau_z^l(\zeta)}{d\zeta^k}\right), \quad \Im\left(\frac{d^l\tau_\zeta'^k(z)}{dz^l}\right) = \Im\left(\frac{d^k\tau_z'^l(\zeta)}{d\zeta^k}\right).
\end{aligned}\right\} \tag{I}$$

$$\int_\alpha d\tau_z^k(\zeta) = 2\pi i\, \Re\left(\frac{d^k w_\alpha(z)}{dz^k}\right), \quad \int_\alpha d\tau_z'^k(\zeta) = 2\pi i\, \Im\left(\frac{d^k w^\alpha(z)}{dz^k}\right). \tag{II}$$

$$\Re\,(\tau_\zeta^k(z)) = \Re\left(\frac{d^k\psi_z(\zeta)}{d\zeta^k}\right), \qquad \Re(\tau_\zeta'^k(z)) = \Im\left(\frac{d^k\psi_z(\zeta)}{d\zeta^k}\right). \tag{III}$$

$$\left.\begin{aligned}
&\Im\left(\int_{z_0}^{z} d\tau_\zeta^k(t)\right) = \Im(\tau_\zeta^k(z)) - \Im(\tau_\zeta^k(z_0)) = \Re\left(\frac{d^k\psi_z'(\zeta)}{d\zeta^k}\right),\\
&\Im\left(\int_{z_0}^{z} d\tau_\zeta'^k(t)\right) = \Im(\tau_\zeta'^k(z)) - \Im(\tau_\zeta'^k(z_0)) = \Im\left(\frac{d^k\psi_z'(\zeta)}{d\zeta^k}\right).
\end{aligned}\right\} \tag{IV}$$

5) WEYL, l. c. 1) p. 112-113.

PROOF. By (1),

$$\Re\left(\frac{d^l\tau_\zeta^k(z)}{dz^l}\right)=\frac{\partial^{l+k}g(z,\zeta)}{\partial x^l\partial\xi^k},\quad \Re\left(\frac{d^k\tau_z^l(\zeta)}{d\zeta^k}\right)=\frac{\partial^{l+k}g(\zeta,z)}{\partial\xi^k\partial x^l}\quad (z=x+iy,\zeta=\xi+i\eta).$$

Since $g(z,\zeta)=g(\zeta,z)$, we have

$$\Re\left(\frac{d^l\tau_\zeta^k(z)}{dz^l}\right)=\Re\left(\frac{d^k\tau_z^l(\zeta)}{d\zeta^k}\right).$$

Similarly we can prove other relations of (I).

From (4) and $g(z,\zeta)=g(\zeta,z)$, we have

$$\int\limits_\alpha\frac{\partial}{\partial\nu}\left(\frac{\partial^k g(\zeta,z)}{\partial x^k}\right)ds_\zeta=\int\limits_\alpha\frac{\partial}{\partial\nu}\left(\frac{\partial^k g(z,\zeta)}{\partial x^k}\right)ds_\zeta=2\pi\Re\left(\frac{d^k w_\alpha(z)}{dz^k}\right).\tag{8}$$

Since

$$\frac{\partial^k g(\zeta,z)}{\partial x^k}+i(\quad)=\tau_z^k(\zeta),$$

the left-hand side of (8) is equal to $\frac{1}{i}\int\limits_\alpha d\tau_z^k(\zeta)$, hence

$$\int\limits_\alpha d\tau_z^k(\zeta)=2\pi i\Re\left(\frac{d^k w_\alpha(z)}{dz^k}\right).$$

Another relation of (II) and relations of (III), (IV) can be proved similarly.

5. Riemann-Roch's theorem

By means of Theorem 8, we can prove easily the following Riemann-Roch's theorem.

THEOREM 9. *Let F be a closed Riemann surface of genus $p\geqq 1$ and $\mathfrak{d}=\frac{\mathfrak{p}_1^{m_1}\cdots\mathfrak{p}_r^{m_r}}{\mathfrak{q}_1^{n_1}\cdots\mathfrak{q}_s^{n_s}}$ $(m_\nu>0, n_\mu>0)$ be a divisor, $m=\sum_{\nu=1}^{r}m_\nu-\sum_{\mu=1}^{s}n_\mu$ being its total order. Let B be the number of (in the complex sense) linearly independent differentials on F, which are multiple of $\mathfrak{d}$ and A be the number of (in the complex sense) linearly independent one-valued analytic functions on F, which are multiple of $1/\mathfrak{d}$, then*

$$A=B+(m+1-p).$$

PROOF. Let $\mathfrak{p}_\nu$ lie on $z=\zeta_\nu$ and $\mathfrak{q}_\mu$ lie on $z=z_\mu$. If ζ_ν or z_μ be a branch point of F, the differentiation in the following means that with respect to the local parameter. Let z_0 be a point of F, which is different from ζ_ν, z_μ. We take off z_0 from F and for the open Riemann surface $F'=F-(z_0)$, we consider the modified Green's function $g(z,\zeta)$ and other potential functions.

Then by Theorem 1, $g(z,\zeta)$ has a logarithmic singularity at z_0 and by the remark of §2, $\tau_\zeta^k(z)$, $\tau_\zeta'^k(z)$ are regular at z_0. Let $f(z)$ be a one-valued analytic function on F, which is multiple of $1/\mathfrak{b}$, then $f(z)$ can be expressed in the form:

$$f(z)=\sum_{\nu=1}^{r}[\alpha_1^\nu \tau_{\zeta_\nu}^1(z)+\beta_1^\nu \tau'^1_{\zeta_\nu}(z)+\cdots+\alpha_{m\nu}^\nu \tau_{\zeta_\nu}^{m\nu}(z)+\beta_{m\nu}^\nu \tau'^{m\nu}_{\zeta_\nu}(z)]+(a+ib), \quad (1)$$

where α_i^ν, β_i^ν, a, b are real constants. Let $\alpha_1, \cdots, \alpha_{2p}$ be a set of canonical ring cuts of F, which makes F into a simply connected surface. We put $w_h(z)=w_{\alpha_h}(z)$, where $w_{\alpha_h}(z)$ is defined by (4) of § 4.

Since $f(z)$ is one-valued, we have $\int_{\alpha_h} df(z)=0$, so that by Theorem 8 (II),

$$\sum_{\nu=1}^{r}\Big[\alpha_1^\nu \Re\Big(\frac{dw_h(\zeta_\nu)}{d\zeta_\nu}\Big)+\beta_1^\nu \Im\Big(\frac{dw_h(\zeta_\nu)}{d\zeta_\nu}\Big)+\cdots+\alpha_{m\nu}^\nu \Re\Big(\frac{d^{m\nu}w_h(\zeta_\nu)}{d\zeta_\nu^{m\nu}}\Big)$$
$$+\beta_{m_\nu}^\nu \Im\Big(\frac{d^{m\nu}w_h(\zeta_\nu)}{d\zeta_\nu^{m\nu}}\Big)\Big]=0, \quad (h=1,2,\cdots,2p). \quad (2)$$

Since $f(z)$ is a multiple of $\mathfrak{q}_1^{n_1}\cdots\mathfrak{q}_s^{n_s}$,

$$\left.\begin{aligned}&\sum_{\nu=1}^{r}[\alpha_1^\nu\Re(\tau_{\zeta_\nu}^1(z_\mu))+\beta_1^\nu\Re(\tau'^1_{\zeta_\nu}(z_\mu))+\cdots+\alpha_{m_\nu}^\nu\Re(\tau_{\zeta_\nu}^{m\nu}(z_\mu))+\beta_{m_\nu}^\nu\Re(\tau'^{m\nu}_{\zeta_\nu}(z_\mu))]+a=0,\\&\sum_{\nu=1}^{r}[\alpha_1^\nu\Im(\tau_{\zeta_\nu}^1(z_\mu)+\beta_1^\nu\Im(\tau'^1_{\zeta_\nu}(z_\mu))+\cdots+\alpha_{m_\nu}^\nu\Im(\tau_{\zeta_\nu}^{m\nu}(z_\mu))+\beta_{m_\nu}^\nu\Im(\tau'^{m\nu}_{\zeta_\nu}(z_\mu))]+b=0,\end{aligned}\right\}(3)$$

$$\left.\begin{aligned}&\sum_{\nu=1}^{r}\Big[\alpha_1^\nu\Re\Big(\frac{d^k\tau_{\zeta_\nu}^1(z_\mu)}{dz_\mu^k}\Big)+\beta_1^\nu\Re\Big(\frac{d^k\tau'^1_{\zeta_\nu}(z_\mu)}{dz_\mu^k}\Big)+\cdots+\alpha_{m\nu}^\nu\Re\Big(\frac{d^k\tau_{\zeta_\nu}^{m\nu}(z_\mu)}{dz_\mu^k}\Big)\\&\qquad\qquad+\beta_{m\nu}^\nu\Re\Big(\frac{d^k\tau'^{m\nu}_{\zeta_\nu}(z_\mu)}{dz_\mu^k}\Big)\Big]=0,\\&\sum_{\nu=1}^{r}\Big[\alpha_1^\nu\Im\Big(\frac{d^k\tau_{\zeta_\nu}^1(z_\mu)}{dz_\mu^k}\Big)+\beta_1^\nu\Im\Big(\frac{d^k\tau'^1_{\zeta_\nu}(z_\mu)}{dz_\mu^k}\Big)+\cdots+\alpha_{m\nu}^\nu\Im\Big(\frac{d^k\tau_{\zeta_\nu}^{m\nu}(z_\mu)}{dz_\mu^k}\Big)\\&\qquad\qquad+\beta_{m\nu}^\nu\Im\Big(\frac{d^k\tau'^{m\nu}_{\zeta_\nu}(z_\mu)}{dz_\mu^k}\Big)\Big]=0.\end{aligned}\right\}(4)$$

$$(\mu=1,2,\cdots s;\ k=1,2\cdots n_\mu-1).$$

By Theorem 8 (I), (III), (IV), (3), (4) can be written in the following forms (3'), (4'), where we normalize, such that

$$\Im(\tau_{\zeta_\nu}^k(z_0))=0,\ \Im(\tau'^k_{\zeta_\nu}(z_0))=0 \qquad (\nu=1,2,\cdots,r;\ k=1,2,\cdots,m_\nu).$$

$$\sum_{\nu=1}^{r}\Big[\alpha_1^\nu \Re\Big(\frac{dw_h(\zeta_\nu)}{d\zeta_\nu}\Big)+\beta_1^\nu \Im\Big(\frac{dw_h(\zeta_\nu)}{d\zeta_\nu}\Big)+\cdots+\alpha_{m\nu}^\nu \Re\Big(\frac{d^{m\nu}w_h(\zeta_\nu)}{d\zeta_\nu^{m\nu}}\Big)$$
$$+\beta_{m\nu}^\nu \Im\Big(\frac{d^{m\nu}w_h(\zeta_\nu)}{d\zeta_\nu^{m\nu}}\Big)\Big]=0. \quad (2)$$

$$\left.\begin{aligned}&\sum_{\nu=1}^{r}\Big[\alpha_1^\nu\Re\Big(\frac{d\psi_{z_\mu}(\zeta_\nu)}{d\zeta_\nu}\Big)+\beta_1^\nu\Im\Big(\frac{d\psi_{z_\mu}(\zeta_\nu)}{d\zeta_\nu}\Big)+\cdots+\alpha_{m_\nu}^\nu\Re\Big(\frac{d^{m_\nu}\psi_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^{m_\nu}}\Big)\\&\qquad\qquad+\beta_{m_\nu}^\nu\Im\Big(\frac{d^{m_\nu}\psi_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^{m_\nu}}\Big)\Big]+a=0,\\&\sum_{\nu=1}^{r}\Big[\alpha_1^\nu\Re\Big(\frac{d\psi'_{z_\mu}(\zeta_\nu)}{d\zeta_\nu}\Big)+\beta_1^\nu\Im\Big(\frac{d\psi'_{z_\mu}(\zeta_\nu)}{d\zeta_\nu}\Big)+\cdots+\alpha_{m_\nu}^\nu\Re\Big(\frac{d^{m_\nu}\psi'_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^{m_\nu}}\Big)\\&\qquad\qquad+\beta_{m_\nu}^\nu\Im\Big(\frac{d^{m_\nu}\psi'_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^{m_\nu}}\Big)\Big]+b=0,\end{aligned}\right\}(3')$$

$$\left.\begin{aligned}&\sum_{\nu=1}^{r}\Big[\alpha_1^\nu\Re\Big(\frac{d\tau^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu}\Big)+\beta_1^\nu\Im\Big(\frac{d\tau^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu}\Big)+\cdots+\alpha_{m_\nu}^\nu\Re\Big(\frac{d^{m_\nu}\tau^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^{m_\nu}}\Big)\\&\qquad\qquad+\beta_{m_\nu}^\nu\Im\Big(\frac{d^{m_\nu}\tau^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^{m_\nu}}\Big)\Big]=0,\\&\sum_{\nu=1}^{r}\Big[\alpha_1^\nu\Re\Big(\frac{d\tau'^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu}\Big)+\beta_1^\nu\Im\Big(\frac{d\tau'^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu}\Big)+\cdots+\alpha_{m_\nu}^\nu\Re\Big(\frac{d^{m_\nu}\tau'^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^{m_\nu}}\Big)\\&\qquad\qquad+\beta_{m_\nu}^\nu\Im\Big(\frac{d^{m_\nu}\tau'^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^{m_\nu}}\Big)\Big]=0,\end{aligned}\right\}(4')$$

$$(h=1,2,\cdots,2p;\ \mu=1,2,\cdots,s;\ k=1,2,\cdots,n_\mu-1).$$

(2), (3′), (4′) is a system of homogeneous linear equations for $2\Big(\sum_{\nu=1}^{r}m+1\Big)$ unkown quantities α_i^ν, β_i^ν, a, b $(\nu=1,2\cdots,r;\ i=1,2\cdots,m_\nu)$. Let R be the rank of the matrix $(\mathfrak{A})$ formed with the coefficients, then the system has (in the real sense)

$$A'=2\Big(\sum_{\nu=1}^{r}m_\nu+1\Big)-R \tag{5}$$

linearly independent solutions, A' is the number of (in the real sense) linearly independent one-valued analytic functions on F, which are multiple of $1/\mathfrak{b}$. Let $(\mathfrak{A}')$ be the transposed matrix of $(\mathfrak{A})$, then $(\mathfrak{A}')$ has the rank R. Hence the following system of homogeneous linear equaltions (7), (8), (9) with the coefficients matrix $(\mathfrak{A}')$ for $2\Big(\sum_{\mu=1}^{s}n_\mu+p\Big)$ unknown quantities a_h, b_μ, b'_μ, c^k_μ, c'^k_μ $(h=1,2,\cdots,2p;\ \mu=1,2,\cdots,s;\ k=1,2,\cdots,n_\mu-1)$ has (in the real sense)

$$B'=2\Big(\sum_{\mu=1}^{s}n_\mu+p\Big)-R \tag{6}$$

linearly independent solutions.

$$\sum_{h=1}^{2p}a_h\Re\Big(\frac{d^\lambda w_h(\zeta_\nu)}{d\zeta_\nu^\lambda}\Big)+\sum_{\mu=1}^{s}\Big[b_\mu\Re\Big(\frac{d^\lambda\psi_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^\lambda}\Big)+b'_\mu\Re\Big(\frac{d^\lambda\psi'_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^\lambda}\Big)\Big]$$
$$+\sum_{\mu=1}^{s}\sum_{k=1}^{n_\mu-1}\Big[c^k_\mu\Re\Big(\frac{d^\lambda\tau^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^\lambda}\Big)+c'^k_\mu\Re\Big(\frac{d^\lambda\tau'^k_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^\lambda}\Big)\Big]=0, \tag{7}$$

$$\sum_{h=1}^{2p} a_h \Im\left(\frac{d^\lambda w_h(\zeta_\nu)}{d\zeta_\nu^\lambda}\right) + \sum_{\mu=1}^{s}\left[b_\mu \Im\left(\frac{d^\lambda \psi_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^\lambda}\right) + b'_\mu \Im\left(\frac{d^\lambda \psi'_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^\lambda}\right)\right]$$
$$+ \sum_{\mu=1}^{s}\sum_{k=1}^{n_\mu - 1}\left[c_\mu^k \Im\left(\frac{d^\lambda \tau_{z_\mu}^k(\zeta_\nu)}{d\zeta_\nu^\lambda}\right) + c_\mu'^k \Im\left(\frac{d^\lambda \tau_{z_\mu}'^k(\zeta_\nu)}{d\zeta_\nu^\lambda}\right)\right] = 0,$$
$$(\nu = 1, 2, \cdots, r;\ \lambda = 1, 2, \cdots, m_\nu). \tag{8}$$

$$\sum_{\mu=1}^{s} b_\mu = 0, \qquad \sum_{\mu=1}^{s} b'_\mu = 0. \tag{9}$$

From (7), (8), we have

$$\sum_{h=1}^{2p} a_h \frac{d^\lambda w_h(\zeta_\nu)}{d\zeta_\nu^\lambda} + \sum_{\mu=1}^{s}\left[b_\mu \frac{d^\lambda \psi_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^\lambda} + b'_\mu \frac{d^\lambda \psi'_{z_\mu}(\zeta_\nu)}{d\zeta_\nu^\lambda}\right]$$
$$+ \sum_{\mu=1}^{s}\sum_{k=1}^{n_\mu - 1}\left[c_\mu^k \frac{d^\lambda \tau_{z_\mu}^k(\zeta_\nu)}{d\zeta_\nu^\lambda} + c_\mu'^k \frac{d^\lambda \tau_{z_\mu}'^k(\zeta_\nu)}{d\zeta_\nu^\lambda}\right] = 0.$$
$$(\nu = 1, 2, \cdots, r;\ \lambda = 1, 2, \cdots, m_\nu). \tag{10}$$

By Theorem 1 and (9), we see that the differential

$$dv(\zeta) = \left[\sum_{h=1}^{2p} a_h \frac{dw_h(\zeta)}{d\zeta} + \sum_{\mu=1}^{s}\left(b_\mu \frac{d\psi_{z_\mu}(\zeta)}{d\zeta} + b'_\mu \frac{d\psi'_{z_\mu}(\zeta)}{d\zeta}\right) + \right.$$
$$\left. \sum_{\mu=1}^{s}\sum_{k=1}^{r_\mu - 1}\left(c_\mu^k \frac{d\tau_{z_\mu}^k(\zeta)}{d\zeta} + c_\mu'^k \frac{d\tau_{z_\mu}'^k(\zeta)}{d\zeta}\right)\right] d\zeta \tag{11}$$

is regular at z_0 and from (10), we see easily that $dv(\zeta)$ is a multiple of $\mathfrak{b}$. Hence B' is the number of (in the real sense) linearly independent differentials, which are multiple of $\mathfrak{b}$.

From (5), (6), we have

$$A' = B' + 2(m + 1 - p), \qquad \left(m = \sum_{\nu=1}^{r} m_\nu - \sum_{\mu=1}^{s} n_\mu\right). \tag{12}$$

Let A be the number of (in the complex sence) linearly independent one-valued analytic functions, which are multiple of $1/\mathfrak{b}$ and B be the number of (in the complex sense) linearly independent differentials, which are multiple of $\mathfrak{b}$, then we can prove easily[6] $A' = 2A$, $B' = 2B$, so that

$$A = B + (m + 1 - p).$$

Hence our theorem is proved.

We remark that, since A' is even number, we see from (5), that R is an even number.

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6) WEYL, l. c. 1) p. 99.