CONVERGENCE CRITERIA FOR FOURIER SERIES

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Let $\varphi(x)$ be even, periodic with period 2π and its Fourier series be

$$\varphi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt.$$

Moreover, we shall confine our attention to the convergency of the series at the origin. Concerning of Young's test, there are following tests.

THEOREM I (Young-Pollard [3]). *If $\varphi(t)$ satisfies*

$$\int_0^t \varphi(u)\, du = o(t) \tag{1}$$

and

$$\int_0^t |d(u\, \varphi(u))| = o(t), \tag{2}$$

then the Fourier series converges to zero at the origin.

THEOREM II (Hardy-Littlewood [2]). *If $\varphi(t)$ satisfies*

$$\int_0^t |\varphi(u)|\, du = o(t/\log 1/t) \tag{3}$$

and

$$\int_0^t |d(u^{\Delta}\, \varphi(u))| = O(t) \tag{4}$$

for some $\Delta > 0$, then the Fourier series converges to zero at the origin.

THEOREM III (Sunouchi [4]). *If $\varphi(t)$ satisfies*

$$\int_0^t \varphi(u)\, du = o(t^{\Delta}) \tag{5}$$

and

$$\int_0^t |d(u^{\Delta}\, \varphi(u))| = O(t) \tag{6}$$

for $\Delta > 1$, then the Fourier series converges to zero at the origin.

On the other hand the condition (2) implies Lebesgue's condition. This fact is due to Pollard [3]. The object of this paper is to establish convergence criteria of Lebesgue type which include Theorem II and III.

THEOREM 1. *If* $\varphi(t)$ *satisfies*

$$\int_0^t \varphi(u)\,du = o(t^{\Delta}) \tag{7}$$

and

$$\lim_{k\to\infty}\limsup_{x\to 0}\int_{(kx)^{1/\Delta}}^{\eta}\left|\frac{\varphi(t)}{t}-\frac{\varphi(t+x)}{t+x}\right|dt = 0 \tag{8}$$

for $\Delta \geqq 1$ *and some fixed* $\eta > 0$, *then the series converges. The condition* (6) *or* (2) *implies* (8).

PROOF. Since the case $\Delta = 1$ is due to Pollard [3], it is sufficient to prove the case $\Delta > 1$. To prove the convergence of the Fourier series is equivalent to prove

$$\lim_{\omega\to\infty}\int_0^{\eta}\varphi(t)\frac{\sin\omega t}{t}\,dt = 0.$$

Let us put

$$\alpha = (\pi k/\omega)^{1/\Delta} \qquad \text{and} \qquad \Phi(t) = \int_0^t \varphi(u)\,du,$$

then

$$\int_0^{\alpha}\varphi(t)\frac{\sin\omega t}{t}\,dt = \left[\Phi(t)\frac{\sin\omega t}{t}\right]_0^{\alpha} - \int_0^{\alpha}\Phi(t)\frac{\omega t\cos\omega t-\sin\omega t}{t^2}\,dt$$
$$= I_1 + I_2,$$

say. Then we have

$$|I_1| = o(\alpha^{\Delta-1}) = o\{(k/\omega)^{(\Delta-1)/\Delta}\} = o(1), \qquad \text{as } \omega\to\infty$$

and

$$|I_2| = o\left(\omega\int_0^{\alpha}t^{\Delta-1}dt\right) = o(\omega\alpha^{\Delta}) = o\{\omega(k/\omega)^{\Delta/\Delta}\} = o(1), \qquad \text{as } \omega\to\infty.$$

It is therefore sufficient to prove that

$$\lim_{k\to\infty}\limsup_{\omega\to\infty} I(\omega) \equiv \lim_{k\to\infty}\limsup_{\omega\to\infty}\int_{\alpha}^{\eta}\varphi(t)\frac{\sin\omega t}{t}\,dt = 0.$$

We can replace the upper limit in the integral $I(\omega)$ by $\eta + \pi/\omega$, and its lower limit by $(k\pi/\omega)^{1/\Delta} + \pi/\omega$, with error $o(1)$ as $\omega\to\infty$, and if we write $t+\pi/\omega$ for t, we obtain

$$I(\omega) = -\int_{\alpha}^{\eta}\frac{\varphi(t+\pi/\omega)}{t+\pi/\omega}\sin\omega t\,dt + o(1).$$

It follows, adding the two expressions for $I(\omega)$, that

$$I(\omega) = \frac{1}{2}\int_{\left(\frac{k\pi}{\omega}\right)^{1/\Delta}} \left\{\frac{\varphi(t)}{t} - \frac{\varphi(t+\pi/\omega)}{t+\pi/\omega}\right\}\sin\omega t\, dt + o(1).$$

From (8), we get $I(\omega)\to 0$, as $\omega\to\infty$ and the convergency is proved.

To prove that (6) implies (8) for $\Delta > 1$, we shall put $\theta(t) = t^{\Delta}\varphi(t)$, then

$$\Theta(t) \equiv \int_0^t |d\theta(u)| \leqq At, \qquad \text{and} \qquad |\theta(t)| \leqq At$$

by (6). If we write $x = \pi/\omega$, then

$$\frac{\varphi(t+x)}{t+x} - \frac{\varphi(t)}{t} = \frac{\theta(t+x)}{(t+x)^{\Delta+1}} - \frac{\theta(t)}{t^{\Delta+1}} = \int_t^{t+x} d\left(\frac{\theta(u)}{u^{\Delta+1}}\right)$$

and

$$\left|\frac{\varphi(t+x)}{t+x} - \frac{\varphi(t)}{t}\right| \leqq \int_t^{t+x}\left|d\left(\frac{\theta(u)}{u^{\Delta+1}}\right)\right| \leqq \int_t^{t+x}\frac{|d\theta(u)|}{u^{\Delta+1}} + (\Delta+1)\int_t^{t+x}\frac{|\theta(u)|}{u^{\Delta+2}}du$$

$$\leqq \frac{\Theta(t+x)}{(t+x)^{\Delta+1}} - \frac{\Theta(t)}{t^{\Delta+1}} + (\Delta+1)\int_t^{t+x}\frac{\Theta(u)}{u^{\Delta+2}}du + (\Delta+1)\int_t^{t+x}\frac{|\theta(u)|}{u^{\Delta+2}}du$$

$$\leqq \frac{\Theta(t+x)}{(t+x)^{\Delta+1}} - \frac{\Theta(t)}{t^{\Delta+1}} + \frac{2(\Delta+1)A}{\Delta}\left(\frac{1}{t^{\Delta}} - \frac{1}{(t+x)^{\Delta}}\right).$$

Integrating this equality, we obtain

$$\int_{(kx)^{1/\Delta}}^{\eta}\left|\frac{\varphi(t+x)}{t+x} - \frac{\varphi(t)}{t}\right|dt \leqq \int_{(kx)^{1/\Delta}}^{\eta}\left[\frac{\Theta(u)}{u^{\Delta+1}} - \frac{2(\Delta+1)A}{\Delta}\frac{1}{u^{\Delta}}\right]_t^{t+x} dt$$

$$= \int_{\eta}^{\eta+x} - \int_{(kx)^{1/\Delta}}^{(kx)^{1/\Delta}+x}\left\{\frac{\Theta(t)}{t^{\Delta+1}} - \frac{c}{t^{\Delta}}\right\}dt \qquad \left(c = \frac{2(\Delta+1)A}{\Delta}\right)$$

$$\leqq \int_{\eta}^{\eta+x}\frac{\Theta(t)}{t^{\Delta+1}}dt + c\int_{(kx)^{1/\Delta}}^{(kx)^{1/\Delta}}\frac{1}{t^{\Delta}}dt$$

$$\leqq A\int_{\eta}^{\eta+x}\frac{1}{t^{\Delta}}dt + c\int_{(kx)^{1/\Delta}}^{(kx)^{1/\Delta}+x}\frac{1}{t^{\Delta}}dt$$

$$\leqq \frac{A}{\Delta-1}\left\{\frac{1}{\eta^{\Delta-1}} - \frac{1}{(\eta+x)^{\Delta-1}}\right\}$$

$$+ \frac{c}{\Delta-1}\left\{\frac{1}{((kx)^{1/\Delta}+x)^{\Delta-1}} - \frac{1}{(kx)^{(\Delta-1)/\Delta}}\right\}.$$

Consequently, we have

$$\lim_{k\to\infty}\limsup_{x\to 0}\int_{(kx)^{1/\Delta}}^{\eta}\left|\frac{\varphi(t+x)}{t+x}-\frac{\varphi(t)}{t}\right|dt$$
$$= o(1)+\frac{c}{\Delta-1}\lim_{k\to\infty}\limsup_{x\to 0}\left\{\frac{1}{(kx)^{(\Delta-1)/\Delta}}\right\}\left[\left\{\frac{1}{1+(kx)^{1/\Delta}}\right\}^{\Delta-1}-1\right]$$
$$= o(1)+\frac{c}{\Delta-1}\lim_{k\to\infty}\limsup_{x\to 0}\left\{\frac{1}{(kx)^{(\Delta-1)/\Delta}}\right\}\left\{(1-\Delta)\frac{x}{(kx)^{1/\Delta}}\right\}$$
$$= o(1)-c\lim_{k\to\infty}\frac{1}{k} = o(1).$$

Thus the theorem is proved.

THEOREM 2. *If $\varphi(t)$ satisfies*

$$\int_0^1 |\varphi(u)|\,du = o(t/\log 1/t) \tag{9}$$

and

$$\lim_{k\to\infty}\limsup_{x\to 0}\int_{(kx)^{1/\Delta}}^{\eta}\left|\frac{\varphi(t)}{t}-\frac{\varphi(t+x)}{t+x}\right|dt = 0 \tag{10}$$

for some $\Delta>0$, then the series converges. The condition (4) *implies* (10).

PROOF. When Δ decreases, the condition (10) becomes stronger, and then we can suppose $\Delta \geqq 1$. If we can prove

$$\int_{\alpha}^{\alpha+\pi/\omega}\frac{\varphi(t)}{t}\sin\omega t\,dt = o(1),\quad \int_{\eta}^{\eta+\pi/\omega}\frac{\varphi(t)}{t}\sin\omega t\,dt = o(1) \tag{11}$$

as $\omega\to\infty$ where $\alpha=(\pi k/\omega)^{1/\Delta}$, then (10) implies

$$\int_{\alpha}^{\eta}\varphi(t)\,\frac{\sin\omega t}{t}\,dt = o(1), \tag{12}$$

similarly as in the proof of Theorem 1. The second of (11) is evident. The absolute value of the first of (11) is less than

$$\int_{\alpha}^{\alpha+\pi/\omega}\frac{|\varphi(t)|}{t}\,dt = \left[\frac{\Phi^*(t)}{t}\right]_{\alpha}^{\alpha+\pi/\omega}-\int_{\alpha}^{\alpha+\pi/\omega}\frac{\Phi^*(t)}{t^2}\,dt$$
$$= o(1)+o\left(\int_{\alpha}^{\alpha+\pi/\omega}\frac{dt}{t\log 1/t}\right)$$

where $\Phi^*(t)=\int_0^t |\varphi(u)|\,du$. Now

$$\int_{\alpha}^{\alpha+\pi/\omega}\frac{dt}{t\log 1/t} = \log\log\left(\frac{\pi k}{\omega}\right)^{1/\Delta}-\log\log\left\{\left(\frac{\pi k}{\omega}\right)^{1/\Delta}+\frac{\pi}{\omega}\right\}$$

which tends to zero as $\omega\to\infty$, for $\Delta\geqq 1$. Thus we have proved (11), and then (12).

Hence it is sufficient to prove that

$$\int_0^{\alpha} \varphi(t) \frac{\sin \omega t}{t} dt = o(1).$$

Since

$$\int_0^{1/\omega} \varphi(t) \frac{\sin \omega t}{t} dt = o(1),$$

it is sufficient to prove

$$\int_{1/\omega}^{\alpha} \varphi(t) \frac{\sin \omega t}{t} dt = o(1). \tag{13}$$

The absolute value of the left hand side is less than

$$\int_{1/\omega}^{\alpha} \frac{|\varphi(t)|}{t} dt = \left[\frac{\Phi^*(t)}{t}\right]_{1/\omega}^{\alpha} + \int_{1/\omega}^{\alpha} \frac{\Phi^*(t)}{t^2} dt$$

$$= o(1) + o\left(\int_{1/\omega}^{\alpha} \frac{dt}{t \log 1/t}\right)$$

where

$$\int_{1/\omega}^{\alpha} \frac{dt}{t \log 1/t} = \left[\log \log \frac{1}{t}\right]_{1/\omega}^{\alpha} = \log \log \omega - \log \log \left(\frac{\omega}{\pi k}\right)^{1/\Delta}$$

$$= -\log \frac{1}{\Delta} + o(1).$$

Thus we get (13) and then the Theorem is proved.

THEOREM 3. *In Theorem* 1 *or* 2, *we can take*

$$\lim_{k\to\infty} \limsup_{x\to 0} \int_{(kx)^{1/\Delta}}^{\eta} \frac{|\varphi(t+x) - \varphi(t)|}{t} dt = 0 \tag{11}$$

in the place of (8) *or* (10).

PROOF. First we assume (11) and (7). It is sufficient to prove

$$S(\omega) \equiv \int_0^{\eta} \frac{\varphi(t)}{t} \sin \omega t \, dt \to 0, \qquad \text{as } \omega \to \infty$$

for a fixed $\eta > 0$. If we put

$$\alpha(\omega, k) \equiv \left\{\int_0^{(kx)^{1/\Delta}} + 2\int_0^{(kx)^{1/\Delta}+x} + \int_0^{(kx)^{1/\Delta}+2x} - 2\int_{\eta}^{\eta+x} - \int_{\eta}^{\eta+2x}\right\} \frac{\varphi(t)}{t} \sin \omega t \, dt$$

then we have

$$\int_{(kx)^{1/\Delta}}^{\eta} \frac{\varphi(t)}{t} \sin \omega t \, dt = -\int_{(kx)^{1/\Delta}}^{\eta} \frac{\varphi(t+x)}{t+x} \sin \omega t \, dt + o(1)$$

by (7).

Let us write

$$4\,S(\omega)-\alpha(\omega,k)=\left\{\int_{(kx)^{1/\Delta}}^{\eta}+2\int_{(kx)^{1/\Delta}+x}^{\eta+x}+\int_{(kx)^{1/\Delta}+2x}^{\eta+2x}\right\}\frac{\varphi(t)}{t}\sin\omega t\,dt$$

$$=2x^2\int_{(kx)^{1/\Delta}}^{\eta}\frac{\varphi(t+x)}{t(t+x)(t+2x)}\sin\omega t\,dt$$

$$+\int_{(kx)^{1/\Delta}}^{\eta}\left\{\frac{\varphi(t+2x)-\varphi(t+x)}{t+2x}-\frac{\varphi(t+x)-\varphi(t)}{t}\right\}\sin\omega t\,dt$$

$$=2\beta(\omega,k)+\gamma(\omega,k)$$

say. In the same way as the proof of Theorem 1, we obtain

$$\lim_{k\to\infty}\limsup_{\omega\to\infty}\alpha(\omega,k)=0,$$

while, since

$$\gamma(\omega,k)=2\int_{(kx)^{1/\Delta}}^{\eta}\frac{|\varphi(t+x)-\varphi(t)|}{t}\,dt+o(1)$$

we get

$$\lim_{k\to\infty}\limsup_{\omega\to\infty}\gamma(\omega,k)=0$$

by (11). On the other hand, since

$$0<g(x,t)\equiv\frac{x^2}{t(t+x)(t+2x)}<\frac{x^2}{t^3}$$

and

$$0<-\frac{\partial g}{\partial t}<\frac{11x^2}{t^4}$$

we obtain by partial integration

$$\beta(\omega,k)=\int_{(kx)^{1/\Delta}}^{\eta}g(x,t)\,\varphi(t+x)\sin\omega t\,dt$$

$$=\Big[\Phi(t+x)g(x,t)\sin\omega t\Big]_{(kx)^{1/\Delta}}^{\eta}-\omega\int_{(kx)^{1/\Delta}}^{\eta}\Phi(t+x)g(x,t)\cos\omega t\,dt$$

$$-\int_{(kx)^{1/\Delta}}^{\eta}\Phi(t+x)\frac{\partial g(x,t)}{\partial t}\sin\omega t\,dt,$$

where

$$\Phi(t)=\int_0^t\varphi(u)\,du.$$

Consequently we have

$$\lim_{k\to\infty}\limsup_{\omega\to\infty}\beta(\omega,k)=0.$$

The case (9) and (11) can be proved analogously.

REMARK. In the case $\Delta=1$, Theorem 3 is more general than Theorem 1.

This is due to Gergen [1]. But in the case $\Delta > 1$, the author could not decide the analogous fact.

References

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