ON THE RIEMANN SUMMABILITY

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The series $\sum_{\nu=1}^{\infty} a_\nu$ is said (R, p)-summable to s if the series

$$\sum_{\nu=1}^{\infty} a_\nu \left(\frac{\sin \nu t}{\nu t}\right)^p \tag{1}$$

converges in some interval $0 < t < t_0$, and if

$$\lim_{t \to 0} \sum_{\nu=1}^{\infty} a_\nu \left(\frac{\sin \nu t}{\nu t}\right)^p = s, \tag{2}$$

where p is a positive integer.

This method of summability has been considered by S. Verblunsky [3], who has shown that, if a series is summable $(C, p - \delta)$ to s, where $\delta > 0$, then it is summable $(R, p + 1)$ to s.

For the case $p = 1$, many results are known. In particular, G. Sunouchi [1] proved the following theorem;

THEOREM 1. *Suppose that*

$$\sum_{\nu=1}^{n} s_\nu = o(n^\alpha),$$

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-\alpha}),$$

where $0 < \alpha < 1$. *Then the series* $\sum_{\nu=1}^{\infty} a_\nu$ *is* $(R, 1)$*-summable to zero.*

As the generalization of the above theorem, H. Hirokawa and G. Sunouchi [2] proved the following theorem;

THEOREM 2. *Let* s_n^β *be the* (C, β)*-sum of* $\sum_{n=1}^{\infty} a_n$.

Then, if

$$s_n^\beta = o(n^{\beta\alpha})$$

and

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-\alpha}),$$

where $0 < \alpha < 1, 0 \leqq \beta$, *the series* $\sum_{n=1}^{\infty} a_n$ *is summable* $(R, 1)$ *to zero.*

The object of this paper is to generalize the above theorems.

THEOREM. *Let s_n^β be the (C, β)-sum of $\sum_{n=1}^{\infty} a_n$.*

If

$$s_n^\beta = o(n^\gamma), \qquad (n \to \infty) \tag{3}$$

for $\beta > \gamma > 0$, $\gamma + 1 > p$, where p is a positive integer, and

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-(1-\delta)}), \qquad (n \to \infty) \tag{4}$$

for $\delta = p(\beta - \gamma)/(\beta + 1 - p)$, $0 < \delta < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is (R, p)-summable to zero.

If we put $p = 1, \beta = 1$, we have Theorem 1 and if we put $p = 1$, then $\delta = \beta - \gamma/\beta$, that is $\gamma = \beta(1 - \delta)$. This is Theorem 2.

PROOF. If we put

$$\varphi(t) = \left(\frac{\sin t}{t}\right)^p$$

we have

$$\varphi^{(k)}(t) = O(1), \qquad (t \to 0) \tag{5}$$

and

$$\varphi^{(k)}(t) = o(t^{-p})^*, \qquad (t \to \infty) \tag{6}$$

for $k = 0, 1, 2, \ldots\ldots$

Firstly, we shall that the series (1) is convergent for all t.

Now

$$\sum_{\nu=0}^{\infty} a_\nu \varphi(\nu t) = \left(\sum_{\nu=0}^{n} + \sum_{\nu=n+1}^{\infty}\right) a_\nu \varphi(\nu t) = \varphi_1 + \varphi_2, \qquad (a_0 = 0),$$

say, where n is to be chosen presently.

By (6) and (4)

$$\varphi_2 = O\left(\sum_{\nu=n+1}^{\infty} \frac{|a_\nu|}{\nu} \nu^{-p+1} t^{-p}\right) = O(t^{-p} n^{-p+\delta}). \tag{7}$$

This shows that for fixed $t > 0$, the series $\sum_{\nu=0}^{\infty} a_\nu \varphi(\nu t)$ converges.

Given a positive integer ε, put

$$n = [(\varepsilon t)^{-\rho}], \tag{8}$$

where $\rho = \dfrac{p}{p - \delta} = \dfrac{\beta + 1 - p}{\gamma + 1 - p}$. Then from (7) it follows that

$$\varphi_2 = O\{t^{-p}(\varepsilon t)^{\rho(p-\delta)}\} = O(\varepsilon^p). \tag{9}$$

Next, if we put $r_n = \sum_{\nu=n}^{\infty} \dfrac{|a_\nu|}{\nu}$, then $|a_n| = n(r_n - r_{n+1})$.

* Cf. M. Obreschkoff [4], Hilfssatz 3.

Since, by (4),

$$\sum_{\nu=1}^{n}|a_\nu| = \sum_{\nu=1}^{n} r_\nu - nr_{n+1} = O\Big(\sum_{\nu=1}^{n} \nu^{-(1-\delta)}\Big) + O(n^\delta) = O(n^\delta),$$

we have

$$s_n = O(n^\delta). \tag{10}$$

Now there is an integer $k \geqq 1$ such that $k-1 < \beta \leqq k$. We suppose that $k-1 < \beta < k$, for the case $\beta = k$ can be easily deduced by the following argument. From (3), (10), using Riesz's convexity theorem, we have

$$\begin{aligned} s_n^\nu &= o(n^{\delta(\beta-\nu)/\beta+\gamma\nu/\beta}), \qquad \nu = 1, 2, \dots,, k-1 \\ s_n^k &= o(n^{k+\gamma-\beta}). \end{aligned} \tag{11}$$

Let $s^0(x) = s(x) = \sum_{\nu=0}^{n} a_\nu$, where $n \leqq x < n+1$, and

$$s^p(x) = \frac{1}{\Gamma(p)}\int_0^x (x-\ )t^{p-1}s(t)dt, \qquad p > 0.$$

Then, by Abel's lemma on partial summation, we have

$$\begin{aligned} \varphi_1 = \sum_{\nu=0}^{n} a_\nu\varphi(\nu t) &= \sum_{\nu=0}^{n-1} s_\nu\{\varphi(\nu t) - \varphi(\overline{\nu+1}t)\} + s_n\varphi(nt) \\ &= -\sum_{\nu=0}^{n-1}\int_\nu^{\nu+1} s^0(x)\frac{d}{dx}\varphi(xt)dx + s_n\varphi(nt) \\ &= -\int_0^n s^0(x)\frac{d}{dx}\varphi(xt)dx + s_n\varphi(nt). \end{aligned}$$

Integrating the first term in the last expression by parts k times, and writing $D_n^\nu = \left[\dfrac{d^\nu}{dx_\nu}\right]_{x=n}$, we get

$$\begin{aligned} \varphi_1 = s_n\varphi(nt) &+ \sum_{\nu=1}^{k-1}(-1)^\nu s^\nu(n)D_n^\nu\varphi(xt) + (-1)^k s^k(n)D_n^k\varphi(xt) \\ &+ (-1)^{k+1}\int_0^n s^k(x)\frac{d^{k+1}}{dx^{k+1}}\varphi(xt)dx = \varphi_3+\varphi_4+\varphi_5+\varphi_6, \text{ say.} \end{aligned} \tag{12}$$

Then, by (10)

$$\varphi_3 = s_n\varphi(nt) = O(n^\delta(nt)^{-p}) = O(t^{-p}n^{-(p-\delta)}) = O(\varepsilon^p). \tag{13}$$

Concerning φ_4,

$$\begin{aligned} s_\nu(n)D_n^\nu\varphi(xt) &= s^\nu(n)t^\nu\left[\frac{d^\nu}{d(xt)^\nu}\varphi(xt)\right]_{x=n} \\ &= o(n^{\delta(\beta-\nu)/\beta+\gamma\nu/\beta}t^\nu(nt)^{-p}) = o\{t^{\nu-p}t^{-p\{\delta(\beta-\nu)+\gamma\nu-p\beta\}/\beta}\}. \end{aligned}$$

Using (8) and (11), the exponent of t is

$$\left[(\nu-p)\beta - \frac{p}{p-\delta}\{\beta(\delta-p)+\nu(\gamma-\delta)\}\right]\Big/\beta$$

$$= \frac{1}{\beta}\left\{\beta\nu - \frac{p}{p-\delta}\nu(\gamma-\delta)\right\} = \frac{\nu}{\beta}\left(\beta - \frac{\beta+1-p}{\beta-1-p}\cdot\frac{\gamma\beta+\gamma-p\beta}{\beta+1-p}\right)$$

$$= \frac{\nu}{\beta(\gamma+1-p)}(\beta-\gamma) > 0,$$

for $\nu = 1, 2, \ldots, k-1$, and these terms appear for $\beta > 1$. Thus we have

$$\varphi_4 = o(1), \text{ as } t \to 0. \tag{14}$$

Next, we obtain

$$\varphi_5 = o(n^{k+\gamma-\beta}t^k(nt)^{-p}) = o(t^{k-p-\rho(k+\gamma-\beta-p)}),$$

by (8) and (11).

The exponent of t is

$$k - p - \rho(k+\gamma-\beta-p) = \frac{1}{p-\delta}\{(p-\delta)(k-p) - p(k+\gamma-\beta-p)\}$$

$$= \frac{1}{p-\delta}\{-k\delta + p(\beta-\gamma+\delta)\} = \frac{1}{p-\delta}\{-k\delta + \delta(\beta+1)\}$$

$$= \frac{\delta}{p-\delta}(\beta+1-k) > 0,$$

for $\delta = p(\beta-\gamma)/(\beta+1-p)$.

Therefore

$$\varphi_5 = o(1), \text{ as } t \to 0. \tag{15}$$

We next take up φ_6. Omitting constant factors, we split up four parts,

$$\varphi_6 = \int_0^n s^k(x)\frac{d^{k+1}}{dx^{k+1}}\varphi(xt)dx = \int_0^n \frac{d^{k+1}}{dx^{k+1}}\varphi(xt)dx\int_0^x (x-u)^{k-\beta-1}s^\beta(u)du$$

$$\begin{aligned} &= \int_0^n s^\beta(u)du\int_u^n (x-u)^{k-\beta-1}\frac{d^{k+1}}{dx^{k+1}}\varphi(xt)dx \\ &= \int_0^{t^{-1}} du\int_u^{u+t^{-1}} dx + \int_{t^{-1}}^{(\epsilon t)^{-\rho}} du\int_u^{u+t^{-1}} dx + \int_0^{(\epsilon t)^{-\rho}-t^{-1}} du\int_{u+t^{-1}}^{(\epsilon t)^{-\rho}} dx \\ &\quad - \int_{(\epsilon t)^{-\rho}-t^{-1}}^{(\epsilon t)^{-\rho}} du\int_{(\epsilon t)^{-\rho}}^{u+t^{-1}} dx = \psi_1+\psi_2+\psi_3-\psi_4 \end{aligned} \tag{16}$$

say. Since $\varphi^{(k)}(t) = O(1)$ for $0 < t \leqq 1$,

$$\begin{aligned} \psi_1 &= \int_0^{t^{-1}} s^\beta(u)du\int_u^{u+t^{-1}} (x-u)^{k-\beta-1}\frac{d^{k+1}}{dx^{k+1}}\varphi(xt)dx \\ &= O\left\{\int_0^{t^{-1}} s^\beta(u)du\int_u^{u+t^{-1}} t^{k+1}(x-u)^{k-\beta-1}dx\right. \\ &= o\left\{t^{k+1}\int_0^{t^{-1}} u^\gamma\Big[(x-u)^{k-\beta}\Big]_u^{u+t^{-1}}du\right\} \\ &= o\left\{t^{k+1}\int_0^{t^{-1}} u^\gamma t^{-(k-\beta)}du\right\} = o(t^{k+1-k+\beta-\gamma-1}) \end{aligned} \tag{17}$$

$= o(t^{\beta-\gamma}) = o(1)$, for $\beta > \gamma$.

Since $1 + \gamma > (\beta+1)/\beta\gamma > p$,

$$\psi_2 = \int_{t^{-1}}^{(\epsilon t)^{-\rho}} s^{\beta}(u)du \int_{u}^{u+t^{-1}} (x-u)^{k-\beta-1} \frac{d^{k+1}}{dx^{k+1}}\varphi(xt)dx$$

$$= O\left\{ t^{k+1} \int_{t_{-1}}^{(\epsilon t)^{-\rho}} s^{\beta}(u)du \int_{u}^{u+t^{-1}} (x-u)^{k-\beta-1}(xt)^{-p}dx \right\}$$

$$= o\left\{ t^{k+t-p} \int_{t^{-1}}^{(\epsilon t)^{-\rho}} u^{\gamma}u^{-p}\left[(x-u)^{k-\beta}\right]_{u}^{u+t^{-1}} du \right\}$$

$$= o\left(t^{k+1-p}t^{-(k-\beta)}\left[u^{\gamma+1-p}\right]_{t^{-1}}^{(\epsilon t)^{-\rho}} \right)$$

$$= o(t^{\beta+1-p-\rho(\gamma+1-p)}) = o(1),$$

as $t \to 0$, by (8). Therefore

(18) $$\psi_2 = o(1).$$

Concerning ψ_3, if we use integration by parts in the inner integral, then

$$\psi_3 = \int_{0}^{(\epsilon t)^{-\rho}-t^{-1}} s^{\beta}(u)du \int_{u+t^{-1}}^{(\epsilon t)^{-\rho}} (x-u)^{k-\beta-1} \frac{d^{k+1}}{dx^{k+1}}\varphi(xt)dx$$

$$= \int_{0}^{(\epsilon t)-\rho-t^{-1}} s^{\beta}(u)du\left\{\left[(x-u)^{k-\beta-1}\frac{d^k}{dx^k}\varphi(xt)\right]_{k+t^{-1}}^{(\epsilon t)^{-\rho}}\right.$$

$$\left. -(k-\beta-1)\int_{u+t^{-1}}^{(\epsilon t)^{-\rho}} (x-u)^{k-\beta-2}\frac{d^k}{dx^k}\varphi(xt)dx\right\}$$

$$= \chi_1 - (k-\beta-1)\chi_2,$$

say. Then

$$\chi_1 = O\left(t^k \int_{0}^{(\epsilon t)^{-\rho}-t^{-1}} s^{\beta}(u)du\left\{ t^{-p}t^{\rho p}((\mathcal{E}t)^{-p} - u)^{k-\beta-1} \right.\right.$$

$$\left.\left. -(u+t^{-1})^{-pt-pt-(k-\beta-1)}\right\}\right)$$

$$= \chi_3 + \chi_4.$$

$$\chi_3 = o(t^{k-p+\rho p}\int_{0}^{(\epsilon t)^{-\gamma}} u^{\gamma}((\mathcal{E}t)^{-\rho} - u)^{k-\beta-1}du) = o(t^{k-p+\rho p-\rho(\gamma+k-\beta)}).$$

Since the exponent of t is

$$k-p+\rho(p-k+\beta-\gamma) = \frac{1}{\gamma+1-p}\{(k-p)(\gamma+1-p)+(\beta+1-p)(p-k+\beta-\gamma)\}$$

$$= \frac{(\beta-\gamma)(\beta+1-k)}{\gamma+1-p} > 0,$$

$\chi_3 = o(1)$, as $t \to 0$.

$$\chi_4 = o\left\{ t^{k-p-(k-\beta-1)} \int_0^{(\epsilon t)^{-\rho}-t^{-1}} u^{\gamma}(u+t^{-1})^{-p} du \right\}$$

$$= o\left\{ t^{\beta+1-p} \int_0^{(\epsilon t)^{-\rho}} u^{\gamma-p}\, du \right\} = o(t^{\beta+1-p-\rho(\gamma+1-p)}) = o(1),$$

as $t \to 0$. Therefore

(20) $$\chi_1 = o(1), \text{ as } t \to 0.$$

Similar estimation gives

$$\chi_2 = O\left\{ t^k \int_0^{(\epsilon t)^{-\rho}-t^{-1}} s^{\beta}(u)du \int_{u+t^{-1}}^{(\epsilon t)^{-\rho}} (xt)^{-p}(x-u)^{k-\beta-2} dx \right.$$

(21)
$$= o\left\{ t^{k-p} \int_0^{(\epsilon t)^{-\rho}-t^{-1}} u^{\gamma-p} \Big[(x-u)^{k-\beta-1} \Big]_{u+t^{-1}}^{(\epsilon t)^{-\rho}} du \right\}$$

$$= o\left\{ t^{k-p} \int_0^{(\epsilon t)^{-\rho}-t^{-1}} u^{\gamma-p} t^{-(k-\beta-1)} du \right\}$$

$$= o(t^{\beta+1-p-\rho(\gamma+1-p)}) = o(1), \text{ as } t \to 0.$$

From (20), (21) and (19), we have

(22) $$\psi_3 = o(1).$$

We have easily

$$\psi_4 = \int_{(\epsilon t)^{-\rho}-t^{-1}}^{(\epsilon t)^{-\rho}} s^{\beta}(u)du \int_{(\epsilon t)^{-\rho}}^{u+t^{-1}} (x-u)^{k-\beta-1} \frac{d^{k+1}}{dx^{k+1}} \varphi(xt)dx$$

$$= O\left\{ \int_{(\epsilon t)^{-\rho}-t^{-1}}^{(\epsilon t)^{-\rho}} s^{\beta}(u)du \int_{(\epsilon t)^{-\rho}}^{u+t^{-1}} (x-u)^{k-\beta-1} t^{k+1}(xt)^{-p} dx \right\}$$

(23)
$$= o\left\{ t^{k+1-p} \int_{(\epsilon t)^{-\rho}-t^{-1}}^{(\epsilon t)^{-\rho}} u^{\gamma} t^{\rho p} \Big[(x-u)^{k-\beta} \Big]_{(\epsilon t)^{-\rho}}^{u+t^{-1}} du \right\}$$

$$= o(t^{k+1-p+\rho p-(k-\beta)} \Big[u^{\gamma+1} \Big]_{(\epsilon t)^{-\rho}-t^{-1}}^{(\epsilon t)^{-\rho}})$$

$$= o(t^{k+1-p-(k-\beta)+\rho(p-\gamma-1)})$$

$$= o(t^{\beta+1-p-\rho(\gamma+1-p)}) = o(1), \text{ as } t \to 0.$$

Summing up (17), (18), (22) and (23) we have

(24) $$\varphi_6 = o(1), \text{ as } t \to 0.$$

From (13), (14), (15) and (24) we have

(25) $$\varphi_1 = o(1), \text{ as } t \to 0.$$

Therefore, from (19) and (25), we obtain

$$\lim_{t \to 0} \sum_{\nu=0}^{\infty} a_\nu \varphi(\nu t) = 0.$$

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