ON THE CESÀRO SUMMABILITY OF FOURIER SERIES

MASAKITI KINUKAWA

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1. Introduction. Let $f(t)$ be an integrable function periodic with period 2π and let

$$\varphi_x(t) = \varphi(t) = f(x+t) + f(x-t) - 2s.$$

We denote by Δ a constant $\geqq 1$.

G. H. Hardy and J. E. Littlewood [3] proved the following theorem:

THEOREM A. *If $\varphi(t)$ satisfies*

$$\int_0^t |\varphi(u)|du = o\left(t/\log \frac{1}{t}\right) \tag{1.1}$$

and

$$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t), \quad 0 \leqq t \leqq \eta, \tag{1.2}$$

then the Fourier series of $f(t)$ converges to s at a point x.

Recently G. Sunouchi [5], [6], [7] proved the following theorems:

THEOREM B. *If $\varphi(t)$ satisfies the condition* (1.2) *and*

$$\int_0^t \varphi(u)du = o(t^\Delta), \tag{1.3}$$

then the Fourier series of $f(t)$ converges to s at a point x.

THEOREM C. *If satisfies the condition* (1.3) *and*

$$\lim_{k\to\infty} \limsup_{u\to 0} \int_{(ku)^{1/\Delta}}^{\eta} \frac{|\varphi(t+u) - \varphi(t)|}{t} dt = 0, \tag{1.4}$$

then the Fourier series of $f(t)$ converges to s at a point x.

THEOREM D. *If $\varphi(t)$ satisfies the conditions* (1.1) *and* (1.4), *then the Fourier series of $f(t)$ converges to s at a point x.*

THEOREM E. *Let us suppose that*

(1.5) *$\theta(x)$ is a positive differentiable function with $\theta'(x) > 0$,*

$$\Theta(x) = \int^x \frac{du}{u\theta(u)} \quad \textit{increases with } x, \tag{1.6}$$

$$\mu(x, c) = 1/\Theta^{-1}(\Theta(x) - c). \tag{1.7}$$

Let $f(t)$ be an even integrable function with mean value zero and its

Fourier series be

$$f(t) \sim \sum_{n=1}^{\infty} a_n \cos nt.$$

If

$$\int_0^t |f(u)|du = o(t/\theta(1/t))$$

and there is a positive constant c *such that* $a_n > -\mu(n, c)$, *then*

$$\sum_{n=1}^{\infty} a_n = 0.$$

As examples of θ and μ, he stated as follows:

(1.8)

Case	1	2	3	4
θ	1	$\log \log x$	$\log x$	$\log x \log \log x$
μ	A/x	$(\log x)^c/x$	$1/x^{e^{-c}}$	$\exp\{-(\log x)^{e^{-c}}\}$

The case $\Delta = 1$ in Theorem B is Young's convergence test. For this case, G. H. Hardy and J. E. Littlewood [2] showed that the conditions (1.2) and (1.3) imply (C, ρ) summability, where ρ is any negative number such that $-1 < \rho \leqq 0$.

Concerning Cesàro summability, J. J. Gergen [1] proved the following theorem:

THEOREM F. *Let* $\varphi_\beta(t)$ *be the* β*-th integral of* $\varphi(t)$. *If*

$$\varphi_\beta(t) = o(t^\beta) \tag{1.9}$$

and

$$\lim_{k\to\infty} \limsup_{u\to 0} u^\rho \int_{ku}^{\pi} \frac{|\Delta_u^{(m)}\varphi(t)|}{t^{1+\rho}} dt = 0, \tag{1.10}$$

then the Fourier series of $f(t)$ *is* (C, ρ) *summable to* s *at a point* x, *where* $\rho > -1$, m *is some positive integer and*

$$\Delta_u^{(m)}\varphi(t) = \sum_{\nu=0}^{m} (-1)^{m+\nu} \binom{m}{\nu} \varphi(t+\nu u). \tag{1.11}$$

Being assumed only the continuity condition of the function, Cesàro summability of positive order was discussed by many writers. For example, S. Izumi and G. Sunouchi [4] proved the following theorem:

THEOREM G. *If* $0 < \beta < \gamma$, $\beta \leqq 1 + \gamma - \beta$ *and* $\varphi_\beta(t) = o(t^\gamma)$, *then the Fourier series of* $f(t)$ *is summable* $(C, \beta/(\gamma - \beta + 1))$ *at a point* x.

Especially if we suppose $\beta = 1$ *and* $\gamma > 1$, *that is,*

$$\int_0^t \varphi(u)du = o(t^\gamma),$$

then the Fourier series of $f(t)$ *is summable* $(C, 1/\gamma)$ *at a point* x.

In this paper, we shall generalize Theorems A, B, C and D. replacing convergence-property by Cesàro summability.

Finally the author has to express his hearty thanks to Prof. S. Izumi, who gave him many valuable remarks and advices.

2. Our resultes are stated as follows:

THEOREM 1. *Let* $\Delta \geqq 1$, $-1 < \rho < 1$ *and*

$$\varepsilon_1 = \Delta - \frac{2\rho(\Delta-1)}{1+\rho}. \tag{2.1}$$

If

$$\int_0^t \varphi(u)du = o(t^{\varepsilon_1}) \tag{2.2}$$

and

$$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t), \tag{2.3}$$

then the Fourier series of $f(t)$ *is* (C, ρ) *summable to* s *at a point* x.

THEOREM 2. *Let* $\Delta \geqq 1$, $-1 < \rho \leqq 0$ *and*

$$\varepsilon_2 = 1 - \frac{\rho(\Delta-1)}{1+\rho}. \tag{2.4}$$

If $\varphi(t)$ *satisfies the conditions* (2.3) *and*

$$\int_0^t |\varphi(u)|du = o\left(t^{\varepsilon_2}/\log \frac{1}{t}\right), \tag{2.5}$$

then the Fourier series of $f(t)$ *is* (C, ρ) *summable to* s *at a point* x.

THEOREM 3. *Let* $\Delta \geqq 1$, $-1 < \rho < 1$ *and*

$$\varepsilon_3 = \Delta - \rho(\Delta - 1). \tag{2.6}$$

If

$$\int_0^t \varphi(u) = o(t^{\varepsilon_3}) \tag{2.7}$$

and

$$\lim_{k\to\infty} \limsup_{u\to 0} u^\rho \int_{(ku)^{1/\Delta}}^{\pi} \frac{|\Delta_u^{(m)}\varphi(t)|}{t^{1+\rho}} dt = 0, \tag{2.8}$$

then the Fourier series of $f(t)$ *is* (C, ρ) *summable to* s *at a point* x, *where* $\Delta_u^{(m)}\varphi(t)$ *is defined by* (1.11).

THEOREM 4. *Let* $\Delta \geqq 1$, $-1 < \rho \leqq 0$ *and*

$$\varepsilon_4 = 1 - \rho(\Delta - 1). \tag{2.9}$$

If $\varphi(t)$ *satisfies the condition* (2.8) *and if*

$$\int_0^t |\varphi(u)|du = o\left(t^{\varepsilon_4}/\log \frac{1}{t} \right), \tag{2.10}$$

then the Fourier series of $f(t)$ is (C, ρ) summable to s at a point x.

3. For the proof of our theorems, we need several known lemmas. Let us denote by $K_n^{(\rho)}(t)$ the n-th Cesàro mean of ordre ρ of the series

$$1/2 + \sum_{k=1}^{\infty} \cos kt.$$

Then we have

LEMMA 1.[1] *If we suppose that* $-1 < \rho \leqq 1$, *then*

$$K_n^{(\rho)}(t) = S_n^{(\rho)}(t) + R_n^{(\rho)}(t), \tag{3.1}$$

where

$$S_n^{(\rho)}(t) = \frac{\cos (A_n t + A)}{A_n^{(\rho)}(2 \sin t/2)^{1+\rho}},$$

$$|R_n^{(\rho)}(t)| < M/nt^2, \quad \left| \frac{d}{dt} R_n^{(\rho)}(t) \right| < \frac{M}{nt^3} + \frac{M}{n^2t^4} \tag{3.2}$$

and

$$A_n = n + (\rho + 1)/2, \quad A = -(\rho + 1)\pi/2. \tag{3.3}$$

LEMMA 2. *If* $-1 < \rho \leqq 1$ *and*

$$\int_0^t \varphi(u)du = o(t),$$

then

$$\lim_{k \to \infty} \limsup_{n \to \infty} \int_0^{k/n} \varphi(t) K_n^{(\rho)}(t)\, dt = 0.$$

Proof runs similarly as in a lemma due to J. J. Gergen ([1], Lemma 11).

LEMMA 3. *If* $\varphi_1(t) = O(t)$, *then we have*

$$\lim_{k \to \infty} \limsup_{n \to \infty} \int_{k/n}^{\pi + \xi y} \varphi(t) R_n^{(\rho)}(t) dt = 0,$$

where ξ is a fixed number and $y = O(k/n)$.

This is due to J. J. Gergen ([1], Lemma 12).

4. **Proof of Theorem 1.** The case $\Delta = 1$ is the Hardy-Littlewood's theorem stated in the introduction. We may then suppose that $\Delta > 1$.

If we denote by $\sigma_n^{(\rho)}(x)$ the n-th Cesàro mean of order ρ of the Fourier series of $f(t)$ at a point x, then

1) Cf. A. Zygmund [8].

$$
\begin{aligned}
\pi\{\sigma_n^{(\rho)}(x) - s\} &= \int_0^\pi \varphi(t)K_n^{(\rho)}(t)\, dt \\
(4.1)\qquad &= \int_0^{k/n} \varphi(t)K_n^{(\rho)}(t)\, dt + \int_{k/n}^\pi \varphi(t)R_n^{(\rho)}(t)dt + \int_{k/n}^\pi \varphi(t)S_n^{(\rho)}(t)\, dt,
\end{aligned}
$$

where, by Lemmas 2 and 3, the first and the second terms on the right side of (4.1) are of $o(1)$. Hence, for the proof of Theorem 1, it is sufficient to show that the third term on the right side of (4.1) is of $o(1)$, which we shall denote by I.

Let $\alpha = (k/n)^\delta$, where $\delta = \dfrac{1+\rho}{\Delta+\rho} = \dfrac{1-\rho}{\varepsilon_1 - \rho}$.

$$
(4.2)\qquad I = \left\{ \int_{k/n}^{\alpha} + \int_{\alpha}^{\pi} \right\} \varphi(t)S_n^{(\rho)}(t)dt = I_1 + I_2,
$$

say.

If we put

$$
\Lambda(t) = -\int_t^\pi \frac{\cos(A_n u + A)}{(2\sin u/2)^{1+\rho+\Delta}}\, du,
$$

then by the second mean value theorem, we have

$$
\Lambda(t) = O(1/nt^{1+\rho+\Delta}).
$$

By integration by parts, we have

$$
\begin{aligned}
I_2 &= \frac{1}{A_n^{(\rho)}} \Big[\varphi(t)(2\sin t/2)^\Delta \Lambda(t) \Big]_\alpha^\pi - \frac{1}{A_n^{(\rho)}} \int_\alpha^\pi \Lambda(t) d\{(2\sin t/2)^\Delta \varphi(t)\} \\
&= I_3 - I_4,
\end{aligned}
$$

say, where

$$
\begin{aligned}
I_3 &= O\left(\frac{1}{n^{1+\rho}} \Big[t^{-\rho-\Delta} \Big]_\alpha^\pi \right) = o(1) + O\left(\frac{1}{k^{\delta(\rho+\Delta)}} \cdot \frac{1}{n^{1+\rho-\delta(\rho+\Delta)}} \right) \\
&= o(1) + O(1/k^{1+\rho}).
\end{aligned}
$$

If we take k sufficiently large, then we have $I_3 = o(1)$. Putting

$$
\theta(t) = (2\sin t/2)^\Delta \varphi(t),\ \Phi(t) = \int_0^t |d\theta(t)|, \text{ we have}
$$

$$
(4.3)\qquad \Phi(t) = O(t),
$$

by the assumption of Theorem 1.

Using (4.3), we get

$$
I_4 = O\left(\frac{1}{n^{1+\rho}} \int_\alpha^\pi t^{-1-\rho-\Delta} |d\theta(t)| \right) = O\left(\frac{1}{n^{1+\rho}} \int_\alpha^\pi t^{-1-\rho-\Delta} d\Phi(t) \right)
$$

$$= O\left(\frac{1}{n^{1+\rho}}\left[\Phi(t)t^{-1-\rho-\Delta}\right]_{\alpha}^{\pi}\right) + O\left(\frac{1}{n^{1+\rho}}\int_{\alpha}^{\pi}\Phi(t)t^{-2-\rho-\Delta}\,dt\right)$$

$$= O(1/n^{1+\rho}) + O\left(\frac{1}{k^{\delta(\rho+\Delta)}}\,\frac{1}{n^{1+\rho-\delta(\Delta+\rho)}}\right) = o(1).$$

It remains to estimate I_1. For this purpose, we put

$$\varphi_1(t) = \int_0^t \varphi(u)\,du = o(t^{\varepsilon_1}).$$

Integrating by parts, we have

$$I_1 = \frac{1}{A_n^{(\rho)}}\left[\varphi_1(t)\frac{\cos(A_n t + A)}{(2\sin t/2)^{1+\rho}}\right]_{k/n}^{\alpha} + \frac{A_n}{A_n^{(\rho)}}\int_{k/n}^{\alpha}\varphi_1(t)\frac{\sin(A_n t + A)}{(2\sin t/2)^{1+\rho}}\,dt$$

$$+ \frac{1+\rho}{2A_n^{(\rho)}}\int_{k/n}^{\alpha}\varphi_1(t)\frac{\cos(A_n t + A)}{(2\sin t/2)^{1+\rho}}\cos t/2\,dt$$

$$= I_5 + I_6 + I_7,$$

say, where

$$I_5 = o\left(\frac{1}{n^{\rho}}\left[t^{\varepsilon_1-1-\rho}\right]_{k/n}^{\alpha}\right) = o(1/n^{\varepsilon_1-1}) + o(1/n^{\rho+\delta(\varepsilon_1-1-\rho)}) = o(1),$$

$$I_6 = o\left(\frac{1}{n^{\rho-1}}\int_{k/n}^{\alpha} t^{\varepsilon_1-1-\rho}\,dt\right) = o(1/n^{\rho-1+\varepsilon_1-\rho}) + o(1/n^{\rho-1+\delta(\varepsilon_1-\rho)}) = o(1)$$

and

$$I_7 = o\left(\frac{1}{n^{\rho}}\int_{k/n}^{\alpha} t^{\varepsilon_1-1-\rho}\,dt\right) = o(1/n^{\rho+\varepsilon_1-\rho}) + o(1/n^{\rho+\delta(\varepsilon_1-\rho)}) = o(1).$$

Thus we obtain $I_1 = o(1)$, which completes the proof.

5. Let us prove Theorem 2. The case $\Delta = 1$ is trivial and the case $\rho = 0$ is Theorem A, and then we may suppose that $\Delta > 1$ and $-1 < \rho < 0$.

As in the proof of Theorem 1, it suffices to show that (4.2) is of $o(1)$.

Let $\alpha = (k/n)^{\delta}$ and $\delta = \dfrac{1+\rho}{\Delta+\rho} = \dfrac{-\rho}{\varepsilon_2-\rho-1}$.

$$I = \left\{\int_{k/n}^{\alpha} + \int_{\alpha}^{\pi}\right\}\varphi(t)S_n^{(\rho)}(t) = J_1 + J_2,$$

say, where $J_2 = o(1)$ as we have $I_2 = o(1)$ in the proof of Theorem 1. In order to estimate J_1, we put

$$\varphi_1^*(t) = \int_0^t |\varphi(u)|\,du = o\left(t^{\varepsilon_2}/\log\frac{1}{t}\right).$$

Then, by integration by parts, we have

$$J_1 = O\left(\frac{1}{n^\rho}\int_{k/n}^{\alpha} |\varphi(t)|t^{-1-\rho}dt\right)$$

$$= O\left(\frac{1}{n^\rho}\Big[\varphi_1^*(t)t^{-1-\rho}\Big]_{k/n}^{\alpha}\right) + O\left(\frac{1}{n^\rho}\int_{k/n}^{\alpha}\varphi_1^*(t)t^{-2-\rho}dt\right)$$

$$= o\left(\frac{1}{n^\rho}\Big[t^{\epsilon_2-1-\rho}\Big]_{k/n}^{\alpha}\right) + o\left(\frac{1}{n^\rho}\int_{k/n}^{\alpha}\Big\{t^{\epsilon_2-1-\rho}/t\log\frac{1}{t}\Big\}dt\right)$$

$$= o(1/n^{\epsilon_2-1}) + o(1/n^{\rho+\delta(\epsilon_2-1-\rho)}) + o\left(\frac{1}{n^{\rho+\delta(\epsilon_2-1-\rho)}}\int_{k/n}^{\alpha}\frac{dt}{t\log 1/t}\right)$$

$$= o(1) + o\left(\int_{k/n}^{\alpha}\frac{dt}{t\log 1/t}\right) = o(1) + o\left(\Big[-\log\log 1/t\Big]_{k/n}^{\alpha}\right)$$

$$= o(1) + o(\log 1/\delta) = o(1),$$

which is the required.

6. For the proof of Theorem 3 we require some lemmas.

In what follows we set $y = \pi/A_n$, A_n being defined by (3.3).

LEMMA 4. *If $\varphi(t)$ satisfies the condition* (2.7), *then*

$$\int_{ky}^{(ky)^{1/\Delta}+vy}\varphi(t)S_n^{(\rho)}(t)\,dt = o(1), \tag{6.1}$$

where v is a positive integer.

PROOF. When k is taken sufficiently large, we may replace the upper limit of the left side integral of (6.1) by $(ky)^{1/\Delta}$:

$$\int_{ky}^{(ky)^{1/\Delta}}\varphi(t)S_n^{(\rho)}(t)\,dt = \frac{1}{A_n^{(\rho)}}\int_{ky}^{(ky)^{1/\Delta}}\varphi(t)\frac{\cos(A_nt+A)}{(2\sin t/2)^{1+\rho}}dt$$

$$= \frac{1}{A_n^{(\rho)}}\Big[\varphi_1(t)\frac{\cos(A_nt+A)}{(2\sin t/2)^{1+\rho}}\Big]_{ky}^{(ky)^{1/\Delta}} - \frac{1}{A_n^{(\rho)}}\int_{ky}^{(ky)^{1/\Delta}}\varphi_1(t)\frac{A_n\sin(A_nt+A)}{(2\sin t/2)^{1+\rho}}dt$$

$$+ \frac{2(1+\rho)}{A_n^{(\rho)}}\int_{ky}^{(ky)^{1/\Delta}}\varphi_1(t)\frac{\cos(A_nt+A)}{(2\sin t/2)^{2+\rho}}\cos\frac{t}{2}\,dt$$

$$= P_1 + P_2 + P_3,$$

say, where

$$P_1 = o\left(\frac{1}{n^\rho}\Big[t^{\epsilon_3-1-\rho}\Big]_{ky}^{(ky)^{1/\Delta}}\right) = o(1) + o(1/n^{(\Delta-1)/\Delta}) = o(1),$$

$$P_2 = o\left(\frac{1}{n^{\rho-1}}\int_{ky}^{(ky)^{1/\Delta}}t^{\epsilon_3-1-\rho}\,dt\right) = o(1/n^{\epsilon_3-1}) + o(1/n^{\rho-1+(\epsilon_3-\rho)\Delta}) = o(1)$$

and

$$P_3 = o\left(\frac{1}{n^\rho}\int_{ky}^{(ky)^{1/\Delta}} t^{\epsilon_3-2-\rho}dt\right) = o(1).$$

Thus we get the required.

LEMMA 5. *If* $\varphi_1(t) = O(t)$, *then*

$$\lim_{k\to\infty}\limsup_{n\to\infty}\frac{1}{A_n^{(\rho)}}\int_{(ky)^{1/\Delta}}^{\pi-my}\varphi(t+\nu y)\,\omega(t,y)\cos(A_nt+A)\,dt = 0.$$

where

$$\omega(t,y) = \frac{2m}{\{\sin(t+\nu y)/2\}^{1+\rho}} - \frac{2m-\nu}{\{\sin t/2\}^{1+\rho}} - \frac{\nu}{\{\sin(t+2my)/2\}^{1+\rho}}$$

and m ***and*** ν ***are integers such that*** $1\leqq\nu\leqq m$.

Proof follows from the inequalities

$$\omega(t) = O(y^2/t^{3+\rho}),\quad \frac{\partial\omega}{\partial t} = O(y^2/t^{4+\rho}),$$

(cf. J. J. Gergen [1], Lemma 13).

LEMMA 6. *Let* $-1<\rho<1$, $\Delta\geqq 1$ *and*

$$\eta_{\rho,\Delta}^{(m)}(u,k) = u^\rho\int_{(ku)^{1/\Delta}}^{\pi}\frac{|\Delta_u^{(m)}\varphi(t)|}{t^{1+\rho}}\,dt.$$

If $\eta_{\rho,\Delta}^{(m)}(u,k) = o(1)$[2], *then* $\eta_{\rho,\Delta}^{(m')}(u,k) = o(1)$, *where* m *and* m' *are positive integers* ***such that*** $m'\geqq m>0$.

Proof runs similarly as Lemma 14 in J. J. Gergen's paper [1].

Using above lemmas, we can now prove Theorem 3:
After J. J. Gergen, we have

$$2^{2m-1}\pi(\sigma_n^{(\rho)}-s) = \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int_0^{ky}\varphi(t)K_n^{(\rho)}(t)\,dt$$

$$+\sum_{\nu=0}^{2m}\binom{2m}{\nu}\int_{ky}^{\pi+(\nu-m)y}\varphi(t)K_n^{(\rho)}(t)\,dt + \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int_{\pi+(\nu-m)y}^{\pi}\varphi(t)K_n^{(\rho)}(t)\,dt$$

$$= Q_1+Q_2+Q_3,$$

say, where $Q_1 = o(1)$ by Lemma 2, and $Q_3 = 0$, since $\varphi(t)K_n^{(\rho)}(t)$ is an even periodic function. Accordingly it is sufficient for the proof to show that $Q_2 = o(1)$:

$$Q_2 = \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int_{ky}^{\pi+(\nu-m)y}\varphi(t)S_n^{(\rho)}(t)\,dt + \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int_{ky}^{\pi+(\nu-m)y}\varphi(t)R_n^{(\rho)}(t)\,dt$$

$$= Q_4+Q_5,$$

say.

2) This means that $\lim_{k\to\infty}\limsup_{n\to 0}\eta_{\rho,\Delta}^{(m)}(u,k)=0$

By Lemma 3 we have $Q_5 = o(1)$. Concerning Q_4, we have

$$(6.2)\qquad Q_4 = \sum_{\nu=0}^{2m} \binom{2m}{\nu} \int_{ky}^{(ky)^{1/\Delta}+\nu y} \varphi(t) S_n^{(\rho)}(t)\, dt + \sum_{\nu=0}^{2m} \binom{2m}{\nu} \int_{(ky)^{1/\Delta}+\nu y}^{\pi+(\nu-m)y} \varphi(t) S_n^{(\rho)}(t)\, dt,$$

where the first part of the right side of (6.2) is of $o(1)$ by Lemma 4, and the second part is

$$\frac{1}{2^{1+\rho} A_n^{(\rho)}} \left\{ \int_{(ky)^{1/\Delta}}^{\pi-my} \frac{\Delta_y^{(2m-1)} \varphi(t+y)}{\{\sin(t+2my)/2\}^{1+\rho}} \cos(A_n t + A)\, dt \right.$$

$$- \int_{(ky)^{1/\Delta}}^{\pi-my} \frac{\Delta_y^{(2m-1)} \varphi(t)}{(\sin t/2)^{1+\rho}} \cos(A_n t + A)\, dt$$

$$\left. + \sum_{\nu=1}^{2m-1} \frac{(-1)^\nu}{2m} \binom{2m}{\nu} \int_{(ky)^{1/\Delta}}^{\pi-my} \varphi(t+\nu y) \omega(t,y) \cos(A_n t + A)\, dt \right\}$$

$$= Q_6 + Q_7 + Q_8,$$

say. By the assumption of the theorem and Lemma 6, we have $Q_6 = o(1)$ and $Q_7 = o(1)$, and $Q_8 = o(1)$ by Lemma 5. Thus we get the theorem.

7. The proof of Theorem 4 is similar as that of Theorem 3, except the estimation of

$$(7.1)\qquad \int_{ky}^{(ky)^{1/\Delta}+\nu y} \varphi(t) S_n^{(\rho)}(t) dt:$$

By integration by parts, we get

$$\left| \int_{ky}^{(ky)^{1/\Delta}+\nu y} \varphi(t) S_n^{(\rho)}(t)\, dt \right| \leqq \int_{ky}^{(ky)^{1/\Delta}+\nu y} |\varphi(t)|\, |S_n^{(\rho)}(t)|\, dt$$

$$= O\left(\frac{1}{n^\rho} \int_{ky}^{(ky)^{1/\Delta}+\nu y} |\varphi(t)| t^{-1-\rho}\, dt \right)$$

$$= O\left(\frac{1}{n^\rho} \Big[\varphi_1^*(t) t^{-1-\rho} \Big]_{ky}^{(ky)^{1/\Delta}} \right) + O\left(\frac{1}{n^\rho} \int_{ky}^{(ky)^{1/\Delta}} \varphi_1^*(t) t^{-2-\rho} dt \right)$$

$$= o\left(\frac{1}{n^\rho} \Big[t^{\epsilon_4 - 1 - \rho} \Big]_{ky}^{(ky)^{1/\Delta}} \right) + o\left(\frac{1}{n^\rho} \int_{ky}^{(ky)^{1/}} t^{\epsilon_4 - 1 - \rho} / t \log \frac{1}{t}\, dt \right)$$

$$= T_1 + T_2,$$

say, where

$$T_1 = o(1/n^{\rho + (\epsilon_4 - 1 - \rho)/\Delta}) = o(1)$$

and

$$T_2 = o\left(\frac{1}{n^{\rho + (\epsilon_4 - 1 - \rho)/\Delta}} \int_{ky}^{(ky)^{1/\Delta}} \frac{1}{t \log 1/t}\, dt \right) = o(1).$$

Thus we can prove the theorem as in the proof of Theorem 3.

8. We can easily generalize Theorem 4 in the following form:

THEOREM 5. *Let* $\theta(x)$ *and* $\mu(x,c)$ *satisfy the conditions* (1.5)~(1.7). *If there are numbers* ε *and* $c>0$ *such that*

$$\varphi_1^*(t)=\int_0^t |\varphi(u)|du = o\left(t^\varepsilon/\theta\left(\frac{1}{t}\right)\right), \qquad \text{as } t\to 0,$$

$$\lim_{k\to\infty}\limsup_{u\to 0} u^\rho \int_{\mu\left(\frac{1}{ku},c\right)}^{\pi} \frac{|\Delta_u^{(m)}\varphi(t)|}{t^{1+\rho}}\,dt=0,$$

(8.1) $\quad \mu(x,c)\leqq kx^{\rho/(\varepsilon-\rho-1)}$, *as* $x\to\infty$, *k is an absolute constant, and* $\varepsilon\geqq 1$, *then the Fourier series of* $f(t)$ *is* (C,ρ) *summable to s at a point x, where* $-1<\rho<0$.

The method of the proof of Theorem 5 is similar to those of Theorem E and Theorem 4:

LEMMA 7. *Under the assumption of the theorem, we have*

$$L=\int_{ky}^{\mu\left(\frac{1}{ky},c\right)+\nu y} \varphi(t)S_n^{(\rho)}(t)\,dt=o(1), \qquad \text{as } n\to\infty, \tag{8.2}$$

where ν *is a positive integer and* $y=\pi/A_n$.

PROOF. In the following we denote $\mu(1/ky,c)$ by μ_n.

$$\begin{aligned}|L| &\leqq \frac{1}{A_n^{(\rho)}}\int_{ky}^{\mu_n+\nu y}|\varphi(t)|\frac{dt}{(2\sin t/2)^{1+\rho}}\\ &= O\left(\frac{1}{n^\rho}\int_{ky}^{\mu_n+\nu y}|\varphi(t)|t^{-1-\rho}\,dt\right)\\ &= O\left(\frac{1}{n^\rho}\Big[\varphi_1^*(t)t^{-1-\rho}\Big]_{ky}^{\mu_n+\nu y}\right)+O\left(\frac{1}{n^\rho}\int_{ky}^{\mu_n+\nu y}\varphi_1^*(t)t^{-2-\rho}\,dt\right)\\ &= V_1+V_2,\end{aligned}$$

say. Using the inequality $\mu(x,c)\geqq 1/x$, we have

$$\begin{aligned}V_1 &= o\left(\frac{1}{n^\rho}\Big[t^{\varepsilon-1-\rho}\Big]_{ky}^{\mu_n+\nu y}\right)=o(1/n^{\varepsilon-1})+o(\mu_n^{\varepsilon-1-\rho}/n^\rho)\\ &= o(1)+o\left(\frac{1}{n^\rho}\left(\frac{1}{ky}\right)^\rho\right)=o(1)\end{aligned}$$

and

$$V_2=o\left(\frac{1}{n^\rho}\int_{ky}^{\mu_n+\nu y} t^{\varepsilon-1-\rho}\Big/t\theta\left(\frac{1}{t}\right)dt\right)=o\left(\frac{(\mu_n+\nu y)^{\varepsilon-1-\rho}}{n^\rho}\int_{ky}^{\mu_n+\nu y}\frac{dt}{t\theta(1/t)}\right)$$

$$= o\left(\int_{ky}^{\mu_n+\nu y} \frac{dt}{t\theta(1/t)}\right).$$

However

$$\int_{ky}^{\mu_n+\nu y} \frac{dt}{t\theta(1/t)} = \int_{1/(\mu_n+\nu y)}^{1/ky} \frac{du}{u\,\theta(u)}$$

$$= \int_{1/(\mu_n+\nu y)}^{1/\mu_n} \frac{du}{u\theta\,(u)} + \int_{1/\mu_n}^{1/ky} \frac{du}{u\,\theta(u)} = \int_{1/\xi\mu_n}^{1/\mu_n} \frac{du}{u\,\theta(u)} + \{\Theta(1/ky) - \Theta(1/\mu_n)\}$$

$$= V_3 + V_4,$$

say, where ξ is a constant such that $1/(\mu_n + \nu y) > 1/\xi\mu_n$.
Since $1/u\theta(u)$ is a monotone decreasing function,

$$V_3 \leqq \frac{\xi\mu_n}{\theta(1/\xi\mu_n)} \int_{1/\xi\mu_n}^{1/\mu_n} du = \frac{\xi\mu_n}{\theta(1/\xi\mu_n)} \frac{1}{\mu_n}(1 - 1/\xi) = O(1)$$

and by (1.6) and (1.7) we have

$$V_4 = \Theta(1/ky) - \Theta(1/ky) + c = O(1).$$

Then we get $V_2 = o(1)$. Thus we have the required.

Moreover we can get some needed lemmas, in which will be replaced $(ky)^{1/\Delta}$ by $\mu(1/ky, c)$ in Lemmas 5 and 6.

Combining these lemmas, as in the proof of Theorem 3 and 4, we can get the theorem.

COROLLARY. *One of the following conditions is sufficient for the* (C, ρ) *summability* $(-1 < \rho < 0)$:

1°.
$$\int_0^t |\varphi(u)|du = o(t)$$

and

$$u^\rho \int_{ku}^{\pi} \frac{|\Delta_u^{(m)}\varphi(t)|}{t^{1+\rho}}\,dt = o(1).$$

2°.
$$\int_0^t |\varphi(u)|du = o(t^{1+\delta}/\log\log 1/t)$$

and

$$u^\rho \int_{ku\left(\log\frac{1}{ku}\right)^c}^{\pi} \frac{|\Delta_u^{(m)}\varphi(t)|}{t^{1+\rho}}\,dt = \bar{o}(1),$$

where $\delta > 0$ *and* $c \geqq 0$.

3°.
$$\int_0^t |\varphi(u)|du = o(t^{\varepsilon_4}/\log 1/t)$$

and

$$u^\rho \int_{(ku)^{1/\Delta}}^{\pi} \frac{|\Delta_u^{(m)}\varphi(t)|}{t^{1+\rho}}\, dt = o(1),$$

where $\mathcal{E}_4$ *is given by* (2.9).

9. For the case $1 > \rho > 0$ in Theorem 1 and 3, we have to give some remarks. As we stated in §1, the conditions (2.2) and (2.7) imply $(C, 1/\mathcal{E}_1)$ and $(C, 1/\mathcal{E}_3)$ summabilities respectively. Therefore if $\rho \geqq 1/\mathcal{E}_i$ $(i = 1, 3)$, then our theorems become trivial. Accordingly Theorem 1 and 3 have the meaning when

$$0 < \rho < 1/(\Delta - 2)$$

and

$$0 < \rho < 1/(\Delta - 1),$$

respectively.

REFERENCES

[1] J. J. GERGEN, Convergence and summability criteria for Fourier series, Quart. Journ. Math., 1(1930), 252–275.

[2] G. H. HARDY and J. E. LITTELWOOD, On Young's convergence criterion for Fourier series, Proc. London Math. Soc., 28(1928), 301–311.

[3] G. H. HARDY and J. E. LITTLEWOOD, Some new convergence criteria for Fourier series, Annali di Pisa (2), 3(1934), 43–62.

[4] S. IZUMI and G. SUNOUCHI, Theorems concerning Cesàro summability, Tôhoku Math. Journ. (2), 1(1950), 313–326.

[5] G. SUNOUCHI, A convergence criterion for Fourier series, Tôhoku Math. Journ. (2), 3(1951), 216–219.

[6] G. SUNOUCHI, Counvergence criteria for Fourier series, Tôhoku Math. Jorn. (2), 4 (1952), 187–193.

[7] G. SUNOUCHI, On the convergence test of Fourier series, Math. Japonicae, 1(1948).

[8] A. ZYGMUND, Trigonometrical series.

MATHEMATICAL INSTITUTE, TÔKYO TORITU UNIVERSITY, TOKYO