A GAP SEQUENCE WITH GAPS BIGGER THAN THE HADAMARD'S

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1. Introduction. In the present note let $f(t)$ be a measurable function satisfying the conditions ;

$$(1.1) \qquad f(t+1)=f(t),\ \int_0^1 f(t)dt = 0 \ \text{ and } \int_0^1 f^2(t)dt = 1.$$

In [2] M. Kac noticed that if $f(t)$ is a function of Lip α or of bounded variation, then it is seen that

$$(1.2) \qquad \lim_{N\to\infty}\left|\left\{t\,;\frac{1}{A_N}\sum_{k=1}^{N}a_k f(n_k t)<\omega\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega} e^{-u^2/2}\,du,$$

where $\{n_k\}$ is a sequence of integers such that

$$(1.3) \qquad \lim_{k\to\infty} n_{k+1}/n_k = +\infty$$

and $\{a_k\}$ is any sequence of real numbers satisfying the following conditions

$$(1.4) \qquad A_N^2=\sum_{k=1}^{N}a_k^2 \to +\infty \ \text{ and } \max_{1\leqq k\leqq N}|a_k| = o(A_N), \qquad \text{as } N\to+\infty.$$

Also in [4] G. Morgenthaler proved that if $f(t)$ is bounded and $\{a_k\}$ satisfies (1.4), then there exists a sequence $\{f(n_k t)\}$ independent of $\{a_k\}$ and (1.2) holds.

On the other hand P. Erdös [2] showed that if $f(t) = \cos 2\pi t + \cos 4\pi t$, then we have

$$\lim_{N\to\infty}\left|\left\{t\,;\frac{1}{\sqrt{N}}\sum_{k=1}^{N}f((2^k-1)t)<\omega\right\}\right| = \frac{1}{\sqrt{\pi}}\int_0^1 dx\int_{-\infty}^{\omega/2|\cos\pi x|} e^{-u^2/2}du.$$

From above facts we see that if (1.2) holds, the properties of n_{k+1}/n_k and the smoothness of $f(t)$ become subjects of considerations (cf. [3]). The purpose of this note is to prove the following

THEOREM. *Let $\{n_k\}$ and $\{a_k\}$ satisfy* (1.3) *and* (1.4) *respectively and for some* $\varepsilon > 0$,

$$(1.5) \qquad \left[\int_0^1 \{f(t)-S_n(t)\}^2dt\right]^{1/2} = O\left(\frac{1}{(\log n)^{1+\epsilon}}\right), \qquad \textit{as } n\to+\infty,$$

where $S_n(t)$ denotes the n-th partial sum of the Fourier series of $f(t)$. Then we have, for any measurable set $E, E \subset [0, 1]$, of positive measure,

$$\lim_{N\to\infty}\frac{1}{|E|}\left|\left\{t\,;\,t\in E,\ \frac{1}{A_N}\sum_{k=1}^{N}a_kf(n_kt)<\omega\right\}\right|=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega}e^{-u^2/2}du.$$

From the above theorem it is seen that under the conditions (1.3) and (1.5), $\sum a_k^2 = +\infty$ implies the almost everywhere divergence of the series $\sum a_kf(n_kt)$.

On the other hand in [1] S. Izumi proved that under the conditions (1.5) and the Hadamard's gap condition $n_{k+1}/n_k > q > 1$, $\sum a_k^2 < \infty$ implies the almost everywhere convergence of the sequence $\lim\limits_{m\to\infty}\sum\limits_{k=1}^{m^2}a_kf(n_kt)$.

2. Proof of the theorem. By (1.3) and (1.4) we can take a sequence of positive integers $\{q_k\}$ such that

$$n_{k+1}/n_k \geqq 4\,q_k \qquad \text{for } k=1,2,3,\ldots\ldots,^{*)} \tag{2.1}$$

and

$$\max_{1\leqq k\leqq N}|q_k^{1/2}a_k| = o(A_N) \text{ and } q_N\to+\infty, \qquad \text{as } N\to+\infty. \tag{2.1'}$$

We put

$$f(t)\sim\sum_{l=1}^{\infty}c_l\cos 2\pi lt \tag{2.2}$$

and, for $k=1,2,\ldots\ldots$,

$$g_k(t)\sim\sum_{l>q_k}c_l\cos 2\pi lt \quad\text{and}\quad R_k=\frac{1}{2}\sum_{l>k}c_l^2. \tag{2.2'}$$

LEMMA. 1. *We have*

$$\lim_{N\to\infty}\int_0^1\left\{\frac{1}{A_N}\sum_{k=1}^{N}a_kg_k(n_kt)\right\}^2dt=0.$$

PROOF. We have, by Parseval's relation for $k > j$,

$$\left|\int_0^1 g_k(n_kt)g_j(n_jt)dt\right|=\left|\frac{1}{2}\sum_{l>q_k}c_ld_l\right|\leqq\left(\frac{1}{2}\sum_{l>q_k}c_l^2\right)^{1/2}\left(\frac{1}{2}\sum_{l>q_k}d_l^2\right)^{1/2},$$

where

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*) The condition (2.1) need not hold for small k, but without loss of generality we may assume that (2.1) holds for all k.

$$d_l = \begin{cases} c_{\frac{n_k l}{n_j}}, & \text{if } n_j \mid n_k l, \\ 0, & \text{if otherwise.} \end{cases}$$

By (2. 1), (2. 2′), (1. 5) and the definition of d_l, we have

$$\left(\frac{1}{2}\sum_{l>q_k} d_l^2\right)^{1/2} \leqq \left(\frac{1}{2}\sum_{l\geqq 4^{k-j}} c_l^2\right)^{1/2} = O\left(\frac{1}{(k-j)^{1+\epsilon}}\right), \qquad \text{as } (k-j)\to+\infty.$$

Hence we have, by above relations

$$\int_0^1 \left\{\frac{1}{A_N}\sum_{k=1}^N a_k g_k(n_k t)\right\}^2 dt = \frac{1}{A_N^2}\left[\sum_{k=1}^N a_k^2 \int_0^1 g_k^2(t)dt + 2\sum_{1\leqq j<k\leqq N} a_j a_k \int_0^1 g_k(n_k t) g_j(n_j t) dt\right]$$

$$= \frac{1}{A_N^2}\sum_{k=1}^N a_k^2 R_{q_k} + O\left(\frac{1}{A_N^2}\sum_{1\leqq j<k\leqq N} a_k a_j R_{q_k}^{1/2}\frac{1}{(k-j)^{1+\epsilon}}\right)$$

$$= \frac{1}{A_N^2}\sum_{k=1}^N a_k^2 R_{q_k} + O\left(\frac{1}{A_N^2}\sum_{r=1}^{N-1}\frac{1}{r^{1+\epsilon}}\sum_{i=1}^{N-r} R_{q_i}^{1/2} a_i a_{i+r}\right)$$

$$= \frac{1}{A_N^2}\sum_{k=1}^N a_k^2 R_{q_k} + O\left[\left(\sum_{i=1}^N \frac{R_{q_i} a_i^2}{A_N^2}\right)^{1/2}\right].$$

Since $R_{q_i}\to 0$ as $i\to+\infty$, by (1. 4) we can prove the lemma.

LEMMA 2. *We have*

$$\max_{1\leqq k\leqq N} |a_k S_{q_k}(n_k t)| = o(A_N) \tag{2. 3}$$

and

$$\int_0^1 \left|\frac{1}{A_N^2}\sum_{k=1}^N a_k^2 S_{q_k}^2(n_k t) - 1\right|^2 dt = o(1), \text{ as } N\to+\infty. \tag{2. 3′}$$

PROOF. By (2. 1′), it follows that

$$\max_{1\leqq k\leqq N} |a_k S_{q_k}(n_k t)| \leqq \max_{1\leqq k\leqq N} |a_k| \sum_{l=1}^{q_k} |c_l| \leqq \max_{1\leqq k\leqq N} 2|a_k| q_k^{1/2} = o(A_N),$$

as $N\to+\infty$.

Further we have, by (1. 4) and (2. 2′),

$$\left|\frac{1}{A_N^2}\sum_{k=1}^N a_k^2 S_{q_k}^2(n_k t) - 1\right|$$

$$\leqq \frac{1}{A_N^2}\sum_{k=1}^N a_k^2 R_{q_k} + \frac{1}{A_N^2}\left|\sum_{k=1}^N a_k^2 \sum_{l=1}^{2q_k} \cos 2\pi n_k l t \sum_{\substack{i-j=l\\ i+j=l}} c_i c_j\right|.$$

By (2. 1) if $k \neq k'$, then for any l, $l'(1 \leqq l \leqq 2q_k$ and $1 \leqq l' \leqq 2q_{k'})$,

$$\int_0^1 \cos 2\pi n_k l t \cos 2\pi n_{k'} l' t dt = 0,$$

and

$$\frac{1}{A_N^2} \sum_{k=1}^N a_k^2 R_{q_k} \to 0, \qquad \text{as } N \to +\infty.$$

Hence for the proof of (2. 3′), it is sufficient to show that

$$I_N = \frac{1}{A_N^4} \sum_{k=1}^N a_k^4 \sum_{l=1}^{2q_k} \Big(\sum_{\substack{i-j=l \\ i+j=l}} c_i c_j \Big)^2 = o(1), \qquad \text{as } N \to +\infty.$$

On the other hand, by (2. 1′) and (1. 4), we have

$$I_N = O\Big(\max_{1 \leqq k \leqq N} \frac{a_k^2 q_k}{A_N^2} \Big) = o(1), \qquad \text{as } N \to +\infty.$$

By Lemma 2, we know that if we put

$$E_N = \Big\{ t; \Big| \frac{1}{A_N^2} \sum_{k=1}^N a_k^2 S_{q_k}^2(n_k t) - 1 \Big| < 1 \Big\}, \tag{2. 4}$$

then we have

$$\lim_{N \to \infty} |E_N| = 1. \tag{2. 4′}$$

For the proof of the theorem it is sufficient, by Lemma 1 and the theorem of Glivenko, for any fixed λ and any interval I, to show that

$$\Phi_N(\lambda, I) = \frac{1}{|I|} \int_I \exp\Big\{ \frac{i\lambda}{A_N} \sum_{k=1}^N a_k S_{q_k}(n_k t) \Big\} dt \to e^{-\frac{\lambda^2}{2}}, \qquad \text{as } N \to +\infty.$$

By (2. 3), (2. 4), (2. 4′) and the fact that $\exp z = (1+z) \exp\Big(\frac{z^2}{2} + O(|z|^3) \Big)$, as $|z| \to 0$, we have, as $N \to +\infty$,

$$\Phi_N(\lambda, I) = \frac{e^{o(1)}}{|I|} \int_{I \cap E_N} \prod_1^N \Big(1 + \frac{i\lambda a_k S_{q_k}(n_k t)}{A_N} \Big) \exp\Big(-\frac{\lambda^2}{2A_N^2} \sum_{k=1}^N a_k^2 S_{q_k}^2(n_k t) \Big) dt.$$

By Lemma 2, (2. 4), and the fact that $|e^x - 1| \leqq |x| e^{|x|}$, we have

$$\Big| \int_{I \cap E_N} \prod_1^N \Big(1 + \frac{i\lambda a_k S_{q_k}(n_k t)}{A_N} \Big) \Big\{ \exp\Big(-\frac{\lambda^2}{2A_N^2} \sum_{k=1}^N a_k^2 S_{q_k}^2(n_k t) \Big) - \exp\Big(-\frac{\lambda^2}{2} \Big) \Big\} dt \Big|$$

$$\leqq \int_{I \cap E_N} \{\exp(\lambda^2)\} \Big| -\frac{\lambda^2}{2A_N^2} \sum_{k=1}^N a_k^2 S_{q_k}^2(n_k t) + \frac{\lambda^2}{2} \Big| dt \to 0, \qquad \text{as } N \to +\infty.$$

Hence for the proof of the theorem it is sufficient to show that

$$(2.5)\qquad \frac{1}{|I|}\int_{I\cap E_N}\prod_1^N\left(1+\frac{i\lambda a_k S_{q_k}(n_k t)}{A_N}\right)dt\to 1,\qquad \text{as } N\to+\infty.$$

LEMMA 3. *We have, for all N,*

$$\int_0^1\left|\prod_1^N\left(1+\frac{i\lambda a_k S_{q_k}(n_k t)}{A_N}\right)\right|^2 dt\leqq M.$$

PROOF. We have

$$\int_0^1\left|\prod_1^N\left(1+\frac{i\lambda a_k S_{q_k}(n_k t)}{A_N}\right)\right|^2=\prod_1^N\left(1+\frac{\lambda^2 a_k^2\sum_{l=1}^{q_k}c_l^2}{2A_N^2}+\frac{\lambda^2 a_k^2 T_k(n_k t)}{2A_N^2}\right)$$

$$=\prod_1^N\left(1+\frac{\lambda^2 a_k^2\sum_{l=1}^{q_k}c_l^2}{2A_N^2}\right)+\Psi_N(t,\lambda)\leqq e^{\lambda^2}+\Psi_N(t,\lambda).$$

$\Psi_N(t,\lambda)$ is the sum of terms of the following form

$$(2.6)\qquad (\text{constant})\times\prod_{i=1}^s\cos 2\pi n_{k_i}l_i t,$$

where

$$(2.6')\qquad 1\leqq k_1<k_2<\cdots\cdots<k_s\leqq N \text{ and } 1\leqq l_i\leqq 2q_{k_i}.$$

(2.6) can be expressed as the sum of the following terms

$$(2.7)\qquad (\text{constant})\times\cos 2\pi(n_{k_s}l_s\pm\cdots\cdots\pm n_{k_1}l_1).$$

On the other hand by (2.1) and (2.6′), we have

$$n_{k_s}l_s-n_{k_{s-1}}l_{s-1}-\cdots\cdots-n_{k_1}l_1\geqq n_{k_s}\left(1-\frac{2}{4}-\frac{2}{4^2}-\cdots\cdots\right)$$

$$\geqq n_{k_s}\left(1-\frac{2/4}{1-1/4}\right)\geqq n_{k_s}/3>0.$$

Hence we have

$$\int_0^1\Psi_N(t,\lambda)dt=0.$$

This completes the proof.

By Lemma 3, and (2.4′) we have

(2. 8)

$$\left|\left(\int_I - \int_{I\cap E_N}\right)\prod_1^N\left(1+\frac{i\lambda a_k S_{q_k}(n_k t)}{A_N}\right)dt\right| \leqq |E_N^c|^{1/2}\left[\int_0^1\left|\prod_1^N\left(1+\frac{i\lambda a_k S_{q_k}(n_k t)}{A_N}\right)\right|^2 dt\right]^{1/2} \to 0$$

as $N \to +\infty$.

LEMMA 4. *We have*

$$\lim_{N\to\infty}\int_I \prod_1^N\left(1+\frac{i\lambda a_k S_{q_k}(n_k t)}{A_N}\right)dt = |I|.$$

PROOF. If we put $\prod_1^N\left(1+\frac{i\lambda a_k S_{q_k}(n_k t)}{A_N}\right) = 1+\theta_N(t,\lambda)$, then $\theta_N(t,\lambda)$ consists of the terms

(2. 9) $$\prod_{j=1}^s\left\{\frac{i\lambda a_{k_j} c_j}{A_N}\cos 2\pi n_{k_j} l_j t\right\} = \prod_{j=1}^s\left(\frac{i\lambda a_{k_j} c_j}{2A_N}\right)\sum \cos 2\pi(n_{k_s} l_s \pm \ldots\ldots \pm n_{k_1} l_1)$$

where $\sum$ denotes the summation over all possible combinations of $\pm$ and

$$1 \leqq k_1 < k_2 < \ldots\ldots < k_s \leqq N \text{ and } 1 \leqq l_j \leqq q_{k_j}.$$

In the same way as that of Lemma 3, we have

(2. 10) $$n_{k_s} l_s \pm n_{k_{s-1}} l_{s-1} \pm \ldots\ldots \pm n_{k_1} l_1 \geqq \frac{2}{3} n_{k_s}.$$

Using (2. 9), (2. 10) and the fact that for $\alpha > 0$ and any interval I,

$$\left|\int_I \cos \alpha t dt\right| \leqq \frac{2}{\alpha},$$

$$\left|\int_I \prod_{j=1}^s\left\{\left(\frac{\lambda a_{k_j} c_j}{A_N}\right)\cos 2\pi n_{k_j} l_j t\right\}dt\right| \leqq \prod_{j=1}^s \frac{|\lambda a_{k_j} c_j|}{A_N} \Big/ n_{k_s}.$$

If we put

$$\int_I \prod_1^N\left(1+\frac{i\lambda a_k S_{q_k}(n_k t)}{A_N}\right)dt = |I| + \Omega_N(\lambda, I),$$

then we have, by (2. 1) and (2. 1′) for $N > N_0$,

$$|\Omega_N(\lambda, I)| \leqq \sum_{k=2}^N\left(\frac{|\lambda a_k|}{n_k A_N}\sum_{l=1}^{q_k}|c_l|\right)\prod_{s=1}^{k-1}\left(1+\sum_{l=1}^{q_s}\frac{|\lambda a_s c_l|}{A_N}\right) + \sum_{l=1}^{q_1}\frac{|\lambda a_1 c_l|}{A_N n_1},$$

$$\leqq \max_{1\leqq k\leqq N}\frac{|2\lambda a_k q_k^{1/2}|}{A_N}\sum_{k=2}^N\frac{1}{n_k}\prod_{s=1}^{k-1}\left(1+\frac{2|\lambda a_s q_s^{1/2}|}{A_N}\right) + \frac{2|\lambda a_1 q_1^{1/2}|}{A_N n_1},$$

$$\leqq \max_{1\leqq k\leqq N} \frac{|\lambda a_k q_k^{1/2}|}{A_N} \sum_{k=1}^{N} \frac{2^k}{4^k} = o(1), \text{ as } N \to +\infty.$$

This completes the proof.

By (2. 5), (2. 8) and Lemma 4, we can prove the theorem.

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