NOTE ON THE n-DIMENSIONAL TEMPERED ULTRA-DISTRIBUTIONS

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(Received July 1, 1960)

In this note, we shall describe explicitly the duality in the space of tempered ultra-distributions of J. Sebastião e Silva in the Euclidean n-space. And, as an application, we shall prove a theorem on the multiplication of tempered ultra-distributions.

I wish to express my hearty thanks to Prof. K. Yosida and Prof. J. Sebastião e Silva for reading the manuscript and giving me valuable remarks.

Notations: Let R^n (resp. C^n) be the real (resp. complex) n-space whose generic points are denoted by $x = (x_1, \ldots\ldots, x_n)$ (resp. $z = (z_1, \ldots\ldots, z_n)$). We shall use the notations: (i) $x + y = (x_1 + y_1, \ldots\ldots, x_n + y_n)$, $\alpha x = (\alpha x_1, \ldots\ldots, \alpha x_n)$; (ii) $x \geqq 0$ means $x_1 \geqq 0, \ldots\ldots, x_n \geqq 0$; (iii) $x \cdot y = \sum_{j=1}^n x_j y_j$ and (iv) $|x| = \sum_{j=1}^n |x_j|$.

Let p be a system of integers $\geqq 0$, $(p_1, \ldots\ldots, p_n)$. We shall denote by $|p|$ the sum $\sum_{j=1}^n p_j$ and by D^p the partial differential operator $\partial^{p_1+\cdots+p_n}/\partial x_1^{p_1} \ldots \ldots \partial x_n^{p_n}$. We put, for any integer $k \geqq 0$, $\partial^k/\partial x^k = \partial^{nk}/\partial x_1^k \ldots\ldots \partial x_n^k$. $p + q$ is the system of integers $(p_1 + q_1, \ldots\ldots, p_n + q_n)$. $p \geqq q$ means $p_1 \geqq q_1, \ldots\ldots, p_n \geqq q_n$. Moreover, $x^p = x_1^{p_1} \ldots\ldots x_n^{p_n}$ and $x^k = x_1^k \ldots\ldots x_n^k$ (k an integer). For $p \geqq q$, put $\binom{p}{q} = \binom{p_1}{q_1} \ldots\ldots \binom{p_n}{q_n}$ with $\binom{p_j}{q_j} = p_j!/q_j!(p_j - q_j)!$.

We shall denote once for all by σ vectors $(\sigma_1, \ldots\ldots, \sigma_n)$ whose components are 0 or 1 and adopt the following conventions: (v) $(-1)^{|\sigma|} = (-1)^{\sigma_1 + \cdots + \sigma_n}$; (vi) $x^\sigma = ((-1)^{\sigma_1} x_1, \ldots\ldots, (-1)^{\sigma_n} x_n)$ for any vector x; (vii) $k^\sigma = ((-1)^{\sigma_1} k, \ldots \ldots, (-1)^{\sigma_n} k)$ for any integer k; (viii) $R^n_\sigma = \{x \in R^n : x^\sigma \geqq 0\}$; (ix) $C^n_{\sigma,\alpha} = \{z \in C^n : (-1)^{\sigma_1} \mathscr{I} z_1 > \alpha, \ldots\ldots, (-1)^{\sigma_n} \mathscr{I} z_n > \alpha\}$ with $\alpha > 0$ and (x) $\Delta_{\sigma,\alpha}$ is the path of integration $(-\infty + i(-1)^{\sigma_1}\alpha, \infty + i(-1)^{\sigma_1}\alpha) \times \ldots\ldots \times (-\infty + i(-1)^{\sigma_n}\alpha, \infty + i(-1)^{\sigma_n}\alpha)$, oriented from $-\infty$ to $+\infty$. Finally V_α denotes the horizontal band in C^n defined by $V_\alpha = \{z \in C^n : |\mathscr{I} z_1| \leqq \alpha, \ldots\ldots, |\mathscr{I} z_n| \leqq \alpha\}$ with $\alpha > 0$.

1. The basic spaces H and Λ_∞. Let H be the space of all C^∞-functions $\varphi(x)$ in R^n such that $\exp(k|x|) D^p \varphi(x)$ is bounded in R^n for any k and p. We define in H semi-norms

(1) $\qquad \|\varphi\|_k = \sup_{0\leqq|p|\leqq k, x} \exp(k|x|)|D^p\varphi(x)|, \ k = 0, 1, 2, \ldots\ldots .$

Then H is a Hausdorff locally convex metrizable space.

Let Γ be a set of continuous functions on R^n such that, for any compact subset $K \subset R^n$, there exists a member $\gamma \in \Gamma$ which never vanishes on K. We say that a function $\varphi \in (\mathscr{E})$ *satisfies the condition of growth defined by* Γ if, for any p and for any $\gamma \in \Gamma$, the function $\gamma(x)D^p\varphi(x)$ is bounded in R^n. Thus the space H consists of all functions in $(\mathscr{E})$ satisfying the condition of growth defined by the class $\Gamma_0 = \{\exp(k|x|): k = 0, 1, 2, \ldots\ldots\}$ or equivalently by the class $\Gamma_0' = \{\exp(k^\sigma \cdot x)\}$ where $k = 0, 1, 2, \ldots\ldots$ and σ varies over all vectors whose components are 0 or 1.

PROPOSITION 1. *The space H is a Fréchet nuclear space and therefore completely reflexive.*

PROOF. The mappings $\Phi_{k\sigma}$ ($k = 0, 1, 2, \ldots\ldots$ and any σ), defined by $\Phi_{k\sigma}(\varphi) = \exp(k^\sigma \cdot x)\varphi(x)$, are continuous linear mappings of H into $(\mathscr{S})$. In fact, if $\varphi_\nu \to 0$ in H, then $\exp((k+1)|x|)D^p\varphi_\nu(x) \to 0$ uniformly in R^n. Thus, for any polynomial $P(x)$, $P(x)\exp(k^\sigma \cdot x)D^p\varphi_\nu(x) \to 0$ uniformly in R^n and therefore $P(x) \to \cdot D^p[\exp(k^\sigma \cdot x)\varphi_\nu(x)] \to 0$ uniformly in R^n. As p is arbitrary, $\Phi_{k\sigma}(\varphi_\nu) \to 0$ in $(\mathscr{S})$.

Now, suppose that $\Phi_{k\sigma}(\varphi_\nu) \to 0$ in $(\mathscr{S})$ for any $k = 0, 1, 2, \ldots\ldots$ and any σ. Then it is easy to show that $\exp(k^\sigma \cdot x)D^p\varphi_\nu(x) \to 0$ uniformly in R^n. Thus $\varphi_\nu \to 0$ in H.

Moreover, if $\varphi \in (\mathscr{E})$ and $\Phi_{k\sigma}(\varphi) \in (\mathscr{S})$ for any $k = 0, 1, 2, \ldots\ldots$ and any σ, then $\varphi \in H$. Hence H is the projective limit of $(\mathscr{S})$ with respect to the mappings $\Phi_{k\sigma}$. Since $(\mathscr{S})$ is nuclear, H is also nuclear by [1; Chap. II, Cor. 1 to Th. 9] or [4; Exposé 18, Cor. 1 to Prop. 7].

Let φ_ν be any Cauchy sequence in H. Then the sequence $\varphi_\nu = \Phi_0(\varphi_\nu)$ converges to some $\varphi(x) \in (\mathscr{S})$ under the topology of $(\mathscr{S})$. It is then easy to see that $\varphi \in H$ and $\varphi_\nu \to \varphi$ in H. Hence H is complete.

In any Fréchet nuclear space, a bounded set B is contained in the closed circular convex envelope of a convergent sequence and therefore relatively compact. Thus the space H is completely reflexive. q. e. d.

PROPOSITION 2. *The space H is of type $\mathcal{H}^\infty$ in the sense of Schwartz* [4], *i. e., it satisfies the following conditions*:

(H_1) *H is the space of $\varphi \in (\mathscr{E})$ satisfying the condition of growth defined by Γ_0.*

(H_2) *H is a Hausdorff complete locally convex space and the injections $(\mathscr{D}) \to H \to (\mathscr{E})$ are continuous.*

(H_3) *A subset $B \subset H$ is bounded if and only if, for any $\gamma \in \Gamma_0$ and p, the set of numbers $\gamma(x)D^p\varphi(x)$, $\varphi \in B$, $x \in R^n$, is bounded.*

(H_4) On any bounded set $B \subset H$, the topology induced by H coincides with the one induced by $(\mathscr{E})$.

PROOF. (H_1) and (H_3) are obvious from the definition of H. The injections $(\mathscr{D}) \to H \to (\mathscr{E})$ are clearly continuous. Thus, in view of Prop. 1, (H_2) is satisfied. If B is bounded and closed in H, then it is compact and therefore compact with respect to the topology induced by $(\mathscr{E})$. Hence the two topologies coincide and (H_4) is fulfilled. q. e. d.

PROPOSITION 3. *The dual of H is the space Λ_∞ of all distributions T of exponential type such that*

$$(2) \qquad T = (\partial^k/\partial x^k)[\exp(k|x|)f(x)],$$

where k is an integer $\geqq 0$ and $f(x)$ is a bounded continuous function. Λ_∞ is a nuclear space of type $\mathcal{H}_c'^\infty$ in the sense of Schwartz [4] *under the strong topology.*

PROOF. It is clear that a distribution T of the form (2) defines a continuous linear functional on H. Conversely, let T be any distribution defining a continuous linear functional on H. Then the set of distributions $\{\exp(-k'|u|) \cdot\tau_u(T_x): u \in R^n\}$ is bounded in $(\mathscr{D}')$ for some $k' > 0$. To see this, we notice firstly that the set of semi-norms (1) is equivalent to the system of semi-norms

$$(1') \qquad \|\varphi\|_k' = \sup_{0\leqq|p|\leqq k, x, \sigma}|D^p[\exp(k^\sigma\cdot x)\varphi(x)]|, \ k = 0, 1, 2, \ldots\ldots .$$

Since T is continuous in H, there exists an integer $k' > 0$ and an $\varepsilon > 0$ such that $\|\varphi\|_l' \leqq \varepsilon$ for $l = 0, 1, \ldots\ldots, k'$, imply $|T(\varphi)| \leqq 1$. For any $\varphi \in (\mathscr{D})$, we have $[\exp(-k'|u|)\tau_u(T_x)]\varphi(x) = \exp(-k'|u|)T_x(\varphi(x+u))$. On the other hand,

$$\begin{aligned}
\|\varphi(x+u)\|_l' &= \sup_{0\leqq|p|\leqq l, x, \sigma}|D^p[\exp(l^\sigma\cdot x)\varphi(x+u)]| \\
&= \sup \exp(-l^\sigma\cdot u)|D^p[\exp(l^\sigma\cdot(x+u))\varphi(x+u)]| \\
&\leqq \exp(l|u|) \sup|D^p[\exp(l^\sigma\cdot x)\varphi(x)]| \\
&= \exp(l|u|)\|\varphi\|_l' \leqq \exp(k'|u|)\|\varphi\|_l' .
\end{aligned}$$

Thus we have

$$\begin{aligned}
|[\exp(-k'|u|)\tau_u(T_x)]\varphi(x)| &= \exp(-k'|u|)|T_x\varphi(x+u)| \\
&\leqq \exp(-k'|u|) \max_{0\leqq l\leqq k'}\{\varepsilon^{-1} \exp(k'|u|)\|\varphi\|_l'\} \\
&\leqq \varepsilon^{-1} \max_{0\leqq l\leqq k'}\{\|\varphi\|_l'\}.
\end{aligned}$$

As φ is arbitrary, this shows that $\{\exp(-k'|u|)\tau_u(T_x): u \in R^n\}$ is bounded in $(\mathscr{D}')$. By a theorem of Schwartz [2 ; Chap. VI, Th. XXII], there exist an integer $m \geqq 0$ and a sufficiently small compact neighborhood K of the origin of

R^n such that, for any $\varphi \in (\mathscr{D}_k^m)$, $\{\exp(-k'|u|)\tau_u(T*\varphi) : u \in R^n\}$ forms a family of bounded continuous functions on some relatively compact open set Ω. Therefore, $\exp(-k'|x|)(T*\varphi)(x)$ is continuous and bounded in R^n. We know that the elementary solution E for the iterated Laplace equation $\Delta^N E = \delta$ is m-times continuously differentiable for large N. Then, for any $\gamma(x) \in (\mathscr{D}_K)$, γE belongs to $(\mathscr{D}_K^m)$ and $\delta = \Delta^N(\gamma E) - \zeta$ where $\zeta \in (\mathscr{D})$. Hence $T = \Delta^N(\gamma E * T) - \zeta * T$ and therefore T is a (finite) sum of distributions of the form $D^p[\exp(k''|x|)f(x)]$, $f(x)$ being bounded and continuous. Now it is easy to show that T can be reduced to the form (2).

Since H is a Fréchet nuclear space, bounded sets and relatively compact sets are the same in H and thus the strong topology τ_b and the topology τ_c of compact-convergence coincide in $H' = \Lambda_\infty$. Thus Λ_∞ is of type $\mathcal{L}_c'^\infty$. As the dual of a Fréchet nuclear space H, the space Λ_∞ is also nuclear. q. e. d.

Let B be a bounded set $\subset \Lambda_\infty$. Then B is equicontinuous and therefore there exists an integer $k' \geqq 0$ such that $\|\varphi\|_l' \leqq \varepsilon$ $(l=0,1,\ldots\ldots, k')$ imply $|T(\varphi)| \leqq 1$ for any $T \in B$. Then, by the argument used above, we have the following

COROLLARY. *A set $B \subset \Lambda_\infty$ is bounded if and only if there exist an integer $k \geqq 0$ and a number $M > 0$ such that B is contained in the set of distributions $T = (\partial^k/\partial x^k)[\exp(k|x|)g(x)]$ with $g(x)$ bounded and continuous satisfying* $\sup_{x\in R^n}|g(x)| \leqq M$.

2. Fourier transform of H and Λ_∞. The spaces $\mathfrak{H}$ and $\mathfrak{U}$. We shall construct the Fourier transform of H. Let $\varphi \in H$ and put

$$f(z) \equiv (\mathfrak{F}\varphi)(z) = (2\pi)^{-n/2}\int \cdots\cdots \int_{R^n} \exp(-iz\cdot x)\varphi(x)dx,$$

for any $z \in C^n$. Since φ satisfies the condition of growth defined by Γ_0, the integral converges uniformly in any horizontal bands V_k, $k = 1, 2, \ldots\ldots$, and therefore $f(z)$ is an entire function. We have

$$z^m f(z) = (2\pi)^{-n/2}\int \cdots\cdots \int_{R^n} i^{mn}(\partial^m/\partial x^m)[\exp(-iz\cdot x)]\varphi(x)dx$$

$$= (2\pi)^{-n/2}(-i)^{mn}\int \cdots\cdots \int_{R^n} \exp(-iz\cdot x)(\partial^m/\partial x^m)\varphi(x)dx$$

which converges uniformly in any bands V_k $(k = 1, 2, \ldots\ldots)$ and

$$(3) \qquad |z^m f(z)| \leqq (2\pi)^{-n/2}\int \cdots\cdots \int_{R^n} \exp(+\mathscr{I}z\cdot x)|(\partial^m/\partial x^m)\varphi(x)|dx$$

$$\leqq (2\pi)^{-n/2}\int \cdots\cdots \int_{R^n} \exp(k|x|)|(\partial^m/\partial x^m)\varphi(x)|dx$$

$$\leqq (2\pi)^{-n/2}\int\cdots\cdots\int_{R^n}\exp(k|x|)\exp(-m'|x|)\|\varphi\|_{m'n}dx$$
$$\leqq 2^n(2\pi)^{-n/2}(m'-k)^{-n}\|\varphi\|_{m'n}$$

for $z \in V_k$, where $m' > \max(m, k)$. As m is arbitrary, $f(z)$ is rapidly decreasing in any horizontal bands.

Conversely, suppose that $f(z)$ is an entire function decreasing rapidly in any horizontal bands V_k and put

$$\varphi(x) \equiv (\overline{\mathfrak{F}}f)(x) = (2\pi)^{-n/2}\int\cdots\cdots\int_{R^n}\exp(ix\cdot u)f(u)du.$$

Then, for any p, we have

$$\begin{aligned}
D^p\varphi(x) &= (2\pi)^{-n/2}D^p\int\cdots\int_{R^n}\exp(ix\cdot u)f(u)du \\
&= (2\pi)^{-n/2}\int\cdots\int_{R^n}\exp(ix\cdot u)(iu)^pf(u)du \\
&= (2\pi)^{-n/2}\int_{-\infty+iv_1}^{\infty+iv_1}\cdots\int_{-\infty+iv_n}^{\infty+iv_n}\exp(ix\cdot\zeta)(i\zeta)^pf(\zeta)d\zeta \\
&= (2\pi)^{-n/2}D_x^p\int\cdots\int_{R^n}\exp(ix\cdot(u+iv))f(u+iv)du \qquad (4)\\
&= (2\pi)^{-n/2}D_x^p\left[\exp(-v\cdot x)\int\cdots\int_{R^n}\exp(ix\cdot u)f(u+iv)du\right] \\
&= (2\pi)^{-n/2}\sum_{0\leqq q\leqq p}\binom{p}{q}D_x^{p-q}\exp(-v\cdot x)D_x^q\int\cdots\int_{R^n}\exp(ix\cdot u)f(u+iv)du \\
&= \exp(-v\cdot x)\left[(2\pi)^{-n/2}\sum_{0\leqq q\leqq p}\binom{p}{q}(-v)^{p-q}\int\cdots\int_{R^n}\exp(ix\cdot u)(iu)^qf(u+iv)du\right].
\end{aligned}$$

Since the function in the bracket is continuous and bounded in R^n, we know, by setting $v = (k_1, \ldots\ldots, k_n)$, $k_j = \pm 1, \pm 2, \ldots\ldots$, that φ satisfies the condition of growth defined by Γ_0, i. e., $\varphi \in H$. Thus we have the first part of

PROPOSITION 4. *The Fourier transform of the space H is the space $\mathfrak{H}$ of entire functions rapidly decreasing in any horizontal bands. The algebraic isomorphism becomes topogical if we define a topology in $\mathfrak{H}$ by semi-norms*

$$p_k(f) = \sup_{z\in V_k}|z^kf(z)|, \quad k = 0, 1, 2, \ldots\ldots. \qquad (5)$$

PROOF OF THE SECOND PART. Suppose that $\varphi_\nu \to 0$ in H. Then $\|\varphi_\nu\|_m \to 0$ for any $m \geqq 0$. Setting $m = k$ and $f(z) = f_\nu(z) = (\mathfrak{F}\varphi_\nu)(z)$ in (3), we have

$$p_k(f_\nu) = \sup_{z\in V_k}|z^k f_\nu(z)| \leqq 2^n(2\pi)^{-n/2}(m' - k)^{-n}\|\varphi_\nu\|_{m'n} \to 0$$

where $m' > k$. Thus, $p_k(f_\nu) \to 0$ for any $k = 0, 1, 2, \ldots\ldots$ and therefore $f_\nu \to 0$ in $\mathfrak{H}$. Now, for any $k = 1, 2, \ldots\ldots$ and any index p with $p_i \leqq k$, we have

$$\left|\int\cdots\int_{R^n} \exp(ix\cdot u)\,(iu)^p f(u + ik^\sigma)du\right| \leqq \int\cdots\int_{R^n} |u^p|\,|f(u + ik^\sigma)|\,du$$

$$\leqq \int\cdots\int_{R^n} |(u + ik^\sigma)^p|\,|f(u + ik^\sigma)|\,du$$

$$= \int\cdots\int_{R^n} |(u + ik^\sigma)^{-2}|\,|(u + ik^\sigma)^{p+2}|\,|f(u + ik^\sigma)|\,du$$

$$\leqq \int\cdots\int_{R^n} \prod_{j=1}^n (u_j^2 + k^2)^{-1}du\cdot p_{k+2}(f) = \pi^n k^{-n} p_{k+2}(f)$$

where $p + 2 = (p_1 + 2, p_2 + 2, \ldots\ldots, p_n + 2)$.
Thus it follows from (4) that

$$|\exp(k^\sigma\cdot x)D^p\varphi(x)|$$

$$\leqq (2\pi)^{-n/2}\left|\sum_{0\leqq q\leqq p}\binom{p}{q}(-k^\sigma)^{p-q}\int\cdots\int_{R^n}\exp(ix\cdot u)\,(iu)^q f(u + ik^\sigma)du\right|$$

$$\leqq (2\pi)^{-n/2}\pi^n k^{-n}\left[\sum_{0\leqq q\leqq p}\binom{p}{q}k^{|p-q|}\right]p_{k+2}(f).$$

Therefore, for $k = 1, 2, \ldots\ldots,$

$$\|\varphi\|_k = \sup_{0\leqq|p|\leqq k, x}|\exp(k|x|)D^p\varphi(x)| \leqq c_k p_{k+2}(f),$$

where c_k is a constant depending only on k and the dimension n. This proves that, if $f_\nu \to 0$ in $\mathfrak{H}$, then $\varphi_\nu \to 0$ in H. Hence the proposition is proved.

Now we shall describe explicitly the dual $\mathfrak{H}'$ of $\mathfrak{H}$. Decompose $T \in \Lambda_\infty$ as follows:

$$T = \sum_\sigma (-1)^{|\sigma|}T^\sigma, \tag{6}$$

where $T^\sigma \in \Lambda_\infty$ and the carrier of T^σ is contained in R^n_σ. For example, if $T = D^p[\exp(k'|x|)g(x)]$, $g(x)$ being bounded and continuous, then we may put $T^\sigma = (-1)^{|\sigma|}D^p[\exp(k'|x|)g(x)Y(x^\sigma)]$ where $Y(x)$ is the n-dimensional Heaviside's function. For any such decomposition, we set

$$F^\sigma(z) = [\mathfrak{F}T^\sigma](z) = (2\pi)^{-n/2} <\exp(iz\cdot x),\ T^\sigma_x>. \tag{7}$$

Then there exists a number $k > 0$ such that each $F^\sigma(z)$ is analytic in $C^n_{\sigma,k}$ and $F^\sigma(z)/z^k$ is bounded and continuous in $\overline{C^n_{\sigma,k}}$.

Conversely, for such a function $F(z)=\sum_{\sigma} F^{\sigma}(z)$, define distributions T^{σ} by

$$(8)\quad T_y^{\sigma}=(\partial^{k+2}/\partial y^{k+2})\Big[(2\pi)^{-n/2}\int\cdots\int_{\Delta_{\sigma,k}}\exp(-iy\cdot\zeta)F^{\sigma}(\zeta)(-i\zeta)^{-(k+2)}d\zeta\cdot Y(y^{\sigma})\Big]$$

and a distribution T by the formula (6). Then $T\in\Lambda_{\infty}$ and the functions $G^{\sigma}(z)$ which are associated, by (7), with this decomposition are nothing but the given functions $F^{\sigma}(z)$. In fact, for $z\in C^n_{\sigma,k}$,

$$\begin{aligned}
G^{\sigma}(z) &\equiv <(2\pi)^{-n/2}\exp(iz\cdot y),\ T_y^{\sigma}>\\
&=(-iz)^{k+2}\Big[(2\pi)^{-n}\int\cdots\int_{\Delta_{\sigma,k}}F^{\sigma}(\zeta)(-i\zeta)^{-(k+2)}d\zeta\Big[\int\cdots\int_{R^n}\exp(-iy\cdot(z-\zeta))dy\Big]\Big]\\
&=(-iz)^{k+2}\Big[(2\pi i)^{-n}\int\cdots\int_{\Delta_{\sigma,k}}F^{\sigma}(\zeta)(-i\zeta)^{-(k+2)}(-1)^{|\sigma|}(\zeta-z)^{-1}d\zeta\Big]\\
&=(-iz)^{k+2}[F^{\sigma}(z)(-iz)^{-(k+2)}]=F^{\sigma}(z).
\end{aligned}$$

If we denote by $\mathcal{A}_{\omega}$ the space of all functions $F(z)$ such that (i) $F(z)$ is analytic in $\{z\in C^n:\ |\mathcal{J}z_1|>k,\ldots\ldots,\ |\mathcal{J}z_n|>k\}$ and (ii) $F(z)/z^k$ is bounded continuous in $\{z\in C^n:\ |\mathcal{J}z_1|\geqq k,\ldots\ldots,|\mathcal{J}z_n|\geqq k\}$, k depending on $F(z)$, then we have shown that the mapping $\mathfrak{F}:\ F(z)\to T$, defined by the formulae (8) and (6), is a mapping of $\mathcal{A}_{\omega}$ onto Λ_{∞}.

Let $F\in\mathcal{A}_{\omega}$ and $T=\mathfrak{F}F$. Then there exists a decomposition (6) such that $F(z)=\sum_{\sigma}F^{\sigma}(z)$ with $F^{\sigma}(z)=[\overline{\mathfrak{F}}T^{\sigma}](z)$ for $z\in C^n_{\sigma,k}$ where k is any integer >0 having the properties (i) and (ii) above. We may write $T_y^{\sigma}=(\partial^k/\partial y^k)\cdot[\exp(k|y|)h^{\sigma}(y)]$ where $h^{\sigma}(y)$ is bounded, continuous in R^n_{σ} and vanishes outside of R^n_{σ}. Then, for any $f\in\mathfrak{H}$,

$$\begin{aligned}
<\overline{\mathfrak{F}}f,T^{\sigma}> &= <(2\pi)^{-n/2}\int\cdots\int_{R^n}\exp(iy\cdot x)f(x)dx,\ T_y^{\sigma}>\\
&=(2\pi)^{-n/2}\Big\langle\int\cdots\int_{\Delta_{\sigma,k+1}}\exp(iy\cdot\zeta)f(\zeta)d\zeta,\ (\partial^k/\partial y^k)[\exp(k|y|)h^{\sigma}(y)]\Big\rangle\\
&=(2\pi)^{-n/2}\int\cdots\int_{R^n_{\sigma}}\Big[\int\cdots\int_{\Delta_{\sigma,k+1}}(-i\zeta)^k\exp(iy\cdot\zeta)f(\zeta)d\zeta\Big]\exp(k|y|)h^{\sigma}(y)dy\\
&=\int\cdots\int_{\Delta_{\sigma,k+1}}f(\zeta)\Big[(2\pi)^{-n/2}(-i\zeta)^k\int\cdots\int_{R^n_{\sigma}}\exp(k|y|+i\zeta\cdot y)h^{\sigma}(y)dy\Big]d\zeta\\
&=\int\cdots\int_{\Delta_{\sigma,k+1}}f(\zeta)F^{\sigma}(\zeta)d\zeta\\
&=\int\cdots\int_{\Delta_{\sigma,k}}f(\zeta)F(\zeta)d\zeta,
\end{aligned}$$

from which follows

$$<f, F> = <f, \overline{\mathfrak{F}}T> = <\overline{\mathfrak{F}}f, T> = \sum_\sigma (-1)^{|\sigma|} <\overline{\mathfrak{F}}f, T^\sigma>$$
$$= \sum_\sigma (-1)^{|\sigma|} \int \cdots \int_{\Delta_{\sigma,k}} f(\zeta) F(\zeta) d\zeta$$
$$= \int \cdots \int_{L_k} f(\zeta) F(\zeta) d\zeta$$

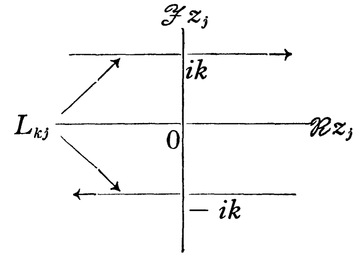

Fig. 1

where L_k is the product of pathes L_{kj} $(j = 1, 2, \ldots\ldots, n)$ defined by Fig. 1.

It is now clear that $<f, F> = 0$ for all $f \in \mathfrak{H}$ if and only if F belongs to the kernel of the mapping $\mathfrak{F} : \mathcal{A}_\omega \to \Lambda_\infty$. Let Π be the kernel of $\mathfrak{F}$. Obviously, any element in $\mathcal{A}_\omega$, which is a polynomial in one of the variables $z_1, \ldots\ldots, z_n$ that is,

$$\sum_s z_j^s G_s(z_1, \ldots\ldots, z_{j-1}, z_{j+1}, \ldots\ldots, z_n)$$

where G_s are functions in $\mathcal{A}_\omega$ with respect to $(z_1, \ldots\ldots, z_{j-1}, z_{j+1}, \ldots\ldots, z_n)$, belongs to Π. Conversely, we can prove readily that Π is the subspace of $\mathcal{A}_\omega$ generated by all such polynomials. Thus, summing up these considerations, we obtain

PROPOSITION 5. *The dual $\mathfrak{H}'$ of $\mathfrak{H}$ is algberaically isomorphic with the space $\mathfrak{U}$ which is the quotient space of $\mathcal{A}_\omega$ by Π. If $F(z)$ is any representative of an element $u \in \mathfrak{U}$ and k is determined by* (i) *and* (ii), *then the duality between $\mathfrak{U}$ and $\mathfrak{H}$ is given by*

$$<f, u> = \int \cdots \int_{L_k} f(\zeta) F(\zeta) d\zeta \quad \text{for } f \in \mathfrak{H}.$$

The distribution $T = \mathfrak{F}u \in \Lambda_\infty$ corresponding to u is then expressed by (8) *and* (6).

Any element in $\mathfrak{U}$ is called a *tempered ultra-distribution* in the n-dimensional space. In view of Cor. to Prop. 3, we have

PROPOSITION 6. *A set $B \subset \mathfrak{U}$ is bounded if and only if there exist an integer $k \geqq 0$ and a number $M > 0$ such that each element $u \in B$ has a representative $F(z)$ satisfying* (i) *$F(z)$ is analytic in $\{z \in C^n : |\mathscr{I} z_1| > k, \ldots\ldots, |\mathscr{I} z_n| > k\}$, continuous in $\{z \in C^n : |\mathscr{I} z_1| \geqq k, \ldots\ldots, |\mathscr{I} z_n| \geqq k\}$ and* (ii) *$|F(z)/z^k| \leqq M$ in the latter set.*

Since the strong dual $\mathfrak{U}$ of $\mathfrak{H}$ is bornological, a convex circular subset of $\mathfrak{U}$ is a neighborhood of the origin if and only if it absorbs all bounded subsets of $\mathfrak{U}$. Thus we get an explicit description of the strong topology for $\mathfrak{U}$. Denoting by B_k the subset of $\mathfrak{U}$, each element of which has a representative $F(z)$ satisfying (i) and (ii) of Prop. 6 with $M = 1$, a basis of neighborhoods of zero in $\mathfrak{U}$, then consists of $\Gamma_{k=1}^{\infty}(\varepsilon_k B_k)$ where ε_k is any sequence of positive numbers $\Gamma_{k=1}^{\infty}$ denotes the convex circular envelope of $\varepsilon_k B_k$'s. In other words, we may say

PROPOSITION 7. *The strong dual $\mathfrak{U}$ is the inductive limit of Banach spaces $\mathfrak{U}_{B_k}$, which are the subspace of $\mathfrak{U}$ generated by B_k with the unit ball B_k.*

REMARK. It is clear that the topological and algebraical structure of $\mathfrak{U}$, stated in Prop. 7, is the same as that of Sebastião e Silva [3] when $n = 1$.

An extension of ultra-distributions of exponential type to n-dimensional space is possible, which was also observed by Prof. Sebastião e Silva in a letter to the author.

3. An application. The multiplication in $\mathfrak{U}$. Since Λ_∞ contains the space $(\mathscr{S}')$ of Schwartz's tempered distributions and $\mathfrak{F}$ is an automorphism of $(\mathscr{S}')$, we may regard $(\mathscr{S}')$ as a subspace of $\mathfrak{U}$, the characterization of which was obtained by Sebastião e Silva [3 ; Prop. 12. 1]. We can obtain a similar characterization for arbitrary n. It is well known that $\alpha \in (\mathscr{O}_M)$ defines a continuous mapping $[\alpha] : S \to \alpha S$ of $(\mathscr{S}')$ into itself. For the mutiplication-operation in $\mathfrak{U}$, we have the following

PROPOSITION 8. *The mapping $[\alpha]$ ($\alpha \in (\mathscr{O}_M)$), defined in $(\mathscr{S}') \subset \mathfrak{U}$, is continuously extendable to a continuous linear mapping of $\mathfrak{U}$ into itself, if and only if $\alpha(x)$ is extendable over C^n as an entire function slowly increasing in any horizontal bands.*

PROOF. The "if"-part was obtained by Sebastião e Silva [3 ; Prop. 15. 1]. Indeed, if $\alpha \in (\mathscr{O}_M)$ can be extended to an entire function $\tilde{\alpha}$ satisfying the condition, then it is easy to see that $f \to \tilde{\alpha} f$ ($f \in \mathfrak{H}$) defines a continuous operation in $\mathfrak{H}$ so that $\tilde{\alpha}$ provides a continuous multiplication in $\mathfrak{H}' = \mathfrak{U}$.

In order to verify the "only if"-part, we notice firstly that $[\alpha]$ is continuously extendable to $\mathfrak{U}$ if and only if the convolution operation $T \to A * T$ ($A = \mathfrak{F}\alpha \in$

($\mathcal{O}_C'$)), defined in $(\mathcal{S}') \subset \Lambda_\infty$, is continuously extendable to a continuous linear mapping of Λ_∞ into itself. Since Λ_∞ is of type $\mathcal{H}_c'^\infty$ by Prop. 3, we may apply a theorem of Schwartz which we state as

LEMMA 1 ([4 ; Exposé 11, Theorem 1]). *Let E be a space of distributions, i. e., $E \subset (\mathcal{D}')$ and the injection of E into $(\mathcal{D}')$ is continuous. Let $\mathcal{H}$ be a space of type $\mathcal{H}^\infty$ and $\mathcal{O}_C'(\mathcal{H}_c' : E)$ the space of continuous linear mappings of $\mathcal{H}_c'$ into E which are convolution operations on $(\mathcal{E}') \subset \mathcal{H}_c'$. Then a distribution A belongs to $\mathcal{O}_C'(\mathcal{H}_c' : E)$ if and only if the function $\vec{A} : y \to \tau_y(A_x)$, $y \in R^n$, belongs to $\widetilde{\mathcal{H}}(E)$, the space of indefinitely differentiable functions $\vec{\varphi}$ with values in E such that $\langle\vec{\varphi}, e'\rangle \in \mathcal{H}$ for any $e' \in E'$.*

Now, suppose that $[\alpha]$ is continuously extendable to $\mathcal{U}$ into itself and therefore that the convolution operation defined by $A = \mathfrak{F}\alpha$ is continuously extendable to Λ_∞, i. e., $A \in \mathcal{O}_C'(\Lambda_\infty : \Lambda_\infty)$. Thus, by Lemma 1, the function $\vec{A} : y \to \tau_y A_t$ must belong to $\widetilde{\mathcal{H}}(\Lambda_\infty)$ where $\mathcal{H} = H$. This means that, for any $k \geqq 0$ and p, the set $\{\exp(k|y|)D_y^p\vec{A}(y) : y \in R^n\}$ is bounded in Λ_∞ by [4 ; Exposé 10, Prop. 4]. Especially for $p = 0$, the set of distributions $\{\exp(k|y|)\tau_y(A_t) : y \in R^n\}$ is bounded in Λ_∞ and à fortiori bounded in $(\mathcal{D}')$. Then, by an argument similar to that used in the proof of Prop. 3, we see that there exist continuous bounded functions $g_1(t)$, $g_2(t)$ and an integer $N > 0$, all depending on A and k, such that

$$A_t = \Delta^N[\exp(-k|t|)g_1(t)] + \exp(-k|t|)g_2(t).$$

It follows that $\langle(2\pi)^{-n/2}\exp(iz\cdot t), A_t\rangle$ has a meaning for any $z \in V_k$ and

$$\begin{aligned}\tilde{\alpha}(z) &\equiv \langle(2\pi)^{-n/2}\exp(iz\cdot t), A_t\rangle \\ &= (2\pi)^{-n/2}\Big\{[-(z_1^2 + \dots\dots + z_n^2)]^N \int\cdots\int_{R^n} \exp(iz\cdot t - k|t|)g_1(t)dt \\ &\qquad + \int\cdots\int_{R^n} \exp(iz\cdot t - k|t|)g_2(t)dt\Big\}.\end{aligned}$$

Since the integrals in the last expression represent analytic functions bounded in $V_{k-\varepsilon}(\varepsilon > 0)$, the function $\tilde{\alpha}(z)$ is analytic and slowly increasing in the bands $V_{k-\varepsilon}(\varepsilon > 0)$. As k is arbitrary, we have shown that $\tilde{\alpha}(z)$ is an entire function slowly increasing in any horizontal bands. Clearly $\tilde{\alpha}(x) = \alpha(x)$ for $x \in R^n$ and this completes the proof.

In the course of the proof of Prop. 8, we have obtained

PROPOSITION 9. *A distribution T belongs to $\mathcal{O}_C'(\Lambda_\infty : \Lambda_\infty)$ if and only if, for any integer $k > 0$, there exists a finite number of bounded continuous functions $g_j(x)$ such that T is a (finite) sum of distribution-derivatives of* $\exp(-k|x|)g_i(x)$.

ADDED IN PROOF. Recently, the following paper has appeared: K. Yoshinaga, "On spaces of distributions of exponential growth," Bulletin of the Kyushu Institute of Technology (Math. & Nat. Sci.), No. 6, 1960. This paper treats, independently of ours, a problem related to the one discussed here, especially to the section 3.

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