ON INFINITESIMAL CONFORMAL AND PROJECTIVE TRANSFORMATIONS OF NORMAL CONTACT SPACES

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Introduction. In the previous paper [4][1] the author discussed some properties of normal contact spaces. However, problems concerning infinitesimal transformations have not been studied. In the present paper such problems are concerned and some satisfactory answers are given.

In § 1, we state the fundamental identities of normal contact spaces.

In § 2, we shall give some preliminary facts concerning infinitesimal transformations for the later use. After these preparations, in § 3, we shall prove that an infinitesimal conformal transformation in normal contact spaces is necessarily concircular. Thus we know that a connected complete normal contact space admitting an infinitesimal non-isometric conformal transformation is isometric with a unit sphere.

It will be shown in § 4 that an infinitesimal projective transformation in a normal contact space has some analogous properties of the one in an Einstein space, for example, that any infinitesimal projective transformation in this space is decomposed as a sum of a Killing vector and an infinitesimal gradient projective transformation.

In § 5, we shall define the notion of η-Einstein spaces and discuss infinitesimal conformal and projective transformations in such spaces.

Finally, we shall devote § 6 to show that one of Sasaki's examples of normal contact spaces is an example of η-Einstein spaces.

1. (ϕ, ξ, η, g)-structure and contact structure. On an n[2]$(= 2m + 1)$-dimensional real differentiable manifold M with local coordinate systems $\{x^i\}$, if there exist a tensor field $\phi_j{}^i$, contravariant and covariant vector fields ξ^i and η_i satisfying the relations

$$\xi^i\eta_i = 1, \tag{1. 1}$$

$$\text{rank } |\phi_j{}^i| = n - 1, \tag{1. 2}$$

$$\phi_j{}^i\xi^j = 0, \tag{1. 3}$$

$$\phi_j{}^i\eta_i = 0, \tag{1. 4}$$

$$\phi_j{}^i\phi_k{}^j = - \delta_k^i + \eta_k\xi^i, \tag{1. 5}$$

1) The numbers in the bracket refer to the bibliography at the end of the paper.

2) In this paper we assume that $n>3$ and that the indices $i, j, k, \cdots = 1, 2, \cdots, n$.

then the set $(\phi_j{}^i, \xi^i, \eta_j)$ is called a (ϕ, ξ, η)-structure. It is well known fact[3] that a manifold M with a (ϕ, ξ, η)-structure always admits a positive definite Riemannian metric tensor g_{ji} such that

$$g_{ji}\xi^j = \eta_i, \tag{1. 6}$$

$$g_{ji}\phi_k{}^j\phi_h{}^i = g_{hk} - \eta_h\eta_k. \tag{1. 7}$$

The aggregate consisting of a (ϕ, ξ, η)-structure together with a Riemannian metric tensor which has above properties is called a (ϕ, ξ, η, g)-structure and the metric tensor g_{ji} is called an associated metric to the (ϕ, ξ, η)-structure.

In this paper, we always consider such a Riemannian metric tensor, so we use a notation η^i in stead of ξ^i.

On the other hand let M be a differentiable manifold with a contact structure $\eta = \eta_i dx^i$. Then if we define ϕ_{ji} by

$$2\phi_{ji} = \partial_j\eta_i - \partial_i\eta_j, \ (\partial_j = \partial/\partial x^j),$$

we can introduce a Riemannian metric g_{ji} such that $\phi_i{}^h = g^{hr}\phi_{ir}$, $\xi^i = g^{ir}\eta_r$, η_i and g_{ji} define a (ϕ, ξ, η, g)-structure.[4]

There are four important tensors $N_{ji}{}^h$, N_{ji}, $N_j{}^i$ and N_j and, as was shown by S.Sasaki and Y.Hatakeyama [7], the vanishing of $N_{ji}{}^h$ yields the vanishing of the other three tensors. The contact manifold with vanishing $N_{ji}{}^h$ is called a *normal* contact space. In the following we only consider a normal contact space.

The following identities[5] are fundamental,

$$\nabla_j\eta_i = \phi_{ji}, \tag{1. 8}$$

$$\nabla_k\phi_{ji} = \eta_j g_{ki} - \eta_i g_{kj}. \tag{1. 9}$$

From the last relation we have

$$\nabla^r\phi_{ir} = (n-1)\eta_i, \tag{1. 10}$$

where and throughout the paper ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols and we put $\nabla^j = g^{jr}\nabla_r$.

In a Riemannian space, a vector field v^i which satisfies

$$\nabla_j v_i + \nabla_i v_j = 0$$

is called an infinitesimal isometry or a Killing vector.

A normal contact space is characterized by the fact that it is a Riemannian space which admits a unit Killing vector η^i satisfying

$$\nabla_k\nabla_j\eta_i = \eta_j g_{ki} - \eta_i g_{kj}.$$

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3) Sasaki, S. [5], Hatakeyama, Y. [2]
4) Sasaki, S. and Hatakeyama, Y. [7]
5) Sasaki, S. and Hatakeyama, Y. [7]

Now, let $R_{kji}{}^{h}$ be Riemannian curvature tensor, i. e.

$$R_{kji}{}^{h} = \partial_k \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ki \end{matrix} \right\} + \left\{ \begin{matrix} r \\ ji \end{matrix} \right\} \left\{ \begin{matrix} h \\ kr \end{matrix} \right\} - \left\{ \begin{matrix} r \\ ki \end{matrix} \right\} \left\{ \begin{matrix} h \\ jr \end{matrix} \right\}$$

and put

$$R_{kjih} = R_{kji}{}^{r} g_{rh},\ R_{ji} = R_{rji}{}^{r},\ R = g^{jk} R_{jk}.$$

We have known the relations[6)]

$$(1.\ 11) \qquad \eta_r R_{kji}{}^{r} = \eta_k g_{ji} - \eta_j g_{ki},$$

$$(1.\ 12) \qquad \phi_{jr} R_l{}^{r} + \frac{1}{2} \phi^{rk} R_{rkjl} = (n-2)\phi_{jl}.$$

From (1. 11) it follows that

$$(1.\ 13) \qquad \eta_r R_j{}^{r} = (n-1)\eta_j,$$

$$(1.\ 14) \qquad R_{ji}\eta^j \eta^i = n-1.$$

Let v_i be an arbitrary vector field. Since $\phi^{kj} = g^{kr}\phi_r{}^{j}$ is anti-symmetric we have by virtue of the Ricci's identity

$$(1.\ 15) \qquad \phi^{kj}\nabla_k \nabla_j v_i = -\frac{1}{2} \phi^{kj} R_{kji}{}^{r} v_r.$$

Next, let us recall the definition of Lie's derivation.[7)] For any vector field v^i and tensor field a_{ji}, we have the following identities.

$$(1.\ 16) \qquad \eta^i \underset{v}{\pounds} \eta_i = \frac{1}{2} \eta^i \eta^j \underset{v}{\pounds} g_{ji},$$

$$(1.\ 17) \qquad \nabla_k \underset{v}{\pounds} a_{ji} - \underset{v}{\pounds} \nabla_k a_{ji} = a_{ri} \underset{v}{\pounds} \left\{ \begin{matrix} r \\ j\,k \end{matrix} \right\} + a_{jr} \underset{v}{\pounds} \left\{ \begin{matrix} r \\ i\,k \end{matrix} \right\},$$

$$(1.\ 18) \qquad \nabla_k \underset{v}{\pounds} \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - \nabla_j \underset{v}{\pounds} \left\{ \begin{matrix} h \\ ki \end{matrix} \right\} = \underset{v}{\pounds} R_{kji}{}^{h}.$$

2. Some infinitesimal transformations. In a Riemannian space, a vector field v^i which satisfies

$$(2.\ 1) \qquad \underset{v}{\pounds} g_{ji} \equiv \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji} \quad ,$$

where ρ is a certain scalar function, is called an infinitesimal conformal transformation or a conformal Killing vector and the function ρ is called an associa-

6) Okumura, M. [4].
7) Yano, K. [9]

ted function of this transformation. As is well known[8], an infinitesimal conformal transformation v^i satisfies

(2. 2) $$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} \equiv \nabla_j\nabla_i v^h + R_{rji}{}^h v^r = \rho_j\delta_i^h + \rho_i\delta_j^h - \rho^h g_{ji},\ \rho_i = \partial_i\rho.$$

According to (1. 18), (2. 1) and (2. 2), we get the following identities,

(2. 3) $$\underset{v}{\pounds} R_{kji}{}^h = \nabla_k\rho_i\delta_j^h - \nabla_k\rho^h g_{ji} - \nabla_j\rho_i\delta_k^h + \nabla_j\rho^h g_{ki},$$

(2. 4) $$\underset{v}{\pounds} R = -2\rho R - 2(n-1)\nabla_r\rho^r.$$

For the non-constant scalar field ρ in a Riemannian space, if there exists a scalar field σ such that

(2. 5) $$\nabla_j\nabla_i\rho = \sigma g_{ji} \quad ,$$

then the scalar field ρ is called a concircular scalar field. Especially if the scalar σ is linear in ρ, i. e. if

(2. 6) $$\nabla_j\nabla_i\rho = (-k\rho + b)g_{ji}$$

is valid with constants k and b, then ρ is called a *special* concircular scalar field in the sense of Y. Tashiro.[9]

About a special concircular scalar field the following theorem has been proved by M.Obata [3] and Y.Tashiro [8] recently.

THEOREM. *Suppose an n-dimensional connected complete Riemannian space admits a special concircular scalar field ρ satisfying* (2.6). *If $k = c^2 > 0$, then the space is isometric with a sphere of radius* $1/c$.

If the associated function ρ is a concircular scalar field, the infinitesimal conformal transformation is called an infinitesimal concircular transformation.

In a Riemannian space, a vector field v^i is called an infinitesimal projective transformation or a projective Killing vector if it satisfies

(2. 7) $$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \rho_j\delta_i^h + \rho_i\delta_j^h$$

where ρ_i is a certain vector field which is called an associated vector of the infinitesimal projective transformation.

Contracting h and i in (2. 7), we can show that the associated vector ρ_j is a gradient vector. So, from (1. 18) and (2. 7) we have for an infinitesimal projective transformation

(2. 8) $$\underset{v}{\pounds} R_{kji}{}^h = \nabla_k\rho_i\delta_j^h - \nabla_j\rho_i\delta_k^h$$

8) Yano, K. [9]
9) Tashiro, Y. [8]

which implies that

$$\underset{v}{\pounds} R_{ji} = -(n-1)\nabla_j\rho_i. \tag{2. 9}$$

In a normal contact space, the vector field v^i satisfying

$$\underset{v}{\pounds}\eta_i = \sigma\eta_i \tag{2. 10}$$

where σ is a certain scalar is called an infinitesimal contact transformation. Especially if σ vanishes identically we call v^i an infinitesimal strict contact transformation.

3. Infinitesimal conformal transformations. In this section, we shall give the following

THEOREM 3.1. *In a normal contact space* ($n>3$), *any infinitesimal conformal transformation is necessarily concircular.*

To prove the theorem, we begin with the following

LEMMA 3.1. *For a scalar field* ρ *if there exist* σ *and* τ *such that*

$$\nabla_j\rho_i = \sigma g_{ji} + \tau\eta_j\eta_i, \qquad \rho_i = \partial_i\rho, \tag{3. 1}$$

then we have $\tau = 0$.

PROOF. Differentiating (3. 1) covariantly and taking account of (1. 8) we have

$$\nabla_k\nabla_j\rho_i = \nabla_k\sigma g_{ji} + \nabla_k\tau\eta_j\eta_i + \tau(\phi_{kj}\eta_i + \phi_{ki}\eta_j). \tag{3. 2}$$

Transvecting the last equation with ϕ^{kj} and making use of (1. 4), (1. 5), (1. 15) we get

$$-\frac{1}{2}\phi^{kj}R_{kjir}\rho^r = \nabla_k\sigma\phi^{kj}g_{ji} + (n-1)\tau\eta_i.$$

Transvecting η^i to this and using (1. 11), we have

$$-\frac{1}{2}\phi^{kj}(\eta_j g_{kr} - \eta_k g_{jr})\rho^r = (n-1)\tau.$$

Since the left hand side of this equation vanishes by virtue of (1. 4), it follows that $\tau = 0$. This completes the proof.

From (1. 16) and (2. 1) we get

LEMMA 3.2. *For a conformal Killing vector* v^i *and its associated function* ρ, *we have*

$$\eta^r\underset{v}{\pounds}\eta_r = \rho. \tag{3. 3}$$

Next, we state the

LEMMA 3. 3. *For a conformal Killing vector v^i and its associated function ρ there exists a scalar function α such that*

$$\eta_r\nabla_j\rho^r = \alpha\eta_j. \tag{3. 4}$$

PROOF. If we recall (1 .11), for any conformal Killing vector we have

$$R_{kji}{}^h\underset{v}{\pounds}\eta_h + \eta_h\underset{v}{\pounds}R_{kji}{}^h = g_{ji}\underset{v}{\pounds}\eta_k + 2\rho g_{ji}\eta_k - g_{ki}\underset{v}{\pounds}\eta_j - 2\rho g_{ki}\eta_j. \tag{3. 5}$$

Substituting (2. 3) into (3. 5), we have

$$R_{kji}{}^h\underset{v}{\pounds}\eta_h =(\underset{v}{\pounds}\eta_k + 2\rho\eta_k + \eta_r\nabla_k\rho^r)g_{ji} -(\underset{v}{\pounds}\eta_j + 2\rho\eta_j + \eta_r\nabla_j\rho^r)g_{ki} + \eta_k\nabla_j\rho_i - \eta_j\nabla_k\rho_i. \tag{3. 6}$$

Transvecting this with ϕ^{kj} and making use of (1. 4), we get

$$(\phi^{kj}R_{kji}{}^r + 2\phi_i{}^r)\underset{v}{\pounds}\eta_r =- 2\eta_r\nabla_h\rho^r\phi_i{}^h. \tag{3. 7}$$

On the other hand, if we transvect (3. 6) with g^{ji} we obtain

$$R_k{}^h\underset{v}{\pounds}\eta_h =(n-1)(\underset{v}{\pounds}\eta_k + 2\rho\eta_k) + (n-2)\eta_r\nabla_k\rho^r + \eta_k\nabla_r\rho^r.$$

Multiplying the last equation with $\phi_j{}^k$ and summing over k, we have

$$\phi_j{}^kR_k{}^h\underset{v}{\pounds}\eta_h = (n-1)\phi_j{}^k\underset{v}{\pounds}\eta_k + (n-2)\eta_r\nabla_k\rho^r\phi_j{}^k.$$

Substituting (1. 12) into the last equation, we get

$$(\phi^{kj}R_{kji}{}^r + 2\phi_i{}^r)\underset{v}{\pounds}\eta_r =- 2(n-2)\eta_r\nabla_k\rho^r\phi_i{}^k. \tag{3. 8}$$

Comparing (3. 7) and (3. 8), we have

$$(n-3)\eta_r\nabla_j\rho^r\phi_i{}^j = 0. \tag{3. 9}$$

Transvecting (3. 9) with $\phi_k{}^i$, we obtain (3. 4). This proves the lemma.

PROOF OF THE THEOREM. Substituting (3. 4) into (3. 6), we get

$$R_{kji}{}^h\underset{v}{\pounds}\eta_h =\{\underset{v}{\pounds}\eta_k + (2\rho+\alpha)\eta_k\}g_{ji} - \{\underset{v}{\pounds}\eta_j + (2\rho+\alpha)\eta_j\}g_{ki} + \eta_k\nabla_j\rho_i - \eta_j\nabla_k\rho_i. \tag{3. 10}$$

Transvecting (3. 10) with η^k and making use of (1. 11), we have

$$(\eta^h g_{ji} - \eta_i\delta_j{}^h)\underset{v}{\pounds}\eta_h =(\eta^r\underset{v}{\pounds}\eta_r + 2\rho + \alpha)g_{ji} -\eta_i\underset{v}{\pounds}\eta_j - 2(\rho+\alpha)\eta_j\eta_i + \nabla_j\rho_i,$$

which implies that

$$-\nabla_j\rho_i =(2\rho+\alpha)g_{ji} - 2(\rho+\alpha)\eta_j\eta_i. \tag{3. 11}$$

Applying Lemma 3.1 to (3. 11), we have $\rho+\alpha=0$. Thus it follows that

$$\nabla_j\rho_i =- \rho g_{ji} \tag{3. 12}$$

which shows that the transformation is concircular. Q. E. D.

By virtue of this theorem and the theorem stated in §2, we have

THEOREM 3.2. *Let M be an $n(>3)$-dimensional connected complete normal contact space. If it admits a non-isometric infinitesimal conformal transformation, then M is isometric with a unit sphere.*

Since ρ is the associated function of an infinitesimal conformal transformation, (2. 1) and (3. 12) show us that

$$\underset{v}{\pounds} g_{ji} = \nabla_j v_i + \nabla_i v_j = -2\nabla_j \rho_i$$

which implies that

$$\nabla_j(v_i + \rho_i) + \nabla_i(v_j + \rho_j) = 0.$$

The last equation shows that the vector $v_j + \rho_j$ is a Killing vector.

Thus we have

THEOREM 3. 3. *If a normal contact space $(n > 3)$ admits an infinitesimal conformal transformation v^j, then v^i is decomposed into*

$$v^i = w^i - \rho^i$$

where w^i is a Killing vector and ρ^i is a gradient[10] *vector defining an infinitesimal conformal transformation.*

REMARK. If we apply Theorem 3.3 to the infinitesimal conformal transformation ρ^i which is the gradient vector of the associated function of v^i, then w^i is a zero vector.

Before describing an application of Theorem 3.3, we must prove the

LEMMA 3. 4. *In a normal contact space, there exists no parallel vector field other than zero vector.*

PROOF. For any parallel vector field v^i, it follows that

$$v^r R_{kjr}{}^h = 0. \tag{3. 13}$$

Transvecting this with $\eta_h \eta^k$ and making use of (1. 11), we have $v_j = (\eta_r v^r)\eta_j$.

On the other hand, transvecting (3. 13) with $\eta^k \delta_h^j$ and making use of (1. 13),we can see that $\eta_r v^r = 0$ easily. Thus the lemma is proved.

Let L_C, L_I and L' be the Lie algebra consisting of all infinitesimal conformal transformations, the Lie algebra consisting of all Killing vector fields and the vector space of the gradient of the associated functon ρ respectively. Then we have the following

THEOREM 3. 4. *In a normal contact space $(n > 3)$, the following relations hold.*

10) This means that the covariant component of ρ^i is a gradient vector.

$$L_C = L_I + L' \ (\textit{direct sum})$$
$$[L_I, L_I] \subset L_I,\ [L_I, L'] \subset L',\ [L', L'] \subset L_I.$$

PROOF. Theorem 3.3 and Lemma 3.4 assert that $L_C = L_I + L'$ holds good. The other parts of the theorem follow from the relations $\underset{[u,v]}{\pounds} = \underset{u}{\pounds}\underset{v}{\pounds} - \underset{v}{\pounds}\underset{u}{\pounds}$ and $[u, v]^i = u^r\nabla_r v^i - v^r\nabla_r u^i$ immediately.

4. Infinitesimal projective transformations. We shall show in this section that an infinitesimal projective transformation in a normal contact space has some analogous properties of the one in an Einstein space.[11]

At first we prove the

LEMMA 4.1. *Let v^i be an infinitesimal projective transformation and ρ_i be its associated vector. If there is a relation*

$$\underset{v}{\pounds} g_{ji} = -\nabla_j\rho_i \tag{4.1}$$

then ρ^i is also an infinitesimal projective transformation.

PROOF. Applying (1.17) to the Riemannian metric tensor, it follows that

$$\nabla_k \underset{v}{\pounds} g_{ji} = g_{ri}\underset{v}{\pounds}\left\{\begin{matrix} r \\ k\,j \end{matrix}\right\} + g_{jr}\underset{v}{\pounds}\left\{\begin{matrix} r \\ k\,i \end{matrix}\right\}. \tag{4.2}$$

By virtue of (2.7) and (4.1), this equation is written as

$$-\nabla_k\nabla_j\rho_i = 2\rho_k g_{ji} + \rho_j g_{ki} + \rho_i g_{jk} \tag{4.3}$$

which implies that

$$\rho_r R_{kji}{}^r = \rho_k g_{ji} - \rho_j g_{ki}. \tag{4.4}$$

According to (4.3) and (4.4), it follows that

$$\underset{\rho}{\pounds}\left\{\begin{matrix} h \\ j\,i \end{matrix}\right\} \equiv \nabla_j\nabla_i\rho^h + R_{rji}{}^h\rho^r = -2(\rho_j\delta_i^h + \rho_i\delta_j^h). \tag{4.5}$$

This proves the lemma.

LEMMA 4.2. *Let v^i be an infinitesimal projective transformation and ρ_i be its associated vector. If there is a relation*

$$\underset{v}{\pounds} g_{ji} = -\nabla_j\rho_i + \beta\eta_j\eta_i, \tag{4.6}$$

then, we have $\beta = 0$.

PROOF. Taking the Lie's differential of the both sides of (1.11), we get

$$R_{kji}{}^h\underset{v}{\pounds}\eta_h + \eta_h\underset{v}{\pounds}R_{kji}{}^h = g_{ji}\underset{v}{\pounds}\eta_k + \eta_k\underset{v}{\pounds}g_{ji} - g_{ki}\underset{v}{\pounds}\eta_j - \eta_j\underset{v}{\pounds}g_{ki}.$$

11) Yano, K. [9] p. 271-272.

Substituting (2. 8) into the last equation, we have

$$R_{kji}{}^{h}\underset{v}{\pounds}\eta_{h} = g_{ji}\underset{v}{\pounds}\eta_{k} + \eta_{k}\underset{v}{\pounds}g_{ji} - g_{ki}\underset{v}{\pounds}\eta_{j} - \eta_{j}\underset{v}{\pounds}g_{ki} - \eta_{j}\nabla_{k}\rho_{i} + \eta_{k}\nabla_{j}\rho_{i} \tag{4.7}$$

from which we have

$$R_{k}{}^{h}\underset{v}{\pounds}\eta_{h} = (n-1)\underset{v}{\pounds}\eta_{k} + \eta_{k}(g^{ji}\underset{v}{\pounds}g_{ji} + \nabla_{r}\rho^{r}) - \eta^{r}(\underset{v}{\pounds}g_{kr} + \nabla_{k}\rho_{r}). \tag{4.8}$$

Substituting (4. 6) into (4. 8), we get

$$R_{k}{}^{h}\underset{v}{\pounds}\eta_{h} = (n-1)\underset{v}{\pounds}\eta_{k}. \tag{4.9}$$

On the other hand, (1. 13) yields that

$$R_{k}{}^{h}\underset{v}{\pounds}\eta_{h} + \eta_{h}\underset{v}{\pounds}R_{k}{}^{h} = (n-1)\underset{v}{\pounds}\eta_{k}. \tag{4.10}$$

Thus we have

$$\eta_{h}\underset{v}{\pounds}R_{k}{}^{h} = \eta_{h}R_{kr}\underset{v}{\pounds}g^{hr} + \eta^{r}\underset{v}{\pounds}R_{kr} = 0. \tag{4.11}$$

Substituting (2. 9) and

$$\underset{v}{\pounds}g^{ji} = \nabla^{j}\rho^{i} - \beta\eta^{j}\eta^{i} \tag{4.12}$$

which is obtained from (4. 6) and $g_{jk}g^{ki} = \delta_{j}^{i}$, into (4. 11) we get

$$\eta_{h}(\nabla^{h}\rho^{r} - \beta\eta^{h}\eta^{r})R_{kr} - (n-1)\eta^{r}\nabla_{k}\rho_{r} = 0.$$

Transvecting the last equation with η^{k} and making use of (1. 13), we have $\beta = 0$. Q. E. D.

Now, we shall show the

THEOREM 4. 1. *Let v^{i} be an infinitesimal projective transformation in a normal contact space. Then its associated vector ρ^{i} is also an infinitesimal projective transformation whose associated vector is $-2\rho_{i}$.*

PROOF. Transvecting (4. 7) with ϕ^{kj} and making use of (1. 4), we have

$$(\phi^{kj}R_{kji}{}^{r} + 2\phi_{i}{}^{r})\underset{v}{\pounds}\eta_{r} = 0. \tag{4.13}$$

On the other hand, if we transvect (4. 8) with $\phi_{j}{}^{k}$, it follows that

$$\phi_{j}{}^{k}R_{k}{}^{h}\underset{v}{\pounds}\eta_{h} = \{(n-1)\underset{v}{\pounds}\eta_{k} - \eta^{r}(\underset{v}{\pounds}g_{rk} + \nabla_{k}\rho_{r})\}\phi_{j}{}^{k}.$$

Substituting (1. 12) into the last equation, we get

$$(\phi^{kj}R_{kji}{}^{r} + 2\phi_{i}{}^{r})\underset{v}{\pounds}\eta_{r} = 2\phi_{i}{}^{k}(\eta^{r}\underset{v}{\pounds}g_{kr} + \eta^{r}\nabla_{k}\rho_{r}). \tag{4.14}$$

Comparing (4. 13) and (4. 14) and regarding (1. 2), we have

$$\eta^{r}\underset{v}{\pounds}g_{kr} + \eta^{r}\nabla_{k}\rho_{r} = \beta\eta_{k} \tag{4.15}$$

for a certain proportional factor β.

Transvecting (4. 7) with η^k, we have the following equation.

$$\underset{v}{\pounds} g_{ji} = -\nabla_j \rho_i + \eta_j(\eta^r \underset{v}{\pounds} g_{ri} + \eta^r \nabla_i \rho_r). \tag{4. 16}$$

Thus, from (4. 15) and (4. 16), it follows that

$$\underset{v}{\pounds} g_{ji} = -\nabla_j \rho_i + \beta \eta_j \eta_i.$$

The last equation, Lemma 4. 1 and 4. 2 show us that the theorem is true.

Writing out (4. 1) explicitly, we find

$$\nabla_j v_i + \nabla_i v_j = -\nabla_j \rho_i.$$

In the same way as in the previous section, we have

THEOREM 4. 2. *If a normal contact space admits an infinitesimal projective transformation v^i, then v^i is decomposed into*

$$v^i = w^i - \frac{1}{2}\rho^i,$$

where w^i is a Killing vector and ρ^i is the associated vector of the infinitesimal projective transformation.

REMARK. By virtue of Theorem 4. 1, we can apply Theorem 4. 2 to the associated vector of an infinitesimal projective transformation. In this case the Killing vector w^i is a zero vector.

THEOREM 4. 3. *In a normal contact space, we have*

$$L_P = L_I + L'' \ (\textit{direct sum})$$

$$[L_I, L''] \subset L'', \ [L'', L''] \subset L_I,$$

where L_P is the Lie algebra consisting of infinitesimal projective transformations and L'' is the vector space of the associated vector.

5. η-Einstein spaces. Let $\hat{R} = \frac{1}{2} R_{ijr}{}^s \phi_s{}^r dx^j \wedge dx^i$. In the previous paper [4], we have seen that the differential form $\hat{R}$ is closed and the following two propositions are valid.

LEMMA 5. 1. *If in a normal contact space we have* ($n > 3$)

$$R_{ji} = a g_{ji} + b \eta_j \eta_i, \tag{5. 1}$$

then a and b are constants.

LEMMA 5. 2. *In order that the two closed 2-forms $\hat{R}$ and $\phi = \phi_{ji} dx^j \wedge dx^i$ are linearly dependent, it is necessary and sufficient that the space has the Ricci tensor of the form* (5. 1).

In this section, we shall discuss a normal contact space with the Ricci tensor of the form (5. 1). In this paper such a space will be called an *η-Einstein space* for brevity. Evidently, any Einstein space is an η-Einstein space. An example of η-Einstein space with $b \neq 0$ will be shown in the next section.

From (5. 1) and (1. 14) we have

$$a + b = n - 1, \tag{5. 2}$$

$$R = an + b. \tag{5. 3}$$

On the other hand by virtue of (1. 12), we get

$$\phi^{rk}R_{rkji} = 2\{(n-2) - a\}\phi_{ji}. \tag{5. 4}$$

In an η-Einstein space, the following theorem holds good.

THEOREM 5.1. *In an η-Einstein space with $b \neq 0$, an infinitesimal isometry is always an infinitesimal strict contact transformation.*

PROOF. For any vector v^i we have

$$\underset{v}{\pounds}R_{ji} = a\underset{v}{\pounds}g_{ji} + b(\eta_i\underset{v}{\pounds}\eta_j + \eta_j\underset{v}{\pounds}\eta_i). \tag{5. 5}$$

For a Killing vector v^i, the last equation becomes

$$b(\eta_i\underset{v}{\pounds}\eta_j + \eta_j\underset{v}{\pounds}\eta_i) = 0. \tag{5. 6}$$

Transvecting this with η^i and using Lemma 3. 2 we get $b\underset{v}{\pounds}\eta_j = 0$. This completes the proof.

Now, let v^i be a conformal Killing vector. From the discussion in § 3, (3. 9) holds good. Consequently, according to (3. 7), it follows that

$$(\phi^{kj}R_{kji}{}^r + 2\phi_i{}^r)\underset{v}{\pounds}\eta_r = 0. \tag{5. 7}$$

Substituting (5. 4) into (5. 7), we have $b\phi_i{}^r\underset{v}{\pounds}\eta_r = 0$ by virtue of (5. 2). From this equation and Lemma 3.2, we get the following equation for $b \neq 0$:

$$\underset{v}{\pounds}\eta_i = \rho\eta_i. \tag{5. 8}$$

Substituting (5. 8) into (3. 6) and making use of Lemma 3. 3 and (1. 11), we have

$$\eta_j\nabla_k\rho_i - \eta_k\nabla_j\rho_i = (2\rho + \alpha)(\eta_k g_{ji} - \eta_j g_{ki}).$$

Transvecting this with g^{ji} and taking account of the fact $\rho + \alpha = 0$, we have

$$\nabla_r\rho^r = - n\rho. \tag{5. 9}$$

On the other hand from (2. 4), (5. 2) and (5. 3), it follows that

(5. 10) $$\nabla_r \rho^r = -(a+1)\rho.$$

Comparing the last two equations we have $\{a-(n-1)\}\rho = 0$. Thus we have the

THEOREM 5. 2. *In an η-Einstein space $(n>3)$ with $b \neq 0$, any infinitesimal conformal transformation is necessarily an infinitesimal isometry.*

Next, let v^i be a projective Killing vector. By means of (2. 9), (4. 1) and (5. 5) we have

(5. 11) $$-(n-1)\nabla_j \rho_i = -a\nabla_j \rho_i + b(\eta_i \underset{v}{\pounds} \eta_j + \eta_j \underset{v}{\pounds} \eta_i).$$

On the other hand (4. 13) and (5. 4) yield

(5. 12) $$\underset{v}{\pounds} \eta_j = \sigma \eta_j$$

where we have put $\sigma = \eta^r \underset{v}{\pounds} \eta_r$.

Substituting (5. 12) into (5. 11), we get

(5. 13) $$-\nabla_j \rho_i = 2\sigma \eta_j \eta_i.$$

Applying Lemma 3.1 to (5. 13), we have $\nabla_j \rho_i = 0$. Thus we have $\underset{v}{\pounds} g_{ji} = 0$ by means of (4. 1).

THEOREME 5. 3. *Let M be an η-Einstein space with $b \neq 0$. Then any infinitesimal projective transformation in M is necessarily an infinitesimal isometry.*

6. An example. In this section we shall show an example of an η-Einstein space which is not an Einstein space.

Let E^{2m+1} be a Euclidean space with cartesian coordinates (x^α, y^α, z) $(\alpha = 1,2, \dots\dots, m)$. We put

(6. 1) $$\eta = \frac{1}{2}\left(dz - \sum_{\alpha=1}^{m} y^\alpha dx^\alpha\right),$$

then η gives a contact structure to E^{2m+1}[12]. If we put

(6. 2) $$x^{\alpha^*} \equiv x^{m+\alpha} = y^\alpha,\ x^\Delta = z,\ \Delta = 2m+1,$$

we have from the definition that

(6. 3) $$\eta_i = \left(-\frac{1}{2}y^\alpha, 0, \frac{1}{2}\right)$$

12) For example, Gray, J. W. [1]. Sasaki, S. [6].

and

$$(6.\ 4) \qquad d\eta = \frac{1}{2}\sum_{\alpha=1}^{m} dx^{\alpha} \wedge dy^{\alpha}.$$

Therefore, the tensor

$$2\phi_{ji} = \partial_j \eta_i - \partial_i \eta_j$$

has the components

$$(6.\ 5) \qquad (\phi_{ji}) = \begin{pmatrix} 0 & \frac{1}{4}\delta_{\alpha\beta} & 0 \\ -\frac{1}{4}\delta_{\alpha\beta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the relations (1. 1) and (1. 3), we have

$$(6.\ 6) \qquad \xi^i = 2\delta^i_\Delta = (0\ ,0,\ 2).$$

Now, we consider a symmetric tensor field in E^{2m+1} defined by

$$(6.\ 7) \qquad (g_{ji}) = \begin{pmatrix} \frac{1}{4}(\delta_{\alpha\beta} + y^{\alpha}y^{\beta}) & 0 & -\frac{1}{4}y^{\alpha} \\ 0 & \frac{1}{4}\delta_{\alpha\beta} & 0 \\ -\frac{1}{4}y^{\alpha} & 0 & \frac{1}{4} \end{pmatrix},$$

then (g_{ji}) defines a positive definite Riemannian metric. The covariant components of the tensor (g_{ji}) are given by

$$(6.\ 8) \qquad (g^{ji}) = \begin{pmatrix} 4\delta^{\alpha\beta} & 0 & 4y^{\alpha} \\ 0 & 4\delta^{\alpha\beta} & 0 \\ 4y^{\beta} & 0 & 4\left(1 + \sum_{\alpha=1}^{m}(y^{\gamma})^2\right) \end{pmatrix}.$$

S.Sasaki [6] proved that thus defined tensors $\phi_{ji}, g_{ji}, \eta_i$ and ξ^i form a (ϕ, ξ, η, g)-structure. Further, he proved that this contact structure η is normal. However, he has not calculated yet the Christoffel symbols, the curvature tensor and the Ricci tensor. In the following, we shall calculate them.

By means of (6. 7) and

$$[ji, r] = \frac{1}{2}\left(\frac{\partial g_{rj}}{\partial x^i} + \frac{\partial g_{ir}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^r}\right),$$

we can verify that

$$(6.9)\quad \begin{cases} [\alpha\beta^*;\gamma] = (\delta_{\gamma\beta}y^\alpha + \delta_{\alpha\gamma}y^\gamma)/8, \; [\alpha\beta,\gamma^*] = -(\delta_{\gamma\alpha}y^\beta + y^\alpha\delta_{\beta\gamma})/8, \\ [\alpha\beta^*,\Delta] = -\delta_{\alpha\beta}/8, \quad [\alpha\Delta,\gamma^*] = \delta_{\alpha\gamma}/8, \quad [\alpha^*\Delta,\gamma] = -\delta_{\alpha\gamma}/8, \\ \text{the other components are zero.} \end{cases}$$

(6. 8), (6. 9) and $\left\{ {h \atop ji} \right\} = g^{hr}[ji.\,r]$ imply that

$$(6.10)\quad \begin{cases} \left\{ {\mu \atop \alpha\beta^*} \right\} = \frac{1}{2}\delta_{\mu\beta}y^\alpha, \; \left\{ {\mu^* \atop \alpha\beta} \right\} = -\frac{1}{2}(\delta_{\alpha\mu}y^\beta + y^\alpha\delta_{\mu\beta}), \; \left\{ {\mu^* \atop \alpha\Delta} \right\} = \frac{1}{2}\delta_{\alpha\mu}, \\ \left\{ {\Delta \atop \alpha\beta^*} \right\} = \frac{1}{2}(y^\alpha y^\beta - \delta_{\alpha\beta}), \; \left\{ {\Delta \atop \Delta\beta^*} \right\} = -\frac{1}{2}y^\beta, \; \left\{ {\mu \atop \alpha^*\Delta} \right\} = -\frac{1}{2}\delta_{\alpha\mu}, \\ \text{the other components are zero.} \end{cases}$$

After straightforward calculations, we obtain the independent components of the curvature tensor R_{kjih} as follows.

$$(6.11)\quad \begin{cases} R_{\delta\gamma\beta\alpha} = (\delta_{\alpha\delta}y^\beta y^\gamma + \delta_{\beta\gamma}y^\alpha y^\delta - \delta_{\alpha\gamma}y^\beta y^\delta - \delta_{\beta\delta}y^\alpha y^\gamma)/16, \\ R_{\delta^*\gamma^*\beta\alpha} = (\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})/16, \\ R_{\delta\gamma\Delta\alpha} = (\delta_{\alpha\gamma}y^\delta - \delta_{\alpha\delta}y^\gamma)/16, \\ R_{\delta\gamma^*\beta^*\alpha} = (\delta_{\beta\gamma}y^\alpha y^\delta - 2\,\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta})/16, \\ R_{\delta\Delta\Delta\alpha} = \delta_{\alpha\delta}/16, \\ R_{\Delta\gamma^*\Delta\alpha^*} = -\delta_{\alpha\gamma}/16, \\ R_{\delta^*\Delta\beta^*\alpha} = \delta_{\beta\delta}y^\alpha/16, \\ \text{the other independent components are zero.} \end{cases}$$

According to (6. 8) and (6. 11), the Ricci tensor has the following components.

$$(6.12)\quad \begin{cases} R_{\delta\alpha} = -(\delta_{\alpha\delta} - my^\alpha y^\delta)/2, \qquad R_{\delta^*\alpha} = 0, \\ R_{\delta^*\alpha^*} = -\delta_{\alpha\delta}/2, \qquad R_{\Delta\Delta} = m/2, \qquad R_{\gamma^*\Delta} = 0, \\ R_{\gamma\Delta} = -my^\gamma/2. \end{cases}$$

Substituting (6. 7) and (6. 3) into (6. 12), we have

$$(6.13)\qquad R_{ji} = -2g_{ji} + 2(m+1)\eta_j\eta_i.$$

Thus the normal contact space defined above is an η-Einstein space and by virtue of (5. 3) and Lemma 5. 1, it is also an example of a space with constant scalar curvature whose Ricci tensor is not parallel.

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