$(\mathfrak{K}, p, \alpha)$ METHODS OF SUMMABILITY

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1. G. Sunouchi [3] has recently introduced some new methods of summability which are regular. These are defined in the following way. A series $\sum_{n=0}^{\infty} a_n$ is said to be summable $(\mathfrak{K}, \alpha)$ to s if the series in

$$f_1(t) = a_0 + \left(\int_0^{\infty} \frac{\sin x}{x^{\alpha+1}}\, dx\right)^{-1} \sum_{n=1}^{\infty} a_n \int_t^{\infty} \frac{\sin nu}{n^{\alpha} u^{\alpha+1}}\, du, \quad 0 < \alpha < 1,$$

converges in some interval $0 < t < t_0$ and $f_1(t) \to s$ as $t \to 0+$. A series $\sum_{n=0}^{\infty} a_n$ is said to be summable $(\mathfrak{K}^*, \alpha)$ to s if the series in

$$f_2(t) = a_0 + \left(\int_0^{\infty} \frac{\sin^2 x}{x^{\alpha+1}}\, dx\right)^{-1} \sum_{n=1}^{\infty} a_n \int_t^{\infty} \frac{\sin^2 nu}{n^{\alpha} u^{\alpha+1}}\, du, \quad 0 < \alpha < 1,$$

converges in some interval $0 < t < t_0$ and $f_2(t) \to s$ as $t \to 0+$.

It is purpose of this paper to obtain information about these Sunouchi's methods of summability and generalization of them. Throughout this paper, p denotes a positive integer and α denotes a real number, not necessarily an integer, such that $0 < \alpha < p$. Let us put

$$C_{p,\alpha} = \int_0^{\infty} \frac{\sin^p x}{x^{\alpha+1}}\, dx,$$

$$\varphi(n, t) \equiv \varphi(nt) \equiv (C_{p,\alpha})^{-1} \int_{nt}^{\infty} \frac{\sin^p x}{x^{\alpha+1}}\, dx = (C_{p,\alpha})^{-1} \int_t^{\infty} \frac{\sin^p nu}{n^{\alpha} u^{\alpha+1}}\, du.$$

Then a series $\sum_{n=0}^{\infty} a_n$ will be said to be summable $(\mathfrak{K}, p, \alpha)$ to s if the series in

$$f(p, \alpha, t) = a_0 + \sum_{n=1}^{\infty} a_n \varphi(nt)$$

converges in some interval $0 < t < t_0$ and $f(p, \alpha, t) \to s$ as $t \to 0+$. Under this definition, the $(\mathfrak{K}, \alpha)$ method and the $(\mathfrak{K}^*, \alpha)$ method are reduced to the

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$(\mathfrak{K}, 1, \alpha)$ method and the $(\mathfrak{K}, 2, \alpha)$ method, respectively. On the other hand, for a series $\sum_{n=0}^{\infty} a_n$, let us write $\sigma_n^\beta = s_n^\beta / A_n^\beta$, where s_n^β and A_n^β are defined by the relations

$$\sum_{n=0}^{\infty} A_n^\beta x^n = (1-x)^{-\beta-1} \quad \text{and} \quad \sum_{n=0}^{\infty} s_n^\beta x^n = (1-x)^{-\beta-1} \sum_{n=0}^{\infty} a_n x^n .$$

Then, if $\sigma_n^\beta \to s$ as $n \to \infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ is summable (C, β) to s. (See, for example, [4].) If $\sigma_n^\beta \to s$ as $n \to \infty$ and $\sum_{n=0}^{\infty} |\sigma_n^\beta - \sigma_{n+1}^\beta| < +\infty$, the series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|C, \beta|$ to s. It is well-known that, if the series $\sum_{n=0}^{\infty} a_n$ is summable (C, β) to 0, then $s_n^\gamma = o(n^\beta)$, $0 \leqq \gamma \leqq \beta$. Our main results in this paper are the following theorems.

THEOREM 1. *Let $0 < \beta < \alpha < p$. Then, if a series $\sum_{n=0}^{\infty} a_n$ is summable (C, β) to s, the series $\sum_{n=0}^{\infty} a_n$ is summable $(\mathfrak{K}, p, \alpha)$ to s.*

THEOREM 2. *Let $0 < \alpha < p$ and let $\lambda_n > 0$ $(n = 1, 2, \cdots)$ and the series $\sum_{n=1}^{\infty} \frac{\lambda_n}{n}$ converge. Then, if*

$$s_n^\alpha - s A_n^\alpha = o(n^\alpha \lambda_n) ,$$

the series $\sum_{n=0}^{\infty} a_n$ is summable $(\mathfrak{K}, p, \alpha)$ to s.

THEOREM 3. *Let $0 < \alpha < p$. Then, if a series $\sum_{n=0}^{\infty} a_n$ is summable $|C, \alpha|$ to s, the series $\sum_{n=0}^{\infty} a_n$ is summable $(\mathfrak{K}, p, \alpha)$ to s.*

2. Some Lemmas.

LEMMA 1. *Let $0 < \alpha < p$ and let $\Delta^m \varphi(n, t)$ denote the m-th difference of $\varphi(n, t)$ with respect to n. Then*

(2. 1) $$\Delta^m \varphi(n,t) = O(n^{-\alpha-1}t^{m-\alpha-1})$$

when m is a positive integer such that $m \leqq p+1$, and

(2. 2) $$\Delta^\mu \varphi(n,t) = O(n^{-\alpha}t^{\mu-\alpha})$$

when μ is a positive integer such that $\mu \leqq p$.

PROOF. By an extension of the mean value theorem in the differential calculus [1; p. 178], we have

(2. 3) $$\Delta^m \varphi(n,t) = (-1)^m (C_{p,\alpha})^{-1} t^m \left[\frac{d^m}{dx^m}\int_x^\infty \frac{\sin^p u}{u^{\alpha+1}}\,du\right]_{x=\theta},$$

where θ is some point such that $nt < \theta < (n+m)t$. Hence, for the proof of (2.1), it is sufficient to prove that

$$\left[\frac{d^m}{dx^m}\varphi(x)\right]_{x=\theta} = O((nt)^{-\alpha-1}),$$

or

(2. 4) $$\frac{d^m}{dx^m}\varphi(x) = O(x^{-\alpha-1}).$$

Now we have to show that, for $m \leqq p+1$,

$$\frac{d^m}{dx^m}\varphi(x) = -\frac{d^{m-1}}{dx^{m-1}}\left(\frac{\sin^p x}{x^{\alpha+1}}\right) = O(x^{-\alpha-1}).$$

An elementary calculation shows that, for $k \leqq p$,

$$\frac{d^k}{dx^k}\sin^p x = \eta_k(x)\sum_{\nu=0}^{k}\frac{1+(-1)^{k+\nu}}{2}\gamma_{k,\nu}\sin^{p-\nu} x,$$

where $\gamma_{k,\nu}$ are constants depending only on k and ν, and

$$\eta_k(x) = 1 \ (k;\ \text{even}),\ = \cos x \ (k;\ \text{odd}).$$

On the other hand

$$\frac{d^k}{dx^k}x^{-\alpha-1} = (-1)^k(\alpha+1)(\alpha+2)\cdots(\alpha+k)x^{-\alpha-k-1} \equiv \delta_k x^{-\alpha-k-1}, \text{ say.}$$

Then, by Leibnitz formula,

$$\frac{d^{m-1}}{dx^{m-1}}\left(\frac{\sin^p x}{x^{\alpha+1}}\right) = \sum_{k=0}^{m-1}\binom{m-1}{k}\delta_{m-k-1}\,\eta_k(x)\sum_{\nu=0}^{k}\frac{1+(-1)^{k+\nu}}{2}\,\gamma_{k,\nu}\,x^{k-m-\alpha}\sin^{p-\nu} x\,.$$

Since $0 \leqq \nu \leqq k \leqq m-1 \leqq p$, we get

$$x^{k-(m-1)}\sin^{p-\nu} x = O(x^{k-(m-1)}\sin^{p-k} x)$$

$$= \begin{cases} O\left(\dfrac{\sin^p x}{x^{m-1}}\left(\dfrac{x}{\sin x}\right)^k\right) = O(1) \quad (0 < x < 1)\,, \\ O\,(x^{k-(m-1)}) = O(1) \quad (x \geqq 1)\,. \end{cases} \tag{2. 5}$$

Then, by $\eta_k(x) = O(1)$, we have (2. 4). Therefore, by (2. 3),

$$\Delta^m\varphi(n,t) = O(t^m\theta^{-\alpha-1}) = O(n^{-\alpha-1}t^{m-\alpha-1})\,.$$

The proof of (2. 2) is similar to that of (2. 1). In this case, it is sufficient to prove that, when $0 \leqq \nu \leqq k \leqq \mu - 1 \leqq p-1$,

$$x^{k-\mu}\sin^{p-\nu} x = O(1)\,.$$

But this is easily proved as in (2. 5). Hence we have (2. 2).

LEMMA 2. *Let $0 < \gamma \leqq \alpha < p$. Then, for non-integral number γ,*

$$G(\gamma, k, t) \equiv \sum_{n=k}^{\infty} A_{n-k}^{-\gamma-1}\,\Delta\varphi(n,t) = O(k^{-\alpha-1}t^{\gamma-\alpha})\,, \tag{2. 6}$$

$$G(\gamma, k, t) = O(k^{[\gamma]-\gamma-\alpha}\,t^{[\gamma]-\alpha+1})^{2)} \tag{2. 7}$$

and

$$G(\gamma-1, k, t) = O(k^{[\gamma]-\gamma-\alpha}\,t^{[\gamma]-\alpha})\,. \tag{2. 8}$$

PROOF. We shall first prove (2. 6). Let $\rho=[1/t]$ and write

$$G(\gamma, k, t) = \left(\sum_{n=k}^{k+\rho-1} + \sum_{n=k+\rho}^{\infty}\right) = g_1(k,t) + g_2(k,t)\,,$$

say. Then, using (2. 1) for $m = 1$,

$$g_2(k,t) = O\left(\sum_{n=k+\rho}^{\infty}(n-k)^{-\gamma-1}\cdot n^{-\alpha-1}\,t^{-\alpha}\right)$$

2) Throughout this paper, $[x]$ denotes the greatest integer less than x.

$$= O\left(k^{-\alpha-1} t^{-\alpha} \sum_{n=k+\rho}^{\infty} (n-k)^{-\gamma-1}\right)$$

$$= O(k^{-\alpha-1} t^{\gamma-\alpha}).$$

By the repeated use of Abel transformation

$$g_1(k, t) = \sum_{n=0}^{\rho-1} A_n^{-\gamma-1} \Delta\varphi(n+k, t)$$

$$= \sum_{n=0}^{\rho-[\gamma]-2} A_n^{[\gamma]-\gamma} \Delta^{[\gamma]+2} \varphi(n+k, t) + \sum_{i=1}^{[\gamma]+1} A_{\rho-i}^{-\gamma+i-1} \Delta^i \varphi(k+\rho-i, t).$$

Since p is an integer, we have $[\gamma] + 2 \leqq p + 1$, and then, by (2.1),

$$g_1(k, t) = O\left(\sum_{n=0}^{\rho} (n+1)^{[\gamma]-\gamma} (n+k)^{-\alpha-1} t^{[\gamma]-\alpha+1}\right)$$

$$+ O\left(\sum_{i=1}^{[\gamma]+1} (\rho-i)^{-\gamma+i-1} (k+\rho-i)^{-\alpha-1} t^{i-\alpha-1}\right)$$

$$= O\left(k^{-\alpha-1} t^{[\gamma]-\alpha+1} \sum_{n=0}^{\rho} (n+1)^{[\gamma]-\gamma}\right) + O(k^{-\alpha-1} t^{\gamma-\alpha})$$

$$= O(k^{-\alpha-1} t^{\gamma-\alpha}).$$

Thus we have (2.6). Next we shall prove (2.7). By the repeated use of Abel transformation, we have, by (2.1),

$$G(\gamma, k, t) = \sum_{n=0}^{\infty} A_n^{-\gamma-1} \Delta\varphi(n + k, t)$$

$$= \sum_{n=0}^{\infty} A_n^{[\gamma]-\gamma} \Delta^{[\gamma]+2} \varphi(n + k, t)$$

$$= O\left(\sum_{n=0}^{k} (n+1)^{[\gamma]-\gamma} (n+k)^{-\alpha-1} t^{[\gamma]-\alpha+1}\right)$$

$$+ O\left(\sum_{n=k+1}^{\infty} n^{[\gamma]-\gamma} (n+k)^{-\alpha-1} t^{[\gamma]-\alpha+1}\right)$$

$$= O\left(k^{-\alpha-1} t^{[\gamma]-\alpha+1} \sum_{n=0}^{k} (n+1)^{[\gamma]-\gamma}\right)$$

$$+ O\left(k^{[\gamma]-\gamma}\, t^{[\gamma]-\alpha+1} \sum_{n=k}^{\infty} (n+k)^{-\alpha-1}\right)$$

$$= O(k^{[\gamma]-\gamma-\alpha}\, t^{[\gamma]-\alpha+1}),$$

which is the required result (2.7). Similarly we have

$$G(\gamma-1, k, t) = \sum_{n=0}^{\infty} A_n^{-\gamma}\, \Delta\varphi(n+k, t)$$

$$= \sum_{n=0}^{\infty} A_n^{[\gamma]-\gamma}\, \Delta^{[\gamma]+1}\, \varphi(n+k, t)$$

$$= O(k^{[\gamma]-\gamma-\alpha}\, t^{[\gamma]-\alpha}),$$

which is the required result (2.8).

3. PROOF OF THEOREM 1. We shall prove theorem when β is non-integral, the case of integral β being easily proved by the method analogous to the following argument. For the proof, we may assume, without loss of generality, that $a_0 = 0$, $s = 0$ and $\alpha - 1 < \beta$. Since

$$\varphi(n, t) = (C_{p,\alpha})^{-1} \int_{nt}^{\infty} \frac{\sin^p u}{u^{\alpha+1}}\, du = O(n^{-\alpha} t^{-\alpha}),$$

we have by Abel transformation and using $s_n^0 = o(n^\alpha)$,

$$f(p, \alpha, t) = \sum_{n=1}^{\infty} a_n\, \varphi(n, t) = \sum_{n=1}^{\infty} s_n^0\, \Delta\varphi(n, t).$$

Therefore, for the proof, it is sufficient to prove that the series

$$\sum_{n=1}^{\infty} s_n^0\, \Delta\varphi(n, t) \tag{3.1}$$

converges in some interval $0 < t < t_0$ and its sum tends to 0 as $t \to 0+$. By a well-known formula

$$s_n^0 = \sum_{k=0}^{n} A_{n-k}^{-\beta-1}\, s_k^\beta,$$

$$\sum_{n=1}^{\infty} s_n^0\, \Delta\varphi(n, t) = \sum_{k=1}^{\infty} s_k^\beta \sum_{n=k}^{\infty} A_{n-k}^{-\beta-1}\, \Delta\varphi(n, t) = \sum_{k=1}^{\infty} s_k^\beta\, G(\beta, k, t),$$

where

$$G(\beta, k, t) = \sum_{n=k}^{\infty} A_{n-k}^{-\beta-1} \Delta \varphi(n, t).$$

Here we must prove that this rearrangement is permissible. For this purpose, it is sufficient to show that, for a fixed $t > 0$,

$$I_N \equiv \sum_{k=1}^{N} s_k^\beta \sum_{n=N+1}^{\infty} A_{n-k}^{-\beta-1} \Delta \varphi(n, t) \to 0 \quad \text{as} \quad N \to \infty .$$

But this is easily proved as following. By (2.1) and $s_n^\beta = o(n^\beta)$,

$$\begin{aligned} I_N &= O\left(\sum_{k=1}^{N} |s_k^\beta| \sum_{n=N+1}^{\infty} (n-k)^{-\beta-1} \cdot n^{-\alpha-1}\right) \\ &= O\left(N^{-\alpha-1} \sum_{k=1}^{N} |s_k^\beta|\right) = o(N^{-\alpha-1} N^{\beta+1}) = o(1). \end{aligned}$$

Let us now write

$$\sum_{k=1}^{\infty} s_k^\beta G(\beta, k, t) = \left(\sum_{k=1}^{\rho-1} + \sum_{k=\rho}^{\infty}\right) = U(t) + V(t),$$

where $\rho = [1/t]$, $0 < t < 1/2$. Then using (2.6) and $s_n^\beta = o(n^\beta)$

$$V(t) = \sum_{k=\rho}^{\infty} s_k^\beta G(\beta, k, t) = o\left(\sum_{k=\rho}^{\infty} k^\beta \cdot k^{-\alpha-1} t^{\beta-\alpha}\right) = o(\rho^{\beta-\alpha} t^{\beta-\alpha}) = o(1),$$

when $t \to 0+$. Now, since $\beta - \alpha < 0$, the series (3.1) converges for every $t > 0$. On the other hand, using Abel transformation, we get, when $t \to 0+$,

$$\begin{aligned} U(t) &= \sum_{k=1}^{\rho-2} s_k^{\beta+1} (G(\beta, k, t) - G(\beta, k+1, t)) + s_{\rho-1}^{\beta+1} G(\beta, \rho-1, t) \\ &= o\left(\sum_{k=1}^{\rho} k^{\beta+1} \cdot k^{-\alpha-1} \cdot t^{\beta-\alpha+1}\right)^{3)} + o(\rho^{\beta+1} \cdot \rho^{-\alpha-1} \cdot t^{\beta-\alpha}) \\ &= o(\rho^{\beta-\alpha+1} t^{\beta-\alpha+1}) + o(\rho^{\beta-\alpha} t^{\beta-\alpha}) = o(1), \end{aligned}$$

in virtue of our assumption $\alpha - 1 < \beta$. Hence the sum of the series (3.1) tends to 0 when $t \to 0+$. Thus the theorem 1 is completely proved.

REMARK. In the proof of the theorem 1 when $\beta = p-1$, for the sake of estimating the sum $\sum_{n=1}^{\rho} s_n^{p-1} \Delta^p \varphi(n, t)$, we use the inequality (2.2).

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3) $G(\beta, k, t) - G(\beta, k+1, t) = O(k^{-\alpha-1} t^{\beta-\alpha+1})$ is proved by the method analogous to that of (2.6).

4. PROOF OF THEOREM 2. We shall prove the theorem in which α is non-integral, the theorem in which α is an integer being easily proved by the method analogous to the following argument. For the proof, we may assume, without loss of generality, that $a_0 = 0$ and $s = 0$. Then, as in the proof of the theorem 1, we have

$$\sum_{n=1}^{\infty} a_n \varphi(n, t) = \sum_{k=1}^{\infty} s_k^{\alpha} G(\alpha, k, t),$$

where

$$G(\alpha, k, t) = \sum_{n=k}^{\infty} A_{n-k}^{-\alpha-1} \Delta\varphi(n, t).$$

Let us now write

$$\sum_{k=1}^{\infty} s_k^{\alpha} G(\alpha, k, t) = \left(\sum_{k=1}^{N-1} + \sum_{k=N}^{\infty}\right) = U(t) + V(t),$$

say, where N is an arbitrary fixed positive integer. By (2. 6) with $\gamma = \alpha$,

$$V(t) = O\left(\sum_{k=N}^{\infty} k^{\alpha} \cdot \lambda_k \cdot k^{-\alpha-1}\right) = O\left(\sum_{k=N}^{\infty} k^{-1}\lambda_k\right).$$

From this, we see that, by the convergence of the series $\sum_{k=1}^{\infty} k^{-1}\lambda_k$, the series

$$\sum_{k=1}^{\infty} s_k^{\alpha} G(\alpha, k, t)$$

converges for every $t > 0$. On the other hand we have

$$\Delta\varphi(n, t) = O\left(t \cdot \frac{\sin^p \theta}{\theta^{\alpha+1}}\right), \quad nt < \theta < (n+1)t,$$

$$(4.\ 1) \qquad = \begin{cases} O(t) & \text{when } \alpha + 1 \leqq p, \\ O(t^{p-\alpha}) & \text{when } \alpha + 1 > p. \end{cases}$$

Hence

$$G(\alpha, k, t) = O\left(\sup_n |\Delta\varphi(n, t)| \cdot \sum_{n=k}^{\infty} |A_{n-k}^{-\alpha-1}|\right)$$

$$= \begin{cases} O(t) & \text{when } \alpha + 1 \leqq p, \\ O(t^{p-\alpha}) & \text{when } \alpha + 1 > p. \end{cases}$$

Therefore, since N is constant,

$$\lim_{t\to 0+} U(t) = 0 .$$

Then we have

$$\limsup_{t\to 0+} \left| \sum_{k=1}^{\infty} s_k^{\alpha} G(\alpha, k, t) \right| = O\left(\sum_{k=N}^{\infty} k^{-1} \lambda_k \right).$$

Since N is arbitrary and the series $\sum_{k=1}^{\infty} k^{-1}\lambda_k$ is convergent, we have

$$\lim_{t\to 0+} \sum_{k=1}^{\infty} s_k^{\alpha} G(\alpha, k, t) = 0 ,$$

and theorem is completely proved.

5. PROOF OF THEOREM 3. We shall prove the theorem in which α is non-integral, the theorem in which α is an integer being easily proved by the method analogous to the following argument. For the proof, we may assume, without loss of generality, that $a_0=0$ and $s=0$. Then, as in the proof of the theorem 1, we have

$$\sum_{n=1}^{\infty} a_n \varphi(n, t) = \sum_{k=1}^{\infty} s_k^{\alpha} G(\alpha, k, t) = \sum_{k=1}^{\infty} (\sigma_k^{\alpha} - \sigma_{k+1}^{\alpha}) U_m(t) ,$$

where

$$G(\alpha, k, t) = \sum_{n=k}^{\infty} A_{n-k}^{-\alpha-1} \Delta \varphi(n, t) \quad \text{and} \quad U_m(t) = \sum_{k=1}^{m} A_k^{\alpha} G(\alpha, k, t) ,$$

provided that

(5. 1) $$U_m(t) = O(1) \quad \text{for} \quad 0 < t < 1 \quad \text{and} \quad m = 1, 2, \cdots .$$

We shall now prove (5. 1). If $mt \leqq 1$, then, by (2 .7),

$$U_m(t) = O\left(\sum_{k=1}^{m} k^{\alpha} \cdot k^{[\alpha]-2\alpha} t^{[\alpha]-\alpha+1} \right)$$
$$= O\left(m^{[\alpha]-\alpha+1} t^{[\alpha]-\alpha+1} \right) = O(1) .$$

On the other hand, if $mt > 1$, then, putting $\rho = [1/t]$, $0 < t < 1/2$, we have, by the modefied Abel transformation ([2; Lemma 3]) and (2. 8),

$$
\begin{aligned}
U_m(t) &= \left(\sum_{k=1}^{\rho-1} + \sum_{k=\rho}^{m}\right) \\
&= O(1) + A_\rho^\alpha G(\alpha-1, \rho, t) - A_m^\alpha G(\alpha-1, m+1, t) + \sum_{k=\rho}^{m-1} A_{k+1}^{\alpha-1} G(\alpha-1, k+1, t) \\
&= O(1) + O(\rho^\alpha \cdot \rho^{[\alpha]-2\alpha} t^{[\alpha]-\alpha}) + O(m^\alpha \cdot m^{[\alpha]-2\alpha} t^{[\alpha]-\alpha}) + O\left(\sum_{k=\rho}^{\infty} k^{\alpha-1} \cdot k^{[\alpha]-2\alpha} t^{[\alpha]-\alpha}\right) \\
&= O(1) + O((\rho t)^{[\alpha]-\alpha}) + O((mt)^{[\alpha]-\alpha}) + O((\rho t)^{[\alpha]-\alpha}) = O(1)\,.
\end{aligned}
$$

Thus $U_m(t)$ is bounded uniformly in $0 < t < 1$ and for all positive integers m. Since the series $\sum_{k=1}^{\infty} |\sigma_k^\alpha - \sigma_{k+1}^\alpha|$ is convergent by our assumption, for an arbitrary small $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$
\left|\sum_{k=N}^{\infty} (\sigma_k^\alpha - \sigma_{k+1}^\alpha)\, U_k(t)\right| = O\left(\sum_{k=N}^{\infty} |\sigma_k^\alpha - \sigma_{k+1}^\alpha|\right) < \varepsilon\,.
$$

Further, using (4.1), we have, for a fixed N,

$$
\lim_{t\to 0+} \sum_{k=1}^{N-1} (\sigma_k^\alpha - \sigma_{k+1}^\alpha)\, U_k(t) = 0\,.
$$

Then we have

$$
\limsup_{t\to 0+} \left|\sum_{k=1}^{\infty} (\sigma_k^\alpha - \sigma_{k+1}^\alpha) U_k(t)\right| \leqq \varepsilon\,.
$$

Since ε is arbitrary, we have

$$
\lim_{t\to 0+} \sum_{k=1}^{\infty} (\sigma_k^\alpha - \sigma_{k+1}^\alpha)\, U_k(t) = 0\,,
$$

and the theorem is proved.

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