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DUAL SPACES OF TENSOR PRODUCTS OF C^*-ALGERAS

TAKATERU OKAYASU AND MASAMICHI TAKESAKI

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We shall use the notations and the terminologies employed in [9] and suppose that C^*-algebras in considerations are all separable. In [5], A. Guichardet studied the quasi-dual space of the tensor product $A_1 \widehat{\otimes}_\alpha A_2$ of C^*-algebras A_1 and A_2 and showed that there is an almost Borel isomorphism $\widetilde{\Pi}$ of $\widetilde{A}_1 \times \widetilde{A}_2$, the cartesian product of quasi-dual spaces $\widetilde{A}_1$ of A_1 and $\widetilde{A}_2$ of A_2, into the quasi-dual space $(A_1 \widehat{\otimes}_\alpha A_2)^\sim$ of $A_1 \widehat{\otimes}_\alpha A_2$. Also he showed that there is an example in which $\widetilde{\Pi}$ is not an onto mapping. In this note we shall show that there is a Borel isomorphism $\widehat{\Pi}_\beta$ of $\widehat{A}_1 \times \widehat{A}_2$, the cartesian product of dual spaces $\widehat{A}_1$ of A_1 and $\widehat{A}_2$ of A_2, into the dual space $(A_1 \widehat{\otimes}_\beta A_2)^\wedge$ of each tensor product $A_1 \widehat{\otimes}_\beta A_2$ of A_1 and A_2 with respect to a B^*-norm $\| \ \|_\beta$ and that $\widehat{\Pi}_\nu$ is an *onto* mapping if and only if one of A_1 and A_2 is of type I (or equivalently a GCR). Combining this and [9; Cor. of Theorem 3], we shall conclude that the dual space $(G_1 \times G_2)^\wedge$ of $G_1 \times G_2$, the direct product of separable locally compact groups G_1 and G_2, is Borel-isomorphic to the cartesian product $\widehat{G}_1 \times \widehat{G}_2$ of dual spaces $\widehat{G}_1$ of G_1 and $\widehat{G}_2$ of G_2 if and only if one of G_1 and G_2 is a group of type I.

For $n = 1, 2, \cdots, \infty$ (countably infinite), let H_n be a fixed n-dimensional Hilbert space. We identify the tensor product $H_n \otimes H_m$ and H_{nm} under some fixed isomorphism. For a separable C^*-algebra A, $\mathrm{Fac}_n(A)$ and $\mathrm{Irr}_n(A)$ are the set of all factor representations on H_n and the set of all irreducible representations on H_n respectively. Put $\mathrm{Fac}(A) = \bigcup_{n=1,2,\cdots,\infty} \mathrm{Fac}_n(A)$ and $\mathrm{Irr}(A) = \bigcup_{n=1,2,\cdots,\infty} \mathrm{Irr}_n(A)$. Each $\mathrm{Fac}_n(A)$ and $\mathrm{Irr}_n(A)$ have the Borel structure induced by the simple convergence topology respectively. The Borel structures in $\mathrm{Fac}(A)$ and $\mathrm{Irr}(A)$ are defined as the unions of Borel spaces $\mathrm{Fac}_n(A)$ and $\mathrm{Irr}_n(A)$, $n = 1, 2, \cdots \infty$, respectively. Of course, $\mathrm{Irr}(A)$ is a Borel subset of $\mathrm{Fac}(A)$ by [2]. The quasi-dual space $\widetilde{A}$ of a C^*-algebra A is the quotient Borel space $\mathrm{Fac}(A)/$"$\approx$" of Fac (A) by the quasi-equivalence relation "$\approx$" and the dual space $\widehat{A}$ of A

is the quotient Borel space $\mathrm{Irr}(A)/$"$\simeq$" of $\mathrm{Irr}(A)$ by the unitary equivalence relation "$\simeq$".

Let A_1 and A_2 be two C^*-algebras. For a B^*-norm $\| \ \|_\beta$ in the $*$-algebraic tensor product $A_1 \odot A_2$ of A_1 and A_2, A_β denote the completion $A_1 \widehat{\otimes}_\beta A_2$ of $A_1 \odot A_2$ under $\| \ \|_\beta$. For a representation π of A_β there exist representations π^1 of A_1 and π^2 of A_2 on the representation space of π such that

$$\pi^1(x_1)\pi^2(x_2) = \pi^2(x_2)\pi^1(x_1) = \pi(x_1 \otimes x_2) \text{ for } x_1 \in A_1 \text{ and } x_2 \in A_2$$

by [5; Prop.1]. We shall call π^1 and π^2 the restrictions of π to A_1 and to A_2 respectively. For each representations π_1 of A_1 and π_2 of A_2 the product representation $\pi_1 \otimes \pi_2$ of $A_1 \odot A_2$ can be extended to a representation of A_β, which is also denoted by $\pi_1 \otimes \pi_2$. Putting $\Pi(\pi_1, \pi_2) = \pi_1 \otimes \pi_2$, Π is a continuous mapping of $\mathrm{Fac}_n(A) \times \mathrm{Fac}_m(A)$ into $\mathrm{Fac}_{nm}(A)$ by [5; Lemme 2]. Moreover the relations $\pi_1 \approx \pi_1$ and $\pi_2 \approx \pi_2$ imply $\Pi(\pi_1, \pi_2) \approx \Pi(\pi_1', \pi_2')$ and the relations $\pi_1 \simeq \pi_2'$ and $\pi_2 \simeq \pi_2$ imply $\Pi(\pi_1, \pi_2) \simeq \Pi(\pi_1', \pi_2')$, so that Π induces naturally a Borel mapping $\widetilde{\Pi}$ of $\widetilde{A}_1 \times \widetilde{A}_2$ into $\widetilde{A}_\beta$ and a Borel mapping $\widehat{\Pi}$ of $\widehat{A}_1 \times \widehat{A}_2$ into $\widehat{A}_\beta$, respectively. If π^1 and π^2 are the restrictions of $\pi_1 \otimes \pi_2$ to A_1 and A_2 respectively then π^1 and π^2 are quasi-equivalent to π_1 and to π_2 respectively, so that $\widetilde{\Pi}$ and $\widehat{\Pi}$ are one-to-one mappings.

LEMMA 1. *A π of $\mathrm{Irr}(A_\beta)$ is unitarily equivalent to $\pi_1 \otimes \pi_2$ for some $\pi_1 \in \mathrm{Irr}(A_1)$ and $\pi_2 \in \mathrm{Irr}(A_2)$ if and only if one of the restrictions π^1 and π^2 of π is of type* I.

PROOF. Suppose $\pi \approx \pi_1 \otimes \pi_2$, $\pi_1 \in \mathrm{Irr}(A_1)$ and $\pi_2 \in \mathrm{Irr}(A_2)$. The unitary operator, which implements the equivalence between π and $\pi_1 \otimes \pi_2$, induces the equivalence between the corresponding restrictions of π and of $\pi_1 \otimes \pi_2$ to A_1 and A_2. Hence π^1 is quasi-equivalent to π_1, so that the irreducibility of π_1 implies our assertion. Similarly π^2 is of type I.

Conversely suppose π^1 is of type I. Let M_i be the von Neumann algebra generated by $\pi^i(A_i)$ for $i=1,2$. Then M_1 and M_2 commute each other and generate $B(H_\pi)$, which is the full operator algebra on the representation space H_π of π. M_1' contains M_2 and M_2' contains M_1, both M_1 and M_2 are factors. By the assumption for π^1, M_1 is a factor of type I, so that $B(H_\pi)$ is isomorphic to $M_1 \otimes M_1'$ under the natural correspondence $\sum_{i=1}^{n} x_i x_i' \longleftrightarrow \sum_{i=1}^{n} x_i \otimes x_i'$, $x_i \in M_1$, $x_i' \in M_1'$ and $i=1,2,\cdots,n$, where $M_1 \otimes M_1'$ means the tensor product of M_1 and M_1' as von Neumann algebras. The von Neumann algebra $R(M_1, M_2)$ generated by M_1 and M_2 is isomorphic to $M_1 \otimes M_2$, because M_2 is containd in M_1'. Hence we get $M_2 = M_1$, so that M_2 is also a factor of type I and $\pi \approx \pi^1 \otimes \pi^2$. Both π^1 and π^2 are factor representations of type I, so that there exist $\pi_1 \in \mathrm{Irr}(A_1)$ and $\pi_2 \in \mathrm{Irr}(A_2)$ such that π_1 and π_2 are quasi-equivalent to π^1 and π^2 respectively. Hence π is quasi-equivalent to $\pi_1 \otimes \pi_2$ by the remark preceeding our lemma. The irreducibilities of both π and $\pi_1 \otimes \pi_2$ and their quasi-equivalence imply their unitary equivalence. This completes the proof.

THEOREM 1. *$\widehat{\Pi}$ is a Borel isomorphism of $\widehat{A}_1 \times \widehat{A}_2$ into $\widehat{A}_\beta$ for each B^*-norm $\| \ \|_\beta$ in $A_1 \odot A_2$.*

PROOF. We shall prove that $\widehat{\Pi}(\widehat{E}_1 \times \widehat{E}_2) = \widehat{E}$ is a Borel subset of A_β for every Borel subsets $\widehat{E}_1$ of $\widehat{A}_1$ and $\widehat{E}_2$ of $\widehat{A}_2$. Let Θ_1, Θ_2 and Θ be the canonical mapping of $\mathrm{Irr}(A_1)$, $\mathrm{Irr}(A_2)$, and $\mathrm{Irr}\ (A_\beta)$ onto $\widehat{A}_1$, $\widehat{A}_2$ and $\widehat{A}_\beta$ respectively, then it suffices to prove that $\Theta^{-1}(\widehat{E})$ is a Borel subset of $\mathrm{Irr}(A_\beta)$. Putting $E_1 = \Theta_1^{-1}(\widehat{E}_1)$, $E_2 = \Theta_2^{-1}(\widehat{E}_2)$ and $E = \Theta^{-1}(\widehat{E})$, we shall prove at first

$$(*) \qquad E = \{\pi \in \mathrm{Irr}(A_\beta); \pi^1 \approx \pi_1,\ \pi^2 \approx \pi_2 \text{ for some } (\pi_1, \pi_2) \in E_1 \times E_2\},$$

where π^1 and π^2 mean the restrictions of π to A_1 and A_2 respectively. Let F be the set of the right side of the above equation. If π belongs to E, then we have $\Theta(\pi) = \widehat{\Pi}(\pi_1, \pi_2)$ for some $(\widehat{\pi}_1, \widehat{\pi}_2) \in \widehat{E}_1 \otimes \widehat{E}_2$. By the definitions of E_1 and E_2 there exists $(\pi_1, \pi_2) \in E_1 \times E_2$ such that $\Theta_1(\pi_1) = \widehat{\pi}_1$ and $\Theta_2(\pi_2) = \widehat{\pi}_2$. From the commutativity of the diagram of mappings

$$\begin{tikzcd}
\mathrm{Irr}(A_1) \times \mathrm{Irr}(A_2) \arrow[r, "\Pi"] \arrow[d, "\Theta_1 \times \Theta_2"] & \mathrm{Irr}(A_\beta) \arrow[d, "\Theta"] \\
\widehat{A}_1 \times \widehat{A}_2 \arrow[r, "\widehat{\Pi}"] & \widehat{A}_\beta
\end{tikzcd}$$

π is unitary equivalent to $\pi_1 \otimes \pi_2$, where $\Theta_1 \times \Theta_2$ is the mapping defined by $\Theta_1 \times \Theta_2(\pi_1, \pi_2) = (\Theta_1(\pi_1), \Theta_2(\pi_2))$. Hence we get $\pi_1 \approx \pi^1$ and $\pi_2 \approx \pi^2$, that is, π belongs to F. Conversely, suppose π belongs to F. That is, π^1 and π^2 are quasi-equivalent to π_1 of E_1 and π_2 of E_2 respectively. The irreducibilities of π_1 and π_2 imply that π^1 and π^2 are factor representations of type I. From Lemma 1 and its proof π is unitarily equivalent to $\pi_1 \otimes \pi_2$, so that we have $\Theta(\pi) = \widehat{\Pi}$ $(\Theta_1(\pi_1), \Theta_2(\pi_2))$. Hence we have $\Theta(\pi) \in \widehat{E}$. The definition of E implies $\pi \in E$. Thus we established the equation (*).

Since E_1 is a Borel subset of $\mathrm{Irr}(A_1)$, E_2 a Borel subset of $\mathrm{Irr}(A_2)$ and these are saturated under the unitary equivalence, the saturations E_1' of E_1 and E_2' of E_2 under the quasi-equivalence are Borel subsets of $\mathrm{Fac}(A_1)$ and $\mathrm{Fac}(A_2)$ respectively by [2; Lemma 5]. Moreover the mapping Π'; $\mathrm{Fac}(A_\beta) \ni \pi \to (\pi^1, \pi^2) \in \mathrm{Fac}(A_1) \times \mathrm{Fac}(A_2)$ is a Borel mapping by [5; Lemme 3]. Hence $E = \Pi'^{-1}(E_1' \times E_2') \cap \mathrm{Irr}(A_\beta)$ is a Borel subset of $\mathrm{Irr}(A_\beta)$.

Let $\mathfrak{B}$ be the family consisting of all subsets $\widehat{E}$ of $\widehat{A}_1 \times \widehat{A}_2$ such that $\widehat{\Pi}(\widehat{E})$ is a Borel subsets in $\widehat{A}_\beta$. Since $\widehat{\Pi}$ is an one-to-one mapping, $\widehat{\Pi}$ preserves all set-theoretic operations, union, intersection and difference. $\mathfrak{B}$ is a σ-ring of subsets of $\widehat{A}_1 \times \widehat{A}_2$. Since $\mathfrak{B}$ contains all product sets of Borel subsets of $\widehat{A}_1$ and $\widehat{A}_2$ as proved above and the Borel structure of $\widehat{A}_1 \times \widehat{A}_2$ is the smallest σ-ring containing all product sets of Borel subsets of $\widehat{A}_1$ and $\widehat{A}_2$, $\mathfrak{B}$ contanis the Borel structure of $\widehat{A}_1 \times \widehat{A}_2$. Thus $\widehat{\Pi}(\widehat{E})$ is a Borel set in $\widehat{A}_\beta$ for every Borel set $\widehat{E}$ in $\widehat{A}_1 \times \widehat{A}_2$, that is, $\widehat{\Pi}$ is an into Borel isomorphism. This completes the proof.

LEMMA 2. *If M_1 and M_2 are von Neumann algebras whose commutators M_1' and M_2' are continuous hyperfinite factors, then there exist normal representations π_1 of M_1 and π_2 of M_2 on the same Hilbert space such that $\pi_1(M_1) = \pi_2(M_2)$ and equivalently $\pi_1(M_1) = \pi_2(M_2)'$.*

PROOF. If M_1 is finite, then it is a continuous hyperfinite factor by [8; Theorem XV]. By the unicity of continuous hyperfinite factors M_1 is isomorphic to M_1' and also to M_2'. Hence there exists an isomorphism π_1 of M_1 onto M_2', so that the couple of π_1 and the identity representation π_2 of M_2 is the desired one. If M_1 is an infinite factor, there exist a factor of type II_1 and an infinite factor N of type I such that $M_1 = M \otimes N$. Hence we may assume $M_1' = M' \otimes \{\lambda 1\}$, representing N as the full operator algebra on a Hilbert space. Hence M_1 is isomorphic to M', so that M is a continuous hyperfinite factor by the finiteness of M. Thus M is isomorphic to M_2. On the other hand, the ampliation $M_2 \ni x_2 \to x_2 \otimes 1 \in M_2 \otimes \{\lambda 1\}$ is an isomorphism and $(M_2 \otimes \{\lambda 1\})' = M_2 \otimes N \cong M \otimes N = M_1$. Taking π_1 as an isomorphism of M_1 onto $M_2 \otimes N$, the couple of of the representation π_1 of M_1 and the representation π_2 of M_2 which is obtained by $\pi_2(x_2) = x_2 \otimes 1$ for $x_2 \in M_2$ is the desired one.

THEOREM 2. *$\widehat{\Pi}$ is a Borel isomorphism of $\widehat{A}_1 \times \widehat{A}_2$ onto $\widehat{A}_\nu$ if and only if one of A_1 and A_2 is of type I (or equivalentely a GCR). In this case the ν-norm in $A_1 \odot A_2$ coincides with the α-norm.*

PROOF. Suppose that neither A_1 nor A_2 is of type I. By the proof of [3; Theorem 1] there exist representations π_1 of A_1 and π_2 of A_2 such that the commutators of $\pi_1(A_1)$ and $\pi_2(A_2)$ are continuous hyperfinite factors respectively. Then there exist normal representations ρ_1 of the von Neumann algebra M_1 generated by $\pi_1(A_1)$ and ρ_2 of the von Neumann algebra M_2 generated by $\pi_2(A_2)$ such that $\rho_1(M_1)$ and $\rho_2(M_2)$ are commutators in each other from Lemma 2. We define a representation π of A_ν as the extension of the representation of $A_1 \odot A_2$ defined by

$$\pi\left(\sum_{k=1}^{n} x_{1,k} \otimes x_{2,k}\right) = \sum_{k=1}^{n} (\rho_1 \circ \pi_1)(x_{1,k})(\rho_2 \circ \pi_2)(x_{2,k}) \quad \text{for} \sum_{k=1}^{n} x_{1,k} \otimes x_{2,k} \in A_1 \odot A_2.$$

Since the von Neumann algebra generated by $\pi(A_\nu)$ contains $\rho_1(M_1)$ and $\rho_2(M_2)$, π becomes an irreducible representation of A_ν. But π can not be represented as a tensor product of representations of A_1 and A_2. Because if π is represented as $\sigma_1 \otimes \sigma_2$, σ_1 and σ_2 representations of A_1 and A_2 respectively, then σ_1 is quasi-equivalent to $\rho_1 \circ \pi_1$, σ_2 to $\rho_2 \circ \pi_2$, and then π must be of type II, which is a contradiction to the irreducibility of π. Hence π does not belong to $\Pi(\mathrm{Irr}(A_1) \times \mathrm{Irr}(A_2))$. Hence $\widehat{\Pi}$ is not an onto mapping. The converse implication is an immediate consequence of Lemma 1. The final assertion is nothing but [10; Theorem 3]. This completes the proof.

Combining our theorem and [9; Cor. of Theorem3], we get the following application to the dual space of direct product of locally compact groups.

COROLLARY. *Let G_1 and G_2 be separable locally compact groups. The natural mvpping $\widehat{\Pi}$ of the cartesian prodhct $\widehat{G}_1 \times \widehat{G}_2$ of the dual spaces $\widehat{G}_1$ of $\widehat{G_1}$ and $\widehat{G}_2$ of G_2 into the dual space $(G_1 \times G_2)^\wedge$ of the direct product group $G_1 \times G_2$ is a Borel isomorphism. $\widehat{\Pi}$ maps $\widehat{G}_1 \times \widehat{G}_2$ onto $(G_1 \times G_2)^\wedge$ if and only if one of G_1 and G_2 is of type* I.

In general, for a locally compact group G, there is a natural mapping of $\widehat{G}$ onto $C^*(G)^\wedge$ which is also a Borel isomorphism, the proof is directly followed from Theorem 2 and [9; Cor. of Theorem 3].

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TÔHOKU UNIVERSITY.