Tôhoku Math. Journ.
Vol. 18, No. 4, 1966

NOTE ON THE INTEGRABILITY CONDITIONS OF (ϕ, ψ)-STRUCTURES

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(Received April 27, 1966)

Some years ago, Professor S. Sasaki proposed an open question on the integrability conditions of (ϕ, ψ)-structure defind in his paper [1][1]. A (ϕ, ψ)-structure on an n-dimensional differentiable manifold M^n is defined by two tensor fields ϕ, ψ of type $(1, 1)$ satisfying the conditions as follows:

$$\operatorname{rank} \phi = l\,, \quad \operatorname{rank} \psi = m\,, \quad l + m = n$$

$$\phi\psi = \psi\phi = 0\,, \quad \varepsilon\phi^2 + \varepsilon'\psi^2 = 1\,,$$

where 1 denotes the unit tensor and $\varepsilon, \varepsilon'$ are plus or minus one. Such structures contain as a special case the almost contact structures [2].

In this short note we intend to show that the integrability conditions of (ϕ, ψ)-structure can be obtained by using a result on the integrability conditions of π-structure by the first author, provided that the structure is of class C^ω. At the same time, we also improve some results by the both authors [4], [5]. It is furthermore shown that the integrability of the structure f satisfying $f^3 + f = 0$ studied by K. Yano and S. Ishihara [6] can also be derived in this way, if the structure is of class C^ω.

1. An n-dimensional manifold is said to be endowed with an r-π-structure if there exist r distributions (differentiable) $T_1, \cdots, T_r$ of (complex) tangent subspaces such that $T_p^c = T_{1p} + \cdots + T_{rp}$ (direct sum) holds at each point, where T_p^c is the complexification of the tangent space at P and T_{tp} is the subspace at P belonging to the distribution T_t; $t = 1, \cdots, r$.

An r-π-structure defined by r distributions T_t is said to be integrable if at each point of the manifold, there is a complex coordinate system such that the subspace T_t of complexified tangent space is represented as $dz^{\bar{\alpha}_t} = 0$, i.e. $dz^i = 0$ except dz^{α_t} where α_t varies from $n_1 + \cdots + n_{t-1}$ to $n_1 + \cdots + n_t$ ($n_t = \dim T_t$, $n_0 = 0$) $t = 1, \cdots, r$.

1) Numbers in brackets refer to the reference at the end of the paper.

It is proved that an r-π-structure of class C^ω is integrable if and only if [3]

$$T(u, v) = -\sum_{\alpha=1}^{r} P_\alpha N(P_\alpha)(u, v) = 0 \tag{1}$$

for any pair (u, v) of vector fields, where $N(P_\alpha)$ denotes the Nijenhuis tensor for the projection tensor field P_α to the α-th distribution given by the r-π-structure.

This is equivalent to the following set of conditions:

$$P_1 N(P_1)(u, v) = 0, \cdots, P_r N(P_r)(u, v) = 0. \tag{2}$$

For example, the first condition can be written as

$$P_1[P_1u, P_1v] - P_1[u, P_1v] - P_1[P_1u, v] + P_1[u, v] = 0. \tag{3}$$

Another expression for this condition is

$$P_1[(P_2 + \cdots + P_r)u, (P_2 + \cdots + P_r)v] = 0, \tag{4}$$

which is also equivalent to the following set:

$$P_1[P_ju, P_kv] = 0; \quad j, k = 2, \cdots, r. \tag{5}$$

These are obtained from (4) by putting P_ju, P_kv in the place of u and v. Thus the integrability conditions (2) is also given by the following set:

$$P_i[P_ju, P_kv] = 0, \ i = 1, \cdots, r \tag{6}$$

$$j, k \text{ being any number in } \{(1, 2, \cdots, r) - (i)\}.$$

2. Now suppose there are given two tensor fields F_1, F_2 of type (1, 1) and of class C^ω such that [4]

$$F_1^2 = \lambda_1^2 I, \ F_2^2 = \lambda_2^2 I \ \text{ and } \ F_1F_2 = F_2F_1 \tag{7}$$

where I also denotes unit tensor field. If we put $F_1F_2 = F_2F_1 = -F_3$, then it follows that $F_3^2 = \lambda_1^2\lambda_2^2 I$, λ_1, λ_2 are non zero complex constants.

It is shown that such structure either defines a 3-π-structure (this case is characterized by $\frac{1}{\lambda_2}F_2 - \frac{1}{\lambda_1}F_1 - \frac{1}{\lambda_1\lambda_2}F_3 = I$ for suitably chosen square roots

λ_1 of λ_1^2 and λ_2 of λ_2^2) or a 4-π-structure. Such structure is said to be integrable if at each point of the manifold, there exists a coordinate system in which the fields F_1, F_2, F_3 have simultaneously numerical components.

It is also known that *such structure is integrable if and only if the corresponding* 3-π-*structure or* 4-π-*structure is integrable* [4].

CASE I. If it defines a 3-π-structure, then we can express them as

$$\begin{aligned} (8) \qquad & F_1 = \lambda_1(P_1-P_2-P_3), \qquad F_2 = \lambda_2(P_1+P_2-P_3), \\ & F_3 = -\lambda_1\lambda_2(P_1-P_2+P_3), \qquad I = P_1+P_2+P_3, \end{aligned}$$

where P_1, P_2, P_3 are three projection tensors to the distributions defined by the structure $\{F_1, F_2\}$.

It is well-known that the integrability condition of the structure defined by F_1 is $[F_1, F_1]=0$, that is

$$(9) \qquad (P_2+P_3)[P_1u, P_1v] = 0 \quad \text{and} \quad P_1[(P_2+P_3)u, (P_2+P_3)v] = 0\,.$$

This latter condition is equivalent to $P_1N(P_1) = 0$.

Therefore, the integrability conditions (6) for the case $r=3$ is equivalent to

$$(10) \qquad [F_1, F_1] = 0\,, \quad [F_2, F_2] = 0 \quad \text{and} \quad [F_3, F_3] = 0\,.$$

Thus we have

PROPOSITION 1. *The integrability condition of the class* C^ω *structure* $\{F_1, F_2\}$ *giving a* 3-π-*distribution is the set* (10).

It can be shown that

PROPOSITION 2. *The integrability condition of the class* C^ω *structure* $\{F_1, F_2\}$ *related to a* 3-π-*structure may also be given by*

$$(11) \qquad [F_1, F_1] = 0, \quad [F_2, F_2] = 0 \quad \text{and} \quad [F_1, F_2] = 0$$

where $[F_1, F_2]$ *is defined by the following*:

$$\begin{aligned} (12) \qquad [F_1, F_2](u, v) = {} & [F_1u, F_2v] - F_1[u, F_2v] - F_2[F_1u, v] + F_1F_2[u, v] \\ & + [F_2u, F_1v] - F_2[u, F_1v] - F_1[F_2u, v] + F_2F_1[u, v]\,. \end{aligned}$$

Since it is trivial that

$$(13)\quad [\alpha F_1+\beta F_2, \lambda F_1+\mu F_2] = 2\alpha\lambda[F_1, F_1] + 2\beta\mu[F_2, F_2] + (\alpha\mu+\beta\lambda)[F_1, F_2],$$

the set (11) is equivalent to the set

$$(14)\qquad [F_1, F_1] = 0, \quad [F_2, F_2] = 0 \quad \text{and} \quad [\alpha F_1+\beta F_2, \alpha F_1+\beta F_2] = 0$$

with $\alpha\beta \neq 0$. From (8) we have $(1/2)(I+\frac{1}{\lambda_1\lambda_2}F_3)=P_2$. As $[I,I]=0$ and $[F,I] = 0$, it follows that

$$(15)\quad [P_2, P_2] = \left[(1/2)\left(I+\frac{1}{\lambda_1\lambda_2}F_3\right),\ (1/2)\left(I+\frac{1}{\lambda_1\lambda_2}F_3\right)\right] = \frac{1}{4\lambda_1{}^2\lambda_2{}^2}[F_3F_3].$$

P_2 can also be expressed as $P_2=(1/2)\left(\frac{1}{\lambda_2}F_2-\frac{1}{\lambda_1}F_1\right)$, therefore

$$(16)\qquad [P_2P_2] = (1/4)\left[\frac{1}{\lambda_2}F_2 - \frac{1}{\lambda_1}F_1,\ \frac{1}{\lambda_2}F_2 - \frac{1}{\lambda_1}F_1\right].$$

The above proposition follows from (15) and (16).

It is to be noted that

PROPOSITION 3. *The condition*

$$(17)\qquad \left[\frac{1}{\lambda_1}F_1 + \frac{1}{\lambda_2}F_2,\ \frac{1}{\lambda_1}F_1 + \frac{1}{\lambda_2}F_2\right] = 0$$

alone also gives the integrability condition of the class C^ω structure $\{F_1, F_2\}$ related to a 3-π-structure.

By (8), this condition may also be written as

$$(18)\qquad [P_1-P_3, P_1-P_3] = [P_1, P_1] + [P_3, P_3] - [P_1, P_3] = 0.$$

As $P_1+P_3 = I-P_2$ we have $[P_2, P_2] = [P_1+P_3, P_1+P_3] = [P_1, P_1] + [P_3, P_3] + [P_1, P_3]$. Thus (17) is written as

$$(19)\qquad 2[P_1, P_1] + 2[P_3, P_3] - [P_2, P_2] = 0.$$

This is equivalent to

$$(20)\qquad 2P_i[P_1,P_1] + 2P_i[P_3,P_3] - P_i[P_2, P_2] = 0\,; \quad i = 1,2,3,$$

from which we have

$$(21)\qquad \begin{cases} P_1[P_2u, P_2v] + 4P_1[P_3u,P_3v] + 2P_1[P_2u,P_3v] + 2P_1[P_3u,P_2v] = 0 \\ P_2[P_1u,P_1v] + P_2[P_3u,P_3v] - P_2[P_1u, P_3v] - P_2[P_3u, P_1v] = 0 \\ 4P_3[P_1u, P_1v] + P_3[P_2u, P_2v] + 2P_3[P_1u,P_2v] + 2P_3[P_2u,P_1v] = 0\,. \end{cases}$$

This system is equivalent to (6) for the case $r=3$.

3. The above result can be used to derive the integrability conditions of the structure defined by a non-null tensor field f of type $(1,1)$ and of class C^ω satisfying $f^3+f=0$ [6]. If the rank of f is r, then in the tangent space at every point, the kernel of f is an $(n-r)$-dimensional subspace and the range of f is the complementary r-dimensional subspace on which f acts as $f^2=-1$. Thus we get a 3-π-structure such that

$$(22)\qquad f = i(P_1-P_3),\quad f^2+1 = P_2,\quad P_1+P_2+P_3 = 1\,,$$

and therefore, (18) is equivalent to $[f,f]=0$. Thus we have:

PROPOSITION 4. *The necessary and sufficient condition for the structure f of class C^ω to be integrable is $[f,f]=0$. By the above mentioned result, this is equivalent to the fact that there exists a coordinate system in which f has constant components.* [6]

4. CASE II. If the structure $\{F_1, F_2, F_3\}$ defines a 4-π-structure then we have

$$(23)\qquad \begin{cases} F_1 = \lambda_1(P_1+P_2-P_3-P_4)\,, & F_2 = \lambda_2(P_1-P_2+P_3-P_4)\,, \\ F_3 = -\lambda_1\lambda_2(P_1-P_2-P_3+P_4)\,, & I = P_1+P_2+P_3+P_4\,, \end{cases}$$

with P_i as the projection tensors to the four distributions. In this case the integrability condition $[F_1, F_1]=0$ for the structure F_1 is written as:

$$(24)\qquad \begin{cases} (P_1+P_2)[(P_3+P_4)u,\ (P_3+P_4)v] = 0\,, \\ (P_3+P_4)[(P_1+P_2)u,\ (P_1+P_2)v] = 0\,. \end{cases}$$

This set is equivalent to the following set.

$$
(25)\qquad \begin{cases} P_1[P_k u, P_l v] = 0\,, \quad P_2[P_k u, P_l v] = 0\,; \quad k, l = 3, 4, \\ P_3[P_i u, P_j v] = 0\,, \quad P_4[P_i u, P_j v] = 0\,; \quad i, j = 1, 2\,. \end{cases}
$$

Thus we have

PROPOSITION 1. *The integrability conditions for the class C^ω structure $\{F_1, F_2, F_3\}$ related to a 4-π-structure is given by*

$$
(26)\qquad [F_1, F_1] = 0\,, \quad [F_2, F_2] = 0 \quad \text{and} \quad [F_3, F_3] = 0\,.
$$

Because this set is equivalent to the integrability conditions (6) for the case $r=4$. This proposition together with Proposition 1 improve a theorem stated in [4] by deleting a redundant condition.

It can also be shown that

PROPOSITION 2′. *The integrability conditions for the class C^ω structure $\{F_1, F_2, F_3\}$ giving a 4-π-structure are* [5]:

$$
(27)\qquad [F_1, F_1] = 0\,, \quad [F_2, F_2] = 0 \quad \text{and} \quad [F_1, F_2] = 0\,.
$$

It is obvious that in the set (27) the condition $[F_1, F_2] = 0$ may be replaced by

$$
(28)\qquad [\alpha F_1 + \beta F_2\,, \alpha F_1 + \beta F_2] = 0 \quad \text{with} \quad \alpha\beta \neq 0\,.
$$

Take $\alpha F_1 + \beta F_2 = (1/2)\left(\frac{1}{\lambda_1} F_1 + \frac{1}{\lambda_2} F_2\right) = P_1 - P_4$, then (28) can be written

$$
(29)\qquad [P_1 - P_4, P_1 - P_4] = [P_1 P_1] + [P_4 P_4] - [P_1 P_4] = 0
$$

which is in turn equivalent to

$$
(30)\qquad P_i([P_1 P_1] + [P_4 P_4] - [P_1 P_4]) = 0\,; \quad i = 1, 2, 3, 4\,.
$$

These conditions are respectively written as

$$
\begin{cases} P_1[(P_2 + P_3 + P_4)u,\ (P_2 + P_3 + P_4)v] + P_1[P_4 u\, P_4 v] \\ \qquad + P_1[(P_2 + P_3 + P_4)u,\ P_4 v] + P_1[P_4 u,\ (P_2 + P_3 + P_4)v] = 0 \end{cases}
$$

$$(31)\quad \begin{cases} P_2[P_1u, P_1v] + P_2[P_4u, P_4v] - P_2[P_1u, P_4v] - P_2[P_4u, P_1v] = 0\,, \\ P_3[P_1u, P_1v] + P_3[P_4u, P_4v] - P_3[P_1u, P_4v] - P_3[P_4u, P_1v] = 0\,, \\ P_4[(P_1+P_2+P_3)u,\ (P_1+P_2+P_3)v] + P_4[P_1u, P_1v] \\ \qquad + P_4[P_1u,\ (P_1+P_2+P_3)v] + P_4[(P_1+P_2+P_3)u, P_1v] = 0\,. \end{cases}$$

These give 26 conditions in all, and which cover all conditions supplied by

$$[F_3F_3] = 0 \text{ (i.e., } [P_1+P_4, P_1+P_4] = 0 \quad \text{or} \quad [P_2+P_3, P_2+P_3] = 0)\,.$$

Since $[P_2-P_3,\ P_2-P_3] = 0$ gives analoguous conditions which together with (31) cover all conditions in (6) for the case $r=4$, we have

PROPOSITION 5. *The following set is also the integrability conditions for the class C^ω structure $\{F_1, F_2, F_3\}$ giving a 4-π-structure*:

$$(32)\quad \begin{aligned} &\Big[(1/2)\Big(\frac{1}{\lambda_1}F_1 + \frac{1}{\lambda_2}F_2\Big),\ (1/2)\Big(\frac{1}{\lambda_1}F_1 + \frac{1}{\lambda_2}F_2\Big)\Big] = [P_1-P_4, P_1-P_4] = 0\,, \\ &\Big[(1/2)\Big(\frac{1}{\lambda_1}F_1 - \frac{1}{\lambda_2}F_2\Big),\ (1/2)\Big(\frac{1}{\lambda_1}F_1 - \frac{1}{\lambda_2}F_2\Big)\Big] = [P_2-P_3, P_2-P_3] = 0\,. \end{aligned}$$

5. We are now going to discuss on the integrability conditions for the (ϕ, ψ)-structures of class C^ω. (ϕ, ψ)-structures can be divided into the following three cases:

1°. $\psi^2-\phi^2 = 1\,,\quad \phi\psi = 0\,,\quad \psi\phi = 0\quad$ and $\quad\psi^3 = \psi\,,\ \phi^3 = -\phi\,;$

2°. $\psi^2+\phi^2 = -1,\quad \phi\psi = 0\,,\quad \psi\phi = 0\quad$ and $\quad\psi^3 = -\psi,\ \phi^3 = -\phi\,;$

3°. $\psi^2+\phi^2 = 1\,,\quad \phi\psi = 0\,,\quad \psi\phi = 0\quad$ and $\quad\psi^3 = \psi\,,\ \phi^3 = \phi\,.$

For the case 1°, put $\psi^2=P_2$ and $-\phi^2 = P_1$. Then $P_1+P_2 = 1$ and $P_1P_2 = P_2P_1=0$. Thus we have two projection tensor fields. The subspaces of complexified tangent space at a point corresponding to P_1 and P_2 are denoted as R_1 and R_2 respectively. Then R_2 consists of the vectors of the form $\psi^2u = P_2u$, R_1 consists of the vectors of the form $-\phi^2u = P_1u$. Since $\psi P_2u=\psi^3u = \psi^2(\psi u)=P_2(\psi u)$, $\psi P_1u=-\psi\phi^2u=0$, and $\psi^2P_2u=P_2^2u=P_2u$, the transformation ψ acts on R_2 as $\psi^2=1$ and kernel $\psi=R_1$. Similarly, $\phi P_2u = \phi\psi^2u=0$,

$\phi P_1 u = -\phi^2(\phi u) = P_1(\phi u)$ and $\phi^2 P_1 u = -P_1^2 u = -P_1 u$, so ϕ acts on R_1 as $\phi^2 = -1$ and kernel $\phi = R_2$.

The proper values of ψ on R_2 are 1 or -1. If ψ has only one proper value 1 on R_2, then $\psi P_2 u = P_2 u$ and $\psi u = \psi(P_1 + P_2)u = \psi P_2 u = P_2 u$ for all u. Thus $\psi = P_2$ and $\psi^2 = P_2^2 = P_2 = \psi$. If ψ has only one proper value -1 on R_2, then $\psi P_2 u = -P_2 u$ and $\psi u = \psi(P_1 + P_2)u = \psi P_2 u = -P_2 u$ for all u. Thus $\psi = -P_2$ and $\psi^2 = P_2^2 = P_2 = -\psi$.

If ψ has both proper values 1 and -1 on R_2, denote the subspaces of R_2 corresponding to 1 and -1 as R_{22} and R_{23}. Let P_{22}, P_{23} be the projection tensors to R_{22} and R_{23}, then $P_2 = P_{22} + P_{23}$ and $P_1 P_{22} = P_{22} P_1 = P_1 P_{23} = P_{23} P_1 = 0$. Then $\psi P_1 u = 0$, $\psi P_{22} u = P_{22} u$, $\psi P_{23} u = -P_{23} u$ and $\psi u = \psi(P_1 + P_{22} + P_{23})u = \psi P_{22} u + \psi P_{23} u = (P_{22} - P_{23})u$. Therefore $\psi = P_{22} - P_{23}$ and $\psi^2 = P_{22} + P_{23} = P_2 \neq \pm\psi$.

Since ϕ acts on R_1 as $\phi^2 = -1$, so R_1 is even dimensional and ϕ has proper values i and $-i$. Denote the subspaces of R_1 corresponding to i and $-i$ as R_{11} and R_{14}, the projection tensors to these subspaces as P_{11} and P_{14}, then $P_1 = P_{11} + P_{14}$ and $P_{11} P_2 = P_2 P_{11} = P_{14} P_2 = P_2 P_{14} = 0$. Then $\phi P_2 u = 0$, $\phi P_{11} u = i P_{11} u$, $\phi P_{14} u = -i P_{14} u$ and $\phi u = \phi(P_{11} + P_{14} + P_2)u = \phi P_{11} u + \phi P_{14} u = i(P_{11} - P_{14})u$. Therefore $\phi = i(P_{11} - P_{14})$ and $\phi^2 = -(P_{11} + P_{14}) \neq \pm\phi$.

Consequently, if $\psi^2 = \pm\psi$, then the (ϕ, ψ)-structure defines a 3-π-structure expressed by $\psi = P_2$ (or $-P_2$), $\phi = i(P_{11} - P_{14})$. If $\psi^2 \neq \pm\psi$, the (ϕ, ψ)-structure defines a 4-π-structure expressed by $\psi = P_{22} - P_{23}$, $\phi = i(P_{11} - P_{14})$.

For the case 2°, if we put $-\psi^2 = P_2$ and $-\phi^2 = P_1$ we can show that both ψ and ϕ act on R_2 and R_1 respectively as $\psi^2 = -1$ and $\phi^2 = -1$. Therefore this case always defines a 4-π-structure which is expressed by $\phi = i(P_{11} - P_{14})$, $\psi = i(P_{22} - P_{23})$.

For the case 3°, if we put $\psi^2 = P_2$ and $\phi^2 = P_1$ we can show that both ψ and ϕ act on R_2 and R_1 respectively as $\psi^2 = 1$ and $\phi^2 = 1$. So as in case 1°, if $\psi^2 = \pm\psi$ and $\phi^2 = \pm\phi$ then the structure defines a 2-π-structure F given by $F = P_1 - P_2$. If $\psi^2 = \pm\psi$ and $\phi^2 \neq \pm\phi$, (or $\psi^2 \neq \pm\psi$ and $\phi^2 = \pm\phi$) then the (ϕ, ψ)-structure defines a 3-π-structure expressed by $\psi = P_2$ (or $-P_2$) and $\phi = P_{11} - P_{14}$ (or $\psi = P_{22} - P_{23}$, $\phi = P_1$ (or $-P_1$)). If both $\psi^2 \neq \pm\psi$ and $\phi^2 \neq \pm\phi$, then the (ϕ, ψ)-structure defines a 4-π-structure expressed by $\psi = P_{22} - P_{23}$, $\phi = P_{11} - P_{14}$.

It is to be noted that for a structure satisfying $\psi^3 = -\psi$, it can not happen that $\psi^2 = \pm\psi$. For, if $\psi^2 = \pm\psi$ then $\psi^3 = \pm\psi^2 = \pm(\pm\psi) = \psi$. Thus from the above argument we have

PROPOSITION 6. *A (ϕ, ψ)-structure defines a 2-π-structure if and only if $\psi^2 = \pm\psi$ and $\phi^2 = \pm\phi$. It defines a 3-π-structure if and only if $\psi^2 = \pm\psi$ and $\phi^2 \neq \pm\phi$ or $\psi^2 \neq \pm\psi$ and $\phi^2 = \pm\phi$. It defines a 4-π-structure if $\psi^2 \neq \pm\psi$ and $\phi^2 \neq \pm\psi$.*

6. If the (ϕ, ψ)-structure defines a 2-π-structure $F=P_1-P_2=2P_1-1$, then $\phi=\pm P_1$ and $\psi=\pm P_2$, so $F=\pm 2\phi-1$. Consequently, the integrability condition is given by $[F,F]=4[\phi,\phi]=0$ (or $[\psi,\psi]=0$.)

If the (ϕ,ψ)-structure defines a 3-π-structure and $\psi^2=\pm\psi$ and $\phi^2\neq\pm\phi$, then $\phi=P_{11}-P_{14}$ or $\phi=i(P_{11}-P_{14})$ according as $\phi^2=1$ or $\phi^2=-1$ on R_1. Thus $[\phi,\phi]=0$ gives $[P_1-P_3, P_1-P_3]=0$ (where $P_1=P_{11}$, $P_3=P_{14}$) and this is the integrability condition as already shown in the proof of Proposition 3. Thus we have:

PROPOSITION 7. *If the (ϕ,ψ)-structure of class C^ω satisfies $\psi^2=\pm\psi$ (or $\phi^2=\pm\phi$) then the integrability condition is given by* $[\phi,\phi]=0$ (*or* $[\psi,\psi]=0$).

Now, if we put $\psi=1+f^2$, $\phi=f$ in the case of structure f satisfying $f^3+f=0$, then we have $\phi\psi=\psi\phi=f^3+f=0$, so it defines a (ϕ,ψ)-structure of case 1° with $\psi^2=\psi$ and the integrability condition is $[\phi,\phi]=[f,f]=0$ as shown above.

A (ϕ,ξ,η)-structure on $(2n+1)$-dimensional manifold is a structure defined by a tensor field ϕ^i_j, a contravariant vetor field ξ^i and a covariant vector field η_j satisfying

$$\xi^i\eta_i=1,\quad \operatorname{rank}\phi=2n,\quad \phi_j{}^i\xi^j=\phi^i{}_j\eta_i=0\quad\text{and}\quad \phi^i_j\phi^j_k=-\delta^i_k+\xi^i\eta_k\,.$$

Therefore, if we put $\psi^i_k=\xi^i\eta_k$, then we have $\psi^2-\phi^2=1$, $\phi\psi=\psi\phi=0$ and $\psi^2=\psi$. This is a special case of the above structure f, and the the integrability condition is also $[\phi,\phi]=0$.

Finally, if the (ϕ,ψ)-structure defines a 4-π-structure (that is, if $\psi^2\neq\pm\psi$ and $\phi^2\neq\pm\phi$), then $\phi=P_{11}-P_{14}$ or $\phi=i(P_{11}-P_{14})$ according as $\phi^2=1$ or $\phi^2=-1$ on R_1; and $\psi=P_{22}-P_{23}$ or $\psi=i(P_{22}-P_{23})$ according as $\psi^2=1$ or $\psi^2=-1$ on R_2. Thus $[\phi,\phi]=0$ gives $[P_1-P_4, P_1-P_4]=0$ (where $P_1=P_{11}, P_4=P_{14}$) and $[\psi,\psi]=0$ gives $[P_2-P_3, P_2-P_3]=0$ (where $P_2=P_{22}$, $P_3=P_{23}$). It is shown in the proof of Proposition 5 that these two conditions give the integrability conditions of the corresponding 4-π-structure. Thus we have

PROPOSITION 8. *If $\psi^2\neq\pm\psi$ and $\phi^2\neq\pm\phi$, then the integrability conditions for the (ϕ,ψ)-structure* of class C^ω *are* $[\phi,\phi]=0$ *and* $[\psi,\psi]=0$.

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