Tôhoku Math. Journ.
20 (1968), 126-136.

ON THE ENESTRÖM-KAKEYA THEOREM

N. K. GOVIL AND Q. I. RAHMAN

(Received September 19, 1967)

The following result is well known in the theory of the distribution of zeros of polynomials.

THEOREM A. (Eneström-Kakeya). *If* $p(z)=\sum_{k=0}^{n} a_k z^k$ *is a polynomial of degree n such that*

$$a_n \geqq a_{n-1} \geqq a_{n-2} \geqq \cdots \geqq a_1 \geqq a_0 > 0, \tag{1}$$

then $p(z)$ *does not vanish in* $|z|>1$.

We may apply this result to $p(z/a)$ to obtain the following more general

THEOREM B. *If* $p(z)=\sum_{k=0}^{n} a_k z^k$ *is a polynomial of degree n such that for some* $a>0$

$$a_n \geqq a a_{n-1} \geqq a^2 a_{n-2} \geqq \cdots \geqq a^{n-1} a_1 \geqq a^n a_0 > 0\,, \tag{2}$$

then $p(z)$ *does not vanish in* $|z|>1/a$.

This is a very elegant result but it is equally limited in scope. The hypothesis is very restrictive and does not seem useful for applications. Our aim is to relax the hypothesis in several ways. In the literature there already exist ([1], [2, Theorem 3], [3]) some extensions of the Eneström-Kakeya theorem. In connection with Theorem A or the more general Theorem B the following questions appear to be very natural to ask.

Q.1. Can we drop the restriction that the coefficients are all positive and instead assume (2) to hold for the moduli of the coefficients ?

Q.2. Can we allow the polynomial of Theorem B to have gaps if the non-vanishing coefficients satisfy (2) ?

As an answer to Q.1 we prove the following

THEOREM 1. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \not\equiv 0$ *be a polynomial of degree* n *with complex coefficients such that for some* $a > 0$

$$|a_n| \geqq a|a_{n-1}| \geqq a^2|a_{n-2}| \geqq \cdots \geqq a^{n-1}|a_1| \geqq a^n|a_0| . \tag{3}$$

Then $p(z)$ *has all its zeros in* $|z| \leqq \left(\frac{1}{a}\right) K_1$, *where* K_1 *is the greatest positive root of the trinomial equation*

$$K^{n+1} - 2K^n + 1 = 0. \tag{4}$$

PROOF. If

$$a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 = p_1(z),$$

then for $|z| = R\left(> \frac{1}{a}\right)$

$$|p_1(z)| \leqq |a_{n-1}| R^{n-1}\left\{1 + \frac{1}{aR} + \cdots + \frac{1}{(aR)^{n-1}}\right\}$$

$$= |a_{n-1}| \frac{(aR)^n - 1}{a^{n-1}(aR-1)} .$$

Hence for every real θ

$$|p(Re^{i\theta})| \geqq |a_n| R^n - |p_1(Re^{i\theta})|$$

$$\geqq |a_n| R^n - |a_{n-1}| \frac{(aR)^n - 1}{a^{n-1}(aR-1)}$$

$$> 0$$

if

$$\frac{|a_n|}{a|a_{n-1}|} > \frac{(aR)^n - 1}{(aR)^n(aR-1)} .$$

Since $|a_n|/(a|a_{n-1}|) \geqq 1$ by hypothesis, we conclude that $p(Re^{i\theta}) \neq 0$ if

$$(aR)^n(aR-1) > (aR)^n - 1.$$

Replacing aR by K we get the result.

The example

$$p(z) = (az)^n - \{(az)^{n-1} + (az)^{n-2} + \cdots + az + 1\}$$

shows that the result is best possible.

It is clear that the conclusion of Theorem 1 remains true if the polynomial has gaps and the non vanishing coefficients $a_n, a_{n_1}, a_{n_2}, \cdots$ satisfy

$$|a_n| \geqq a^{n-n_1}|a_{n_1}| \geqq a^{n-n_2}|a_{n_2}| \geqq \cdots.$$

Next we prove the following generalization of Theorem A.

THEOREM 2. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \not\equiv 0$ *be a polynomial of degree* n *with complex coefficients such that*

$$|\arg a_k - \beta| \leqq \alpha \leqq \pi/2, \qquad k = 0, 1, 2, \cdots, n$$

for some real β, *and*

$$|a_n| \geqq |a_{n-1}| \geqq |a_{n-2}| \geqq \cdots \geqq |a_0|, \tag{5}$$

then $p(z)$ *has all its zeros on or inside the circle*

$$|z| = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{k=0}^{n-1} |a_k|.$$

For $\alpha = \beta = 0$, *this reduces to the Eneström-Kakeya theorem.*

PROOF OF THEOREM 2. We may plainly assume $\beta = 0$. By geometrical considerations it is very easy to prove that for $k = 1, 2, \cdots, n$

$$|a_k - a_{k-1}| \leqq (|a_k| - |a_{k-1}|)\cos\alpha + (|a_k| + |a_{k-1}|)\sin\alpha. \tag{6}$$

To verify it analytically let $\arg a_k = \alpha_k$, $\arg a_{k-1} = \alpha_{k-1}$. Then

$$\begin{aligned}|a_k - a_{k-1}|^2 &= \left||a_k|e^{i\alpha_k} - |a_{k-1}|e^{i\alpha_{k-1}}\right|^2 \\ &= |a_k|^2 + |a_{k-1}|^2 - 2|a_k||a_{k-1}|ccs(\alpha_k - \alpha_{k-1}) \\ &\leqq |a_k|^2 + |a_{k-1}|^2 - 2|a_k||a_{k-1}|\cos 2\alpha \\ &= (|a_k| - |a_{k-1}|)^2\cos^2\alpha + (|a_k| + |a_{k-1}|)^2\sin^2\alpha \\ &\leqq \{(|a_k| - |a_{k-1}|)\cos\alpha + (|a_k| + |a_{k-1}|)\sin\alpha\}^2\end{aligned}$$

and (6) follows.

Now consider

$$\begin{aligned}g(z) = (1-z)\,p(z) &= -a_n z^{n+1} + \sum_{k=1}^{n}(a_k - a_{k-1})z^k + a_0 \\ &= -a_n\, z^{n+1} + P(z) \qquad \text{say.}\end{aligned}$$

For $|z| = 1$, we have

$$\begin{aligned}|P(z)| &\leqq \sum_{k=1}^{n}|a_k - a_{k-1}| + |a_0| \\ &\leqq \sum_{k=1}^{n}(|a_k| - |a_{k-1}|)\cos\alpha + \sum_{k=1}^{n}(|a_k| + |a_{k-1}|)\sin\alpha + |a_0| \quad \text{(by (6))} \\ &= |a_n|(\cos\alpha + \sin\alpha) + 2\Big(\sum_{k=0}^{n-1}|a_k|\Big)\sin\alpha - |a_0|(\cos\alpha + \sin\alpha - 1) \\ &\leqq |a_n|(\cos\alpha + \sin\alpha) + 2\Big(\sum_{k=0}^{n-1}|a_k|\Big)\sin\alpha.\end{aligned}$$

Hence also

$$\left|z^n P\left(\frac{1}{z}\right)\right| \leqq |a_n|(\cos\alpha + \sin\alpha) + 2\Big(\sum_{k=0}^{n-1}|a_k|\Big)\sin\alpha \tag{7}$$

for $|z| = 1$. By the maximum modulus theorem (7) holds inside the unit circle as well. If $R > 1$ then $\frac{1}{R}e^{-i\theta}$ lies inside the unit circle for every real θ and from (7) it follows that

$$|P(Re^{i\theta})| \leqq \{|a_n|(\cos\alpha + \sin\alpha) + 2\Big(\sum_{k=0}^{n-1}|a_k|\Big)\sin\alpha\}R^n$$

for every $R \geqq 1$ and θ real.

Thus for $|z| = R > 1$

$$\begin{aligned} |g(z)| &= |-a_n z^{n+1} + P(z)| \\ &\geqq |a_n| R^{n+1} - \{|a_n|(\cos\alpha + \sin\alpha) + 2\left(\sum_{k=0}^{n-1} |a_k|\right)\sin\alpha\} R^n \\ &> 0 \end{aligned}$$

if

$$R > (\cos\alpha + \sin\alpha) + \frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1} |a_k| .$$

From this the theorem follows.

If $a_k = 0$ for $k = 0, 1, \cdots, m$ then arg a_k is not defined for each of these a_k. However, it is clear that the theorem holds if $|\arg a_k - \beta| \leqq \alpha \leqq \pi/2$ for the non-vanishing coefficients and (5) is satisfied.

We may apply Theorem 2 to $z^n p(1/z)$ to get the following

COROLLARY 1. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \not\equiv 0$ *be a polynomial of degree* n *with complex coefficients such that*

$$|\arg a_k - \beta| \leqq \alpha \leqq \pi/2, \quad k = 0, 1, 2, \cdots, n$$

for some real β, *and*

$$|a_0| \geqq |a_1| \geqq |a_2| \geqq \cdots \geqq |a_n| ,$$

then $p(z)$ *does not vanish in*

$$|z| < \{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{k=1}^{n} |a_k|\}^{-1}.$$

We shall briefly indicate how we can prove the following more general result.

THEOREM 3. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k \not\equiv 0$ *be analytic in* $|z| \leqq 1$. *If*

$$|\arg a_k| \leqq \alpha \leqq \pi/2, \quad k = 0, 1, 2, \cdots$$

and

$$|a_0| \geqq |a_1| \geqq |a_2| \geqq \cdots,$$

then $f(z)$ does not vanish in

$$|z| < \{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|} \sum_{k=1}^{\infty} |a_k|\}^{-1}.$$

PROOF OF THEOREM 3. It is clear that $\lim_{k\to\infty} a_k = 0$. Consider

$$\begin{aligned} h(z) = (1-z)f(z) &= a_0 - \{(a_0 - a_1)z + (a_1 - a_2)z^2 + \cdots\} \\ &= a_0 - zF(z) \qquad \text{say.} \end{aligned}$$

We wish to show that $h(z_0) \neq 0$ if $|z_0| < \{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|} \sum_{k=1}^{\infty} |a_k|\}^{-1}$. Obviously $h(0) \neq 0$. Hence we may suppose that $z_0 \neq 0$ and consider

$$\begin{aligned} \frac{h(z_0)}{z_0} &= \frac{a_0}{z_0} - \{(a_0 - a_1) + (a_1 - a_2)z_0 + \cdots\} \\ &= \frac{a_0}{z_0} - F(z_0) \qquad \text{say}. \end{aligned}$$

From (6) it follows that

$$\begin{aligned} |F(z_0)| &\leqq \sum_{k=0}^{\infty} (|a_k| - |a_{k+1}|)\cos\alpha + \sum_{k=0}^{\infty} (|a_k| + |a_{k+1}|)\sin\alpha \\ &= |a_0|(\cos\alpha + \sin\alpha) + 2\Big(\sum_{k=1}^{\infty} |a_k|\Big)\sin\alpha. \end{aligned}$$

Hence

$$\begin{aligned} \left|\frac{h(z_0)}{z_0}\right| = \left|\frac{a_0}{z_0} - F(z_0)\right| &\geqq \left|\frac{a_0}{z_0}\right| - \{|a_0|(\cos\alpha + \sin\alpha) + 2\Big(\sum_{k=1}^{\infty} |a_k|\Big)\sin\alpha\} \\ &> 0 \end{aligned}$$

if

$$|z_0| < |a_0| / \{|a_0|(\cos\alpha + \sin\alpha) + 2\Big(\sum_{k=1}^{\infty} |a_k|\Big)\sin\alpha\}.$$

From this the conclusion follows immediately.

We also prove

THEOREM 4. *Let* $p(z)=\sum_{k=0}^{n} a_k z^k \not\equiv 0$ *be a polynomial of degree* n. *If* $\operatorname{Re} a_k = \alpha_k$, $\operatorname{Im} a_k = \beta_k$ *for* $k = 0, 1, 2, \cdots, n$ *and*

$$\alpha_n \geqq \alpha_{n-1} \geqq \cdots \geqq \alpha_1 \geqq \alpha_0 \geqq 0, \qquad \alpha_n > 0, \tag{8}$$

then $p(z)$ *does not vanish in*

$$|z| > 1 + \frac{2}{\alpha_n}\sum_{k=0}^{n} |\beta_k| .$$

This is clearly a generalization of the Eneström-Kakeya theorem.

PROOF OF THEOREM 4. Geometrically it is obvious that

$$|a_k - a_{k-1}| \leqq (\alpha_k - \alpha_{k-1}) + (|\beta_{k-1}| + |\beta_k|), \qquad k = 1, 2, \cdots, n. \tag{9}$$

Hence again, if

$$\begin{aligned} g(z) = (1-z)p(z) &= -a_n z^{n+1} + \sum_{k=1}^{n} (a_k - a_{k-1}) z^k + a_0 \\ &= -\alpha_n z^{n+1} + p^*(z) \end{aligned}$$

then for $|z| = 1$

$$\begin{aligned} |p^*(z)| &\leqq |\beta_n| + \sum_{k=1}^{n} \{(\alpha_k - \alpha_{k-1}) + (|\beta_{k-1}| + |\beta_k|)\} + \alpha_0 + |\beta_0| \\ &= \alpha_n + 2\sum_{k=0}^{n} |\beta_k| \end{aligned}$$

and as in the proof of Theorem 2

$$|p^*(z)| \leqq (\alpha_n + 2\sum_{k=0}^{n} |\beta_k|) R^n$$

for $|z| = R \geqq 1$. Consequently

$$\begin{aligned} |g(Re^{i\theta})| &\geqq \alpha_n R^{n+1} - |p^*(Re^{i\theta})| \\ &\geqq \alpha_n R^{n+1} - (\alpha_n + 2\sum_{k=0}^{n} |\beta_k|) R^n \\ &> 0 \end{aligned}$$

if

$$R > 1 + \frac{2}{\alpha_n} \sum_{k=0}^{n} |\beta_k|,$$

and the theorem follows.

Refinement of Eneström-Kakeya theorem: In a recent paper Rubinstein [4, Theorem 1] has proved the following interesting result.

THEOREM C. *Let the function* $h(z) = \sum_{k=0}^{\infty} a_k z^k$ *be analytic in* $|z| \leqq 1$. *If*

$$D = \frac{\max_{|z|=1} |h'(z)|}{\max_{|z|=1} |h(z)|}$$

and $\max_{|z|=1} |h(z)| = |h(e^{i\alpha})|$ *then* $h(z)$ *does not vanish in the disc* $|z - De^{-i\alpha}/(D+1)| < 1/(D+1)$.

Whereas it is obvious that $h(z)$ does not vanish near the point $e^{i\alpha}$ it is not at all clear that $h(z) \neq 0$ in the region indicated in Theorem C. This is what makes the above theorem interesting. Although not explicitly mentioned before it is immediate that $h(z)$ does not vanish in the disc $\left| z - \frac{De^{i\alpha}}{(D+1)} \right| < \frac{1}{(D+1)}$ as well. To see this we consider the function $H(z) = h(ze^{i\alpha})$ which satisfies the hypothesis of Theorem C with $\alpha = 0$. Hence $H(z) \neq 0$ in $\left| z - \frac{D}{(D+1)} \right| < 1/(D+1)$ and so $h(z) = H(ze^{-i\alpha}) \neq 0$ in $|ze^{-i\alpha} - D/(D+1)| < 1/(D+1)$. This is equivalent to our assertion.

Rubinstein applied Theorem C to obtain the following [4, Corollary 1] refinement of Theorem A.

THEOREM D. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a polynomial of degree n. If*

$$a_n \geqq a_{n-1} \geqq \cdots \geqq a_1 \geqq a_0 > 0,$$

then all the zeros of $p(z)$ *lie in the complement of the open disc*

$$\left| z - \frac{\sum_{k=0}^{n} k a_k}{\sum_{k=0}^{n} (k+1) a_k} \right| < \frac{\sum_{k=0}^{n} a_k}{\sum_{k=0}^{n} (k+1) a_k}$$

with respect to the unit disc.

If $\max_{|z|\leq 1}|p(z)|$ is attained at $z=1$ then $\max_{|z|\leq 1}|p(z^2)|$ is attained at $z=1$ and also at $z=-1$. We can use this observation to prove that $p(z^2)$ does not vanish if

$$\left| z - \frac{\sum_{k=0}^{n} 2k a_k}{\sum_{k=0}^{n} (2k+1) a_k} \right| < \frac{\sum_{k=0}^{n} a_k}{\sum_{k=0}^{n} (2k+1) a_k}.$$

From this a certain zero free region for $p(z)$ can be obtained. Then we may use the fact that $\max_{|z|\leq 1}|p(z^3)|$ is attained at the points $e^{2\pi i/3}, e^{4\pi i/3}, 1$. This will give two other regions in which $p(z)$ does not vanish. This argument can be extended in the obvious way.

Another refinement of the Eneström-Kakeya theorem which we note is the following:

THEOREM 5. *If*

$$a_n \geqq a_{n-1} \geqq a_{n-2} \geqq \cdots \geqq a_1 \geqq a_0 \geqq 0,$$

then the polynomial $p(z) = \sum_{k=0}^{n} a_k z^k \not\equiv 0$ *does not vanish in a region containing the sector* $|\arg z| < \pi/(n+1)$ *and the two regions* D_1 *and* D_2 *given by*

$$\{|z| < 1\} \cap \left\{ \left| z - \frac{1}{2} \right| > \frac{1}{2} \right\} \cap \left\{ \frac{\pi}{n+1} \leqq \arg z \leqq \frac{3\pi}{2(n+1)} \right\}$$

and

$$\{|z| < 1\} \cap \left\{ \left| z - \frac{1}{2} \right| > \frac{1}{2} \right\} \cap \left\{ -\frac{3\pi}{2(n+1)} \leqq \arg z \leqq -\frac{\pi}{n+1} \right\}$$

respectively.

PROOF. Since

$$\mathrm{Re}\ \{p(re^{i\theta})\} = a_0 + a_1 r\cos\theta + a_2 r^2 \cos 2\theta + \cdots + a_n r^n \cos n\theta,$$

$\mathrm{Re}\ p(z) > 0$ and so $p(z) \neq 0$ for $|\arg z| < \pi/\{2(n+1)\}$. On the other hand

$$|\mathrm{Im}\ \{p(re^{i\theta})\}| = |a_1 r\sin\theta + a_2 r^2 \sin 2\theta + \cdots + a_n r^n \sin n\theta| > 0$$

if $0 < |\arg z| < \pi/(n+1)$. Hence $p(z) \neq 0$ for $|\arg z| < \pi/(n+1)$. Now let

$$q(z) = z^n p(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

and consider $(1-z)q(z)$. It is clear that

$$(1-z)q(z) = \sum_{k=1}^{n} (a_{n-k} - a_{n-k+1})(z^k - 1) - a_0(z^{n+1} - 1).$$

Hence $\mathrm{Re}\ \{(1-z)\,q(z)\} > 0$ if $\mathrm{Re}\,(z^k - 1) < 0$ for $k = 1, 2, \cdots, n+1$. This certainly holds if $\pi/(n+1) \leqq |\arg z| \leqq 3\pi/\{2(n+1)\}$ and $\mathrm{Re}\, z < 1$. This implies that $p(z)$ does not vanish in the two regions D_1, D_2 defined by

$$\{|z| < 1\} \cap \left\{\left|z - \frac{1}{2}\right| > \frac{1}{2}\right\} \cap \left\{\pi/(n+1) \leqq \arg z \leqq 3\pi/\{2(n+1)\}\right\},$$

and

$$\{|z| < 1\} \cap \left\{\left|z - \frac{1}{2}\right| > \frac{1}{2}\right\} \cap \left\{-3\pi/\{2(n+1)\} \leqq \arg z \leqq -\pi/(n+1)\right\},$$

respectively. Thus we have shown that in addition to the exterior of the unit circle, $p(z)$ does not vanish in the shaded portion of the unit circle indicated in Fig.1. It is clear that the inequalities

$$\mathrm{Re}(z^k - 1) < 0, \quad k = 1, 2, \cdots, n+1,$$

hold also if z lies in the half plane $\mathrm{Re}\, z < 1$, and $\mathrm{Re}(z^{n+1}) < 1$. It follows that if $z = re^{i\theta}$ then $p(z)$ does not vanish in the regions defined by $r > \{\cos(n+1)\theta\}^{1/(n+1)}, \left|z - \frac{1}{2}\right| > \frac{1}{2}, r < 1.$

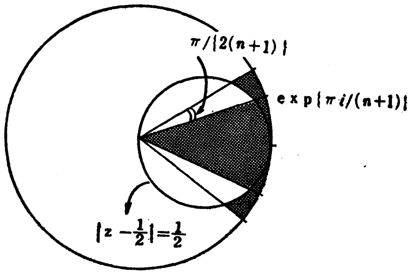

Fig. 1

REFERENCES

[1] G. T. CARGO AND O. SHISHA, Zeros of polynomials and fractional order differences of their coefficients, Journ. Math. Anal. Appl., 7 (1963), 176–182.
[2] A. JOYAL, G. LABELLE AND Q. I. RAHMAN, On the location of zeros of polynomials, Canad. Math. Bull., 10 (1967), 53–63.
[3] P. V. KRISHNAIAH, On Kakeya's theorem, Journ. London Math. Soc., 30 (1955), 314–319.
[4] Z. RUBINSTEIN, Some results in the location of zeros of polynomials, Pacific Journ. Math., 15 (1965), 1391–1395.

MATHEMATICS DEPARTMENT
UNIVERSITÉ DE MONTRÉAL
CANADA