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EXCISION THEOREMS ON THE PAIR OF MAPS

HIDEO ANDO

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Introduction. Let $E_f \longrightarrow X \xrightarrow{f} Y$ be an extended fibration. Then the 1-1 and onto correspondence $\mathcal{E}_f^{-1}: \pi_1(V,f) \to \pi(V, E_f)$ is defined easily (section 2). Moreover, let Ψ be the pair-map $(g_1, g_2): f_1 \to f_2$ in (1.3); then the 1-1 and onto correspondence $\mathcal{E}_\Psi^{-1}: \pi_2(V, \Psi) \to \pi_1(V, f_{1,2})$ is defined (section 3). The object of this paper is to establish excision theorems on the pair of maps by applying $\mathcal{E}_f^{-1}$ and $\mathcal{E}_\Psi^{-1}$. These excision theorems are described in section 5.

1. Preliminaries. Throughout this paper we consider the category of spaces of the homotopy type of CW-complexes with base points denoted by $*$, and all maps and homotopies are assumed to preserve base points.

PX is the space of paths in X emanating from $*$, and ΩX is the loop space. If $f: X \to Y$ is any map, $Y \cup_f CX$ is the space obtained by attaching to Y the reduced cone over X by means of f. X is embedded in CX by $x \to (x, 1)$, and ΣX is the reduced suspension. $X \times Y$ is the Cartesian product and $X \vee Y = X \times * \cup * \times Y$. Then the smash product $X \# Y$ is the quotient space $X \times Y / X \vee Y$.

By applying the mapping track functor, any map $f: X \to Y$ is converted into a homotopy equivalent fibre map $p: E \to Y$,

$$\text{(1.1)} \qquad \begin{array}{ccccc} E_f & \xrightarrow{j_f} & X & \xrightarrow{f} & Y \\ \Vert & \simeq & \downarrow h & \text{comm.} & \Vert \\ E_f & \xrightarrow{i} & E & \xrightarrow{p} & Y, \end{array}$$

where $E = \{(x, \eta) \in X \times Y^I | f(x) = \eta(1)\}$, $p(x, \eta) = \eta(0)$,
$E_f = \{(x, \eta) \in X \times PY | f(x) = \eta(1)\}$, i = the inclusion map,
$j_f(x, \eta) = x$, $h(x) = (x, \eta_x)$ and $\eta_x(t) = f(x)$ for $t \in I$,
$\simeq$ in the left diagram means homotopy commutativity.

Then the sequence $E_f \xrightarrow{j_f} X \xrightarrow{f} Y$ is called the extended fibration.

Dually, by applying the mapping cylinder functor, any map f is converted into a homotopy equivalent cofibre map $q: X \to M_f$.

(1.1˙)
$$\begin{array}{ccccc} X & \xrightarrow{q} & M_f & \xrightarrow{j} & C_f \\ \| & \text{comm.} & \downarrow k & \simeq & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{i_f} & C_f , \end{array}$$

where $M_f =$ the mapping cylinder of f, $q(x) = (x, 0)$,
$k(x, t) = f(x)$ for $(x, t) \in X \times I$ and $k(y) = y$ for $y \in Y$,

then the sequence $X \xrightarrow{f} Y \xrightarrow{i_f} C_f$ is called the extended cofibration.

The join $X * Y$ of X and Y is the quotient space obtained from $X \times I \times Y$ by factoring out the relation: $(x, 0, y_1) \sim (x, 0, y_2)$ for all $y_1, y_2 \in Y$ and $(x_1, 1, y) \sim (x_2, 1, y)$ for all $x_1, x_2 \in X$. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and let $r: E \cup_i CF \to B$ be given by $r|E = p$ and $r(CF) = *$.

Then we obtain

PROPOSITION 1.2. [2; Theorem 1.1]. *There exists a weak homotopy equivalence* $w: F * \Omega B \to F_r$, *where* F_r *is the fibre of* r *and given by* $F_r = \{(a, \beta) \in E \times PB \mid p(a) = \beta(1)\} \cup (CF \times \Omega B)$.

We shall denote by j_0 the composite of w with the projection $F_r \to E \cup_i CF$; then the triple $F * \Omega B \xrightarrow{j_0} E \cup_i CF \xrightarrow{r} B$ may be regarded as a fibration [2]. We consider the diagram

(1.3)
$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ \downarrow g_1 & & \downarrow g_2 \\ X & \xrightarrow{f_2} & Y , \end{array}$$

which is homotopy commutative (commutative). Such a pair of maps (g_1, g_2) is called a transformation of f_1 to f_2. If (1.3) is commutative, (g_1, g_2) is called the pair-map and we write it as $(g_1, g_2): f_1 \to f_2$.

2. The correspondence $\mathcal{E}_f^{-1}: \pi_1(V, f) \to \pi(V, E_f)$. Let (g_1, g_2) be a transformation of f_1 to f_2, and let $F_t: g_2 \circ f_1 \simeq f_2 \circ g_1$ be a fixed homotopy. And we consider the following diagram

$$(2.1)\qquad \begin{array}{ccccc} E_{f_1} & \xrightarrow{j_{f_1}} & A & \xrightarrow{f_1} & B \\ \Big\downarrow g_{1,2} & & \Big\downarrow g_1 & \simeq & \Big\downarrow g_2 \\ E_{f_2} & \xrightarrow{j_{f_2}} & X & \xrightarrow{f_2} & Y \end{array},$$

where $g_{1,2}: E_{f_1} \to E_{f_2}$ is defined by $g_{1,2}(a, \beta) = (g_1(a), \beta')$ for $a \in A$, $\beta \in PB$ with $f_1(a) = \beta(1)$, and $\beta' \in PY$ is given by

$$\beta'(s) = \begin{cases} g_2\beta(2s) & \text{for} \quad 0 \leqq s \leqq 1/2\,, \\ F_{2s-1}(a) & \text{for} \quad 1/2 \leqq s \leqq 1\,. \end{cases}$$

Then the left diagram in (2.1) is commutative.

Now let $E_f \xrightarrow{j_f} X \xrightarrow{f} Y$ be an extended fibration. Then we consider the correspondence $\varepsilon_f^{-1}: \pi_1(V, f) \to \pi(V, E_f)$ defined as follows:

For any element $\{(a_1, a_2)\} \in \pi_1(V, f)$, $\bar{a}_2: V \to PY$ is defined by $\bar{a}_2(v)(s) = a_2(v, s)$; then $\varepsilon_f^{-1}\{(a_1, a_2)\} = \{U_{(a_1, a_2)}\}$, $U_{(a_1, a_2)}(v) = (a_1(v), \bar{a}_2(v))$. Thus defined ε_f^{-1} is well defined, and 1-1 and onto.

PROPOSITION 2.2. *Let (g_1, g_2) be a transformation of f_1 to f_2 with a fixed homotopy F_t or a pair-map. If g_1 and g_2 are homotopy equivalences then there exists a* 1-1 *and onto correspondence $\pi_1(V, f_1) \to \pi_1(V, f_2)$.*

PROOF. We consider the sequence

$$\pi_1(V, f_1) \xrightarrow{\varepsilon_{f_1}^{-1}} \pi(V, E_f) \xrightarrow{g_{1,2*}} \pi(V, E_{f_2}) \xrightarrow{\varepsilon_{f_2}} \pi_1(V, f_2)\,,$$

where ε_{f_2} is the inverse correspondence of $\varepsilon_{f_2}^{-1}$. Since $g_{1,2}$ is the homotopy equivalence by [8; Lemma 6], $g_{1,2*}$ is 1-1 and onto. Hence $\varepsilon_{f_2} \circ g_{1,2*} \circ \varepsilon_{f_1}^{-1}$ is the desired correspondence. If (g_1, g_2) is the pair-map then we have $(g_1, g_2)_* = \varepsilon_{f_2} \circ g_{1,2*} \circ \varepsilon_{f_1}^{-1}$.

If we now consider the pair-map $(1, g_2): f_1 \to f_2 = g_2 \circ f_1$ and $(g_1, 1): f_1 = f_2 \circ g_1 \to f_2$;

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ \Big\downarrow f_1 & & \Big\downarrow f_2 = g_2 \circ f_1 \\ B & \xrightarrow{g_2} & Y \end{array}, \qquad \begin{array}{ccc} A & \xrightarrow{g_1} & X \\ \Big\downarrow f_1 = f_2 \circ g_1 & & \Big\downarrow f_2 \\ B & \xrightarrow{1} & B \end{array},$$

then we have

COROLLARY 2.3. *If $g_2: B \to X$ is a homotopy equivalence, then*

$$(1, g_2)_*: \pi_1(V, f_1) \to \pi_1(V, g_1 \circ f_1)$$

is 1-1 *and onto.*

COROLLARY 2.4. *If $g_1: A \to X$ is a homotopy equivalence, then*

$$(g_1, 1)_*: \pi_1(V, f_2 \circ g) \to \pi_1(V, f_2)$$

is 1-1 *and onto.*

COROLLARY 2.5. *If $f_1 \simeq f_2: A \to B$, then there exists a* 1-1 *and onto correspondence $\pi_1(V, f_1) \to \pi_1(V, f_2)$.*

REMARK. (i) Corollary 2.3 and 2.4 are extensions of Proposition 2.2 and 2.3 in [1], respectively.

(ii) We may define the dual 1-1 and onto correspondence $\mathcal{E}_f'^{-1}: \pi_1(f, W) \to \pi(C_f, W)$.

3. The correspondence $\mathcal{E}_\Psi^{-1}: \pi_2(V, \Psi) \to \pi_1(V, f_{1,2})$. We consider the pair-map $\Psi = (g_1, g_2): f_1 \to f_2$. Then any element $\left\{\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix}\right\} \in \pi_2(V, \Psi)$ is represented by the commutative diagram

(3.1)
$$\begin{tikzcd}
V \arrow[rrr, "a_1"] \arrow[dr, "\iota"] \arrow[dd, "\iota"'] & & & A \arrow[dr] \arrow[dd] & \\
& CV \arrow[rrr, "a_2"] \arrow[dd, "\iota C"] & & & B \arrow[dd] \\
CV \arrow[r, phantom, "\Phi\Downarrow"] \arrow[rrr, "b_1"] \arrow[dr, "C\iota"'] & & & X \arrow[r, phantom, "\Downarrow\Psi"] \arrow[dr] & \\
& C^2V \arrow[rrr, "b_2"] & & & Y
\end{tikzcd}$$

where $\iota C: CV \to C^2V$ and $C\iota: CV \to C^2V$ are given by $(v, t) \to (v, t, 1)$ and $(v, s) \to (v, 1, s)$, respectively. Maps $\bar{b}_1: V \to PX$ and $\bar{b}_2: CV \to PX$ are defined by $\bar{b}_1(v)(s) = b_1(v, s)$ and $\bar{b}_2(v, t)(s) = b_2(v, t, s)$, respectively. The correspondences

$$\mathcal{E}_\Psi^{-1}: \pi_2(V, \Psi) \longrightarrow \pi_1(V, f_{1,2}), \quad f_{1,2}: E_{g_1} \longrightarrow E_{g_2},$$

are defined as follows: $\mathcal{E}_\Psi^{-1}\left\{\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix}\right\} = \{(U_{(a_1 b_1)}, U_{(a_2 b_2)})\} \in \pi_1(V, f_{1,2})$, where $U_{(a_1 b_1)}(v) = (a_1(v), \bar{b}_1(v))$, $U_{(a_2 b_2)}(v, t) = (a_2(v, t), \bar{b}_2(v, t))$. Then the definition of $\mathcal{E}_\Psi^{-1}$ is well defined, and if $\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix} \simeq \begin{pmatrix} a_1' a_2' \\ b_1' b_2' \end{pmatrix}$ then $\mathcal{E}_\Psi^{-1}\left\{\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix}\right\} = \mathcal{E}_\Psi^{-1}\left\{\begin{pmatrix} a_1' a_2' \\ b_1' b_2' \end{pmatrix}\right\}$.

PROPOSITION 3.2. *$\mathcal{E}_\Psi^{-1}: \pi_2(V, \Psi) \to \pi_1(V, f_{1,2})$ is 1-1 and onto.*

PROOF. First we shall prove that $\mathcal{E}_\Psi^{-1}$ is onto. For any $\{(f, g)\} \in \pi_1(V, f_{1,2})$ we consider the commutative diagram

$$\begin{tikzcd}
V \arrow[rr, "f"] \arrow[dr, "\iota"'] & & E_{g_1} \arrow[rr, "i_{g_1}"] \arrow[dr, "f_{1,2}"] \arrow[dd, "j_{g_1}"] & & PX \arrow[dr, "Pf_2"] \arrow[dd, "p_X"] & \\
& CV \arrow[rr, "g"] & & E_{g_2} \arrow[rr, "i_{g_2}"] \arrow[dd, "j_{g_2}"] & & PY \arrow[dd, "p_Y"] \\
& & A \arrow[rr, "g_1"] \arrow[dr, "f_1"'] & & X \arrow[dr, "f_2"] & \\
& & & B \arrow[rr, "g_2"] & & Y
\end{tikzcd}$$

where $i_{g_1}(a, \omega) = \omega$ for $a \in A$, $\omega \in PX$ and $i_{g_2}(b, \eta) = \eta$ for $b \in B$, $\eta \in PY$.

Then maps $a_1: V \to A$, $a_2: CV \to B$, $b_1: CV \to X$ and $b_2: C^2V \to Y$ are defined as follows:

$$a_1 = j_{g_1} \circ f, \quad a_2 = j_{g_2} \circ g,$$
$$b_1(v, s) = (i_{g_1} \circ f(v))(s)$$
and
$$b_2(v, t, s) = (i_{g_2} \circ g(v, t))(s).$$

Then we may show that the diagram (3.1) is commutative, and we obtain $\mathcal{E}_\Psi^{-1}\left\{\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix}\right\} = \{(f, g)\}$. Thus $\mathcal{E}_\psi^{-1}$ is onto.

Next we shall see easily that $\mathcal{E}_\Psi^{-1}$ is 1-1.

REMARK. Also we may define the dual 1-1 and onto correspondence $\mathcal{E}_\Psi'^{-1}: \pi_2(\Psi, W) \to \pi_1(f_{1,2}', W)$, where $f_{1,2}': C_{g_1} \to C_{g_2}$.

4. Transposition. We recall the notation of the transpose of a map [1; p. 291]. In the diagram (1.3), the transpose of the pair-map $\psi = (g_1, g_2): f_1 \to f_2$ is the map $\Psi^T = (f_1, f_2): g_1 \to g_2$. Then the 1-1 correspondence between maps $\Phi \to \Psi$ and maps $\Phi^T \to \Psi^T$ is given by the transposition $\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix} \to \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}$, where $\Phi = (\iota, \iota C)$ (c.f. (3.1)).

This correspondence induce a 1-1 and onto correspondence $\tau_0: \pi(\Phi, \Psi) \to \pi(\Phi^T, \Psi^T)$. Let $u: C^2V \to C^2V$ be the homeomorphism given by $u(v, t, s) = (v, s, t)$; then $\begin{pmatrix} 1\,1 \\ 1\,u \end{pmatrix}$ induces a 1-1 and onto correspondence $\tau_1: \pi(\Phi, \Psi) \to \pi(\Phi^T, \Psi)$, where $\Phi = (\iota, \iota C)$. And we get the 1-1 and onto correspondence $\tau = \tau_0 \tau_1 : \pi_2(V, \Psi) \to \pi_2(V, \Psi^T)$ given by $\tau\left\{\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix}\right\} = \left\{\begin{pmatrix} a_1 b_1 \\ a_2 b_2 u \end{pmatrix}\right\}$ for $\left\{\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix}\right\}$

$\in \pi_2(V, \Psi)$.

Now we consider the following diagram induced from (1. 3):

$$\begin{tikzcd}
E_{g_{1,2}} \arrow[d, "j_{g_{1,2}}"] & E_{f_{1,2}} \arrow[r, "j_{f_{1,2}}"] & E_{g_1} \arrow[r, "f_{1,2}"] \arrow[d, "j_{g_1}"] & E_{g_2} \arrow[d, "j_{g_2}"] \\
E_{f_1} \arrow[rr, "j_{f_1}"] \arrow[d, "g_{1,2}"] & & A \arrow[r, "f_1"] \arrow[d, "g_1"] & B \arrow[d, "g_2"] \\
E_{f_2} \arrow[rr, "j_{f_2}"] & & X \arrow[r, "f_2"] & Y
\end{tikzcd},$$

where
$$E_{f_{1,2}} = \{((a, \omega), (\beta, \rho)) \in E_{g_1} \times PE_{g_2} \mid g_1(a) = \omega(1),\ f_1(a) = \beta(1),\ f_2(\omega(s)) = \rho(1, s) \text{ and } \rho(s, 1) = g_2(\beta(s))\},$$
$$E_{g_{1,2}} = \{((a, \beta), (\omega, \bar{\rho})) \in E_{f_1} \times PE_{f_2} \mid g_1(a) = \omega(1),\ f_1(a) = \beta(1),\ f_2(\omega(s)) = \bar{\rho}(s, 1) \text{ and } \bar{\rho}(1, s) = g_2(\beta(s))\},$$

$\rho, \bar{\rho}: I\#I \to Y_2$, and maps set as follows:

$$f_{1,2}(a, \omega) = (f_1(a), \omega') \quad \text{for} \quad a \in A,\ \omega \in PX \quad \text{with} \quad \omega'(s) = f_2(\omega(s)),$$
$$g_{1,2}(a, \beta) = (g_1(a), \beta') \quad \text{for} \quad a \in A,\ \beta \in PB \quad \text{with} \quad \beta (s) = g_2(\beta(s)),$$
$$j_{f_1}(a, \beta) = a,\ j_{f_2}(x, \eta) = x,\ j_{f_{1,2}}((a, \omega), (\beta, \rho)) = (a, \omega),$$
$$j_{g_1}(a, \omega) = a,\ j_{g_2}(b, \eta) = b,\ j_{g_{1,2}}((a, \beta), (\omega, \bar{\rho})) = (a, \beta).$$

Maps $d: E_{f_{1,2}} \to E_{g_{1,2}}$ and $d': E_{g_{1,2}} \to E_{f_{1,2}}$ defined by

$$d((a, \omega), (\beta, \rho)) = ((a, \beta), (\omega, \rho\sigma)) \quad \text{and} \quad d'((a, \beta), (\omega, \bar{\rho})) = ((a, \omega), (\beta, \bar{\rho}\sigma)),$$

respectively, are homeomorphisms, where $\sigma: I\#I \to I\#I$ is defined by $\sigma(s, t) = (t, s)$.

PROPOSITION 4.1. *$d_*: \pi(V, E_{f_{1,2}}) \to \pi(V, E_{g_{1,2}})$ is equivalent to $\tau: \pi_2(V, \Psi) \to \pi_2(V, \Psi^T)$ in the sense that the diagram*

$$\begin{tikzcd}
\pi(V, E_{f_{1,2}}) \arrow[r, "d_*"] & \pi(V, E_{g_{1,2}}) \\
\pi_1(V, f_{1,2}) \arrow[u, "\varepsilon_{f_{1,2}}^{-1}"] & \pi_1(V, g_{1,2}) \arrow[u, "\varepsilon_{g_{1,2}}^{-1}"'] \\
\pi_2(V, \Psi) \arrow[u, "\varepsilon_{\Psi}^{-1}"] \arrow[r, "\tau"] & \pi_2(V, \Psi^T) \arrow[u, "\varepsilon_{\Psi^T}^{-1}"']
\end{tikzcd}$$

is commutative.

PROOF. For any element $\left\{\begin{pmatrix} a_1a_2 \\ b_1b_2 \end{pmatrix}\right\} \in \pi_2(V, \Psi)$ we see that

$$\begin{aligned}\mathcal{E}_{g_{1,2}}^{-1} \circ \mathcal{E}_{\Psi^T}^{-1} \circ \tau \circ \left\{\begin{pmatrix} a_1a_2 \\ b_1b_2 \end{pmatrix}\right\} &= \mathcal{E}_{g_{1,2}}^{-1} \circ \mathcal{E}_{\Psi^T}^{-1} \left\{\begin{pmatrix} a_1b_1 \\ a_2b_2u \end{pmatrix}\right\} \\ &= \mathcal{E}_{g_{1,2}}^{-1}\{(U_{(a_1,a_2)}, U_{(b_1,b_2u)})\} \\ &= \{U_{(U_{(a_1,a_2)}, U_{(b_1,b_2u)})}\},\end{aligned}$$

where $U_{(U_{(a_1,a_2)}, U_{(b_1,b_2u)})}(v) = (U_{(a_1a_2)}(v), \overline{U}_{(b_1b_2u)}(v))$, $\overline{U}_{(b_1,b_2u)}(v)(s) = (b_1(v,s), \overline{b_2u}(v,s))$, and $\overline{b_2u}(v,s)(t) = b_2(v,t,s)$.

On the other hand,

$$\begin{aligned}d_* \circ \mathcal{E}_{f_{1,2}}^{-1} \circ \mathcal{E}_{\Psi}^{-1} \left\{\begin{pmatrix} a_1a_2 \\ b_1b_2 \end{pmatrix}\right\} &= d_* \circ \mathcal{E}_{f_{1,2}}^{-1}\{(U_{(a_1,b_1)}, U_{(a_2,b_2)})\} \\ &= d_*\{U_{(U_{(a_1,b_1)}, U_{(a_2,b_2)})}\} \\ &= \{d \circ U_{(U_{(a_1,b_1)}, U_{(a_2,b_2)})}\},\end{aligned}$$

and $U_{(U_{(a_1,b_1)}, U_{(a_2,b_2)})}(v) = (U_{(a_1,b_1)}(v), \overline{U}_{(a_2,b_2)}(v)) = ((a_1(v), \overline{b}_1(v)), (\overline{a}_2(v), \overline{\overline{b}}_2(v)))$ where $\overline{b}_1 : V \to PX$, $\overline{a}_2 : V \to PB$, $\overline{\overline{b}}_2 : V \to P(PY)$ and $\overline{b}_2 : CV \to PY$ are maps such that $\overline{b}_1(v)(s) = b_1(v,s)$, $\overline{a}_2(v)(t) = a_2(v,t)$, $\overline{\overline{b}}_2(v)(t) = \overline{b}_2(v,t)$ and $\overline{b}_2(v,t)(s) = b_2(v,t,s)$.

There exists a homeomorphism $\theta : Y^{I\#I} \approx (Y^I)^I$ defined by $\theta(f)(t)(s) = f(t,s)$, and hence $\overline{\overline{b}}_2(v)$ may be replaced by $\theta^{-1}\overline{\overline{b}}_2(v)$ such that $(\theta^{-1}\overline{\overline{b}}_2(v))(t,s) = (\overline{\overline{b}}(v))(t)(s)$. Thus we have

$$\begin{aligned}d \circ U_{(U_{(a_1,b_1)}, U_{(a_2,b_2)})}(v) &= d((a_1(v), \overline{b}_1(v)), (\overline{a}_2(v), \theta^{-1}\overline{\overline{b}}_2(v))) \\ &= ((a_1(v), \overline{a}_2(v)), (\overline{b}_1(v), \theta^{-1}\overline{\overline{b}}_2(v)\sigma))\end{aligned}$$

such that $(\theta^{-1}\overline{\overline{b}}_2(v)\sigma)(s,t) = (\theta^{-1}\overline{\overline{b}}_2(v))(t,s) = \overline{\overline{b}}_2(v)(t)(s) = \overline{b}_2(v,t)(s) = b_2(v,t,s)$. Therefore we have the desired result.

5. The excision theorems. In this section we consider the excision theorems on pair of maps. Let Ψ, Ψ' be pair-maps

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\Downarrow\Psi"] & B \arrow[d, "g_2"] \\ X \arrow[r, "f_2"] & Y \end{tikzcd}, \qquad \begin{tikzcd} A' \arrow[r, "f_1'"] \arrow[d, "g_1'"'] \arrow[dr, phantom, "\Downarrow\Psi'"] & B' \arrow[d, "g_2'"] \\ X' \arrow[r, "f_2'"] & Y' \end{tikzcd},$$

and let $\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix}$ be a map from Ψ to Ψ'.

PROPOSITION 5.1. *If* $\Lambda = (l_1, l_2): f_1 \to f_1'$ *and* $\Theta = (m_1, m_2): f_2 \to f_2'$, *and* l_1, l_2, m_1 *and* m_2 *are homotopy equivalences, then* $(\Lambda, \Theta)_* = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix}_*$: $\pi_2(V, \Psi) \to \pi_2(V, \Psi')$ *is* 1-1 *and onto.*

PROOF. We consider the following commutative diagram

$$\begin{array}{ccc} E_{g_1} & \xrightarrow{n_1} & E_{g_1'} \\ \Big\downarrow f_{1,2} & & \Big\downarrow f'_{1,2} \\ E_{g_2} & \xrightarrow{n_2} & E_{g_2'} \end{array},$$

where $f_{1,2}$ and $f'_{1,2}$ are defined as before by f_1, f_2 and f_1', f_2', respectively, and n_1 and n_2 are defined as follows:

$$n_1(a, \omega) = (l_1(a), \bar{\omega}) \quad \text{for} \quad a \in A,\ \omega \in PX \quad \text{with} \quad \bar{\omega}(s) = m_1(\omega(s)),$$

$$n_2(b, \eta) = (l_2(b), \bar{\eta}) \quad \text{for} \quad b \in B,\ \eta \in PY \quad \text{with} \quad \bar{\eta}(s) = m_2(\eta(s)).$$

Then n_1 and n_2 are homotopy equivalences by the assumptions and we obtain the commutative diagram

$$\begin{array}{ccc} \pi_1(V, f_{1,2}) & \xrightarrow{(n_1, n_2)_*} & \pi_1(V, f'_{1,2}) \\ \Big\uparrow \mathcal{E}_\Psi^{-1} & & \Big\uparrow \mathcal{E}_{\Psi'}^{-1} \\ \pi_2(V, \Psi) & \xrightarrow{(\Lambda, \Theta)_*} & \pi_2(V, \Psi') \end{array},$$

and $(n_1, n_2)_*$ is 1-1 and onto by Proposition 2.2; hence $(\Lambda, \Theta)_*$ is 1-1 and onto.

COROLLARY 5.2. *In Proposition* 5.1, *if* l_1, l_2 *are the identity maps and* m_1, m_2 *are homotopy equivalences, then* $(1, \Theta)_*$: $\pi_2(V, \Psi) \to \pi_2(V, \Theta \circ \Psi)$ *is* 1-1 *and onto.*

Similarly we have

COROLLARY 5.3. *In Proposition* 5.1, *if* m_1, m_2 *are the identity maps and* l_1, l_2 *are homotopy equivalences, then* $(\Lambda, 1)_*$: $\pi_2(V, \Psi' \circ \Lambda) \to \pi_2(V, \Psi')$ *is* 1-1 *and onto.*

REMARK. Corollary 5.2 and 5.3 are extensions of Proposition 6.2 and 6.3 in [1].

Let Ψ be a weak fibration (i.e., g_1 and g_2 are fibrations) with fibre $f_{X,Y}$:

$$\begin{tikzcd} F_X \arrow[r, "f_{X,Y}"] \arrow[d, "j_X"'] \arrow[dr, phantom, "\Downarrow\Pi"] & F_Y \arrow[d, "j_Y"] \\ A \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\Downarrow\Psi"] & B \arrow[d, "g_2"] \\ X \arrow[r, "f_2"] & Y \end{tikzcd} .$$

Then there are excision correspondences

$$\mathcal{E}_{1,\Psi} : \pi_1(V, f_{X,Y}) \longrightarrow \pi_2(V, \Psi) ,$$

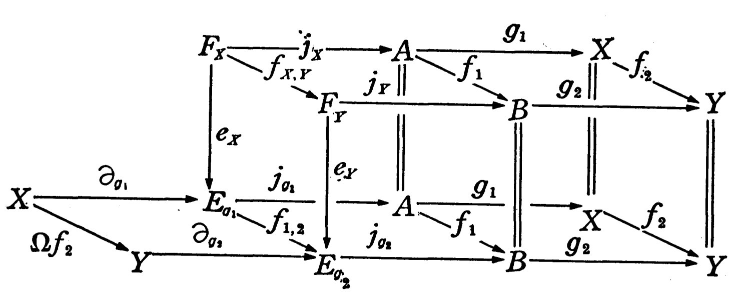
$$\mathcal{E}_{2,\Pi} : \pi_2(V, \Pi) \longrightarrow \pi_2(V, f_2)$$

defined as follows: Let Π_* and Ψ^* be pair-maps such that

$$\begin{tikzcd} F_X \arrow[r, "f_{X,Y}"] \arrow[d] \arrow[dr, phantom, "\Downarrow\Pi_*"] & F_Y \arrow[d] \\ * \arrow[r] & * \end{tikzcd} , \quad \begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\Psi^*"] & * \arrow[d] \\ X \arrow[r, "f_2"] & Y \end{tikzcd} .$$

For any element $\left\{\begin{pmatrix} a_1 a_2 \\ * \ * \end{pmatrix}\right\} \in \pi_2(V, \Pi_*) = \pi_1(V, f_{X,Y})$, $\mathcal{E}_{1,\Psi}\left\{\begin{pmatrix} a_1 a_2 \\ * \ * \end{pmatrix}\right\}$ $=\left\{\begin{pmatrix} j_X a_1 & j_Y a_2 \\ * & * \end{pmatrix}\right\} \in \pi_2(V, \Psi)$, and for any element $\left\{\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix}\right\} \in \pi_2(V, \Pi)$, $\mathcal{E}_{2,\Pi}\left\{\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix}\right\}$ $=\left\{\begin{pmatrix} * & * \\ g_1 b_1 & g_2 b_2 \end{pmatrix}\right\} \in \pi_2(V, \Psi^*) = \pi_2(V, f_2)$.

We consider the following diagram



where $e_X(a) = (j_X(a), *)$ for $a \in F_X$, $e_Y(b) = (j_Y(b), *)$ for $b \in F_Y$,
$\partial_{g_1}(\omega) = (*, \omega)$ for $\omega \in \Omega X$, $\partial_{g_2}(\eta) = (*, \eta)$ for $\eta \in \Omega Y$,

and e_X, e_Y are homotopy equivalences [2]. Then we have the commutative diagrams

$$\begin{array}{ccc} \pi_1(V, f_{X,Y}) & \xrightarrow{(e_X, e_Y)_*} & \pi_1(V, f_{1,2}) \\ \downarrow{\scriptstyle \mathcal{E}_{1,\Psi}} & & \uparrow{\scriptstyle \mathcal{E}_{\Psi}^{-1}} \\ \pi_2(V, \Psi) & = & \pi_2(V, \Psi) \end{array}, \qquad \begin{array}{ccc} \pi_2(V, \Pi) & \xrightarrow{(\Lambda, 1)_*} & \pi_1(V, \Pi') \\ \downarrow{\scriptstyle \mathcal{E}_{2,\Pi}} & & \uparrow{\scriptstyle \mathcal{E}_{1,\Pi'}} \\ \pi_2(V, f_2) & = & \pi_1(V, \Omega f_2) \end{array}.$$

where $\Lambda = (e_X, e_Y): f_{X,Y} \to f_{1,2}$ and $\Pi' = (j_{g_1}, j_{g_2}): f_{1,2} \to f_1$.

In the above diagrams, since $\mathcal{E}_{\Psi}^{-1}$ and $(e_X, e_Y)_*$ are 1-1 and onto we get that $\mathcal{E}_{1,\Psi}$ is 1-1 and onto, and since Π' is a weak fibration with fibre Ωf_2 the excision correspondence $\mathcal{E}_{1,\Pi'}$ is 1-1 and onto, and also $(\Lambda, 1)_*$ is 1-1 and onto by Corollary 5.3. Hence $\mathcal{E}_{2,\Pi}$ is 1-1 and onto.

The results obtained above are summarized as follows:

THEOREM 1. *If Ψ is a weak fibration as before, then the excision correspondences*

$$\mathcal{E}_{1,\Psi}: \pi_1(V, f_{X,Y}) \longrightarrow \pi_2(V, \Psi),$$
$$\mathcal{E}_{2,\Pi}: \pi_2(V, \Pi) \longrightarrow \pi_2(V, f_2)$$

are 1-1 *and onto.*

REMARK 1. Theorem 1 is an extension of the dual Theorem 6.5* in [1].

REMARK 2. Note that Theorem 1 and results in the preceding sections can be dualized.

Eckmann and Hilton defined homology groups of maps and pair-maps [1], [3]. If f and Ψ are a map and a pair-map, $H_q(f)$ and $H_q(\Psi)$ are defined and Abelian for all q.

Now let $P \xrightarrow{f} Q \xrightarrow{p} F$ be a cofibration; then the homology excision homomorphism $\mathcal{E}_{2,f}^H: H_q(f) \to H_q(F)$ is given by

$$\mathcal{E}_{2,f}^H(x, y) = py \quad \text{for} \quad x \in C_{q-1}(P),\ y \in C_q(Q),$$

hereafter we use the same symbol for a map and the chain map which it induces. It is well known that $\varepsilon^H_{2,f}$ is an isomorphism for all q.

Next let $P \xrightarrow{f} Q \xrightarrow{i_f} C_f$ be the extended cofibration, where f is any map. Then by (1.1′) we obtain the commutative diagram

$$\begin{tikzcd}
H_q(P) \arrow[r, "f_\#"] \arrow[d, equal] & H_q(Q) \arrow[r, "J_f"] & H_q(f) \\
H_q(P) \arrow[r, "q_{f\#}"] \arrow[d, equal] & H_q(M_f) \arrow[r, "J_{q_f}"] \arrow[u, "k_\#"'] \arrow[d, "k_\#"] & H_q(q_f) \arrow[u, "(1,k)_\#"'] \arrow[d, "\varepsilon^H_{2,q_f}"] \\
H_q(P) \arrow[r, "f_\#"] & H_q(Q) \arrow[r, "i_{f\#}"] & H_q(C_f) \;,
\end{tikzcd}$$

where each row is homology exact sequence and $k_\#$ is an isomorphism and $P \xrightarrow{q_f} M_f \xrightarrow{j_{q_f}} C_f$ is the cofibration. By the five lemma we deduce that $(1,k)_\#$ is an isomorphism, and we have easily

$$\bar{\varepsilon}^H_{2,f} = \varepsilon^H_{2,q_f} \circ (1,k)^{-1}_\# : H_q(f) \cong H_q(C_f) \quad \text{for all} \quad q\,,$$

where $\bar{\varepsilon}^H_{2,f}$ is defined by $\bar{\varepsilon}^H_{2,f}(x,z) = i_f z$ for $x \in C_{q-1}(P)$, $z \in C_q(Q)$.

Particularly, let $P \xrightarrow{f} Q \xrightarrow{p} F$ be the cofibration as before; then $\bar{\varepsilon}^H_{2,f} = \varepsilon^H_{2,q_f} \circ (1,k)^{-1}_\# = \tilde{k}^{-1}_\# \circ \varepsilon^H_{2,f}$:

$$\begin{tikzcd}
H_q(f) \arrow[r, "\varepsilon^H_{2,f}"] & H_q(F) \\
H_q(q_f) \arrow[r, "\varepsilon^H_{2,q_f}"] \arrow[u, "(1,k)_\#"'] & H_q(C_f) \;, \arrow[u, "\tilde{k}_\#"']
\end{tikzcd}$$

where $\tilde{k}$ is determined by 1 and k, and a homotopy equivalence [3; Corollary 3.7′].

Let $\Psi = (g_1, g_2) : f_1 \to f_2$ be a weak cofibration with cofibre $\bar{f}_{X,Y}$ or an extended weak cofibration with cofibre f_c (Ψ is any pair-map):

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "g_1"] & B \arrow[d, "g_2"] \\
X \arrow[r, "f_2"] \arrow[d, "i_1"] & Y \arrow[d, "i_2"] \\
\bar{F}_{g_1} \arrow[r, "\bar{f}_{X,Y}"] & \bar{F}_{g_2}\;,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "g_1"] & B \arrow[d, "g_2"] \\
X \arrow[r, "f_2"] \arrow[d, "i_{g_1}"] & Y \arrow[d, "i_{g_2}"] \\
C_{g_1} \arrow[r, "f_c"] & C_{g_2}\;,
\end{tikzcd}$$

where $i_{g_1}(i_{g_2})$ is an inclusion map and f_c is given by $f_c(x)=f_2(x)$ for $x\in X\subset C_{g_1}$ and $f_c(a,t)=(f_1(a),t)$ for $a\in A$.

If Ψ is the weak cofibration, the homology excision homomorphism

$$\mathcal{E}^H_{2,\Psi}: H_q(\Psi) \longrightarrow H_q(\bar{f}_{X,Y})$$

is defined by $\mathcal{E}^H_{2,\Psi}(a,b,x,y)=(i_1x, i_2y)$ for $a\in C_{q-2}(A)$, $b\in C_{q-1}(B)$, $x\in C_{q-1}(X)$, $y\in C_q(Y)$.

If Ψ is the extended weak cofibration, the excision homomorphism

$$\bar{\mathcal{E}}^H_{2,\Psi}: H_q(\Psi) \longrightarrow H_q(f_c)$$

is defined by $\bar{\mathcal{E}}^H_{2,\Psi}(a,b,x,y)=(i_{g_1}x, i_{g_2}y)$ for $a\in C_{q-2}(A)$, $b\in C_{q-1}(B)$, $x\in C_{q-1}(X)$, $y\in C_q(Y)$.

THEOREM 2. (i) *If $\Psi=(g_1,g_2): f_1\to f_2$ is the weak cofibration with cofibre $\bar{f}_{X,Y}$ then*

$$\mathcal{E}^H_{2,\Psi}: H_q(\Psi)\cong H_q(\bar{f}_{X,Y}) \text{ for all } q.$$

(ii) *If Ψ is the weak cofibration with cofibre f_c then*

$$\bar{\mathcal{E}}^H_{2,\Psi}: H_q(\Psi)\cong H_q(f_c) \text{ for all } q.$$

PROOF. (i) Consider the commutative diagram

$$\begin{tikzcd}
\longrightarrow H_q(g_1) \arrow[r, "(f_1,f_2)_\#"] \arrow[d, "\cong"', "\mathcal{E}^H_{2,g_1}"] & H_q(g_2) \arrow[r, "J"] \arrow[d, "\cong"', "\mathcal{E}^H_{2,g_2}"] & H_q(\Psi^T) \arrow[r, "\partial"] \arrow[d, "\mathcal{E}^H_{2,\Psi^T}"] & H_{q-1}(g_1) \longrightarrow \arrow[d, "\cong"', "\mathcal{E}^H_{2,g_1}"] \\
\longrightarrow H_q(\bar{F}_{g_1}) \arrow[r, "\bar{f}_{X,Y\#}"] & H_q(\bar{F}_g) \arrow[r, "J"] & H_q(\bar{f}_{X,Y}) \arrow[r, "\partial"] & H_{q-1}(\bar{F}_{g_1}) \longrightarrow
\end{tikzcd},$$

where the upper and lower rows are exact sequences of Ψ^T and $\bar{f}_{X,Y}$, respectively, and $\mathcal{E}^H_{2,\Psi^T}$ is defined by $\mathcal{E}^H_{2,\Psi^T}(a,x,b,y)=(i_1x, i_2y)$ for $a\in C_{q-2}(A)$, $x\in C_{q-1}(X)$, $b\in C_{q-1}(B)$, $y\in C_q(Y)$. Then by using the five lemma we obtain that $\mathcal{E}^H_{2,\Psi^T}$ is an isomorphism for all q. The chain map $\tau: C_q(\Psi)\to C_q(\Psi^T)$ is defined by $\tau(a,b,x,y)=(-a,x,b,y)$ and a chain isomorphism; hence τ induces a homology isomorphism $\tau_\#: H_q(\Psi)\cong H_q(\Psi^T)$ for all q (see [1], [3]). Since $\mathcal{E}^H_{2,\Psi}=\mathcal{E}^H_{2,\Psi^T}\circ\tau_\#$, $\mathcal{E}^H_{2,\psi}$ is an isomorphism for all q.

(ii) We consider the commutative diagram

$$\begin{array}{ccccccccccc}
\longrightarrow H_q(g_1) & \xrightarrow{(f_1,f_2)_\#} & H_q(g_2) & \xrightarrow{J} & H_q(\Psi^T) & \xrightarrow{\partial} & H_{q-1}(g_1) & \xrightarrow{(f_1,f_2)_\#} & H_{q-1}(g_2) \longrightarrow \\
\cong\downarrow \bar{\mathcal{E}}^H_{2,g_1} & & \cong\downarrow \bar{\mathcal{E}}^H_{2,g_2} & & \downarrow \bar{\mathcal{E}}^H_{2,\Psi^T} & & \cong\downarrow \bar{\mathcal{E}}^H_{2,g_1} & & \cong\downarrow \bar{\mathcal{E}}^H_{2,g_2} \\
\longrightarrow H_q(C_{g_1}) & \xrightarrow{f_{c\#}} & H_q(C_{g_2}) & \xrightarrow{J} & H_q(f_c) & \xrightarrow{\partial} & H_{q-1}(C_{g_1}) & \xrightarrow{f_{c\#}} & H_{q-1}(C_{g_2}) \longrightarrow,
\end{array}$$

where the upper and lower rows are exact sequences of Ψ^T and f_c, respectively, and $\bar{\mathcal{E}}^H_{2,\Psi^T}$ is defined similarly as $\mathcal{E}^H_{2,\Psi^T}$. Then $\bar{\mathcal{E}}^H_{2,g_1}$ and $\bar{\mathcal{E}}^H_{2,g_2}$ are isomorphisms, and hence by the five lemma $\bar{\mathcal{E}}^H_{2,\Psi^T}$ is isomorphism for all q. And since $\bar{\mathcal{E}}^H_{2,\Psi} = \bar{\mathcal{E}}^H_{2,\Psi^T} \circ \tau_\#$, $\bar{\mathcal{E}}^H_{2,\Psi}$ is an isomorphism for all q.

By Theorem 2 (ii) we have easily

COROLLARY 5.4. *If Ψ is the extended weak cofibration, then the sequence*

$$\longrightarrow H_q(f_1) \xrightarrow{(g_1,g_2)_\#} H_q(f_2) \xrightarrow{(i_{g_1},i_{g_2})_\#} H_q(f_c) \xrightarrow{\partial_{fc}} H_{q-1}(f_1) \longrightarrow$$

is exact, where $\partial_{f_c} = \partial_\Psi \circ \bar{\mathcal{E}}^{H^{-1}}_{2,\Psi}$.

The Whitehead theorem [5; p. 167] may be rewritten as follows:

LEMMA 5.5 (Whitehead). *In the sequence $E_f \longrightarrow X \xrightarrow{f} Y \longrightarrow C_f$,* (i) *if X and Y are arcwise connected and E_f is $(n-1)$-connected $(n>0)$, then C_f is homology n-connected.* (ii) *If X and Y are simply connected and C_f is homology n-connected, then E_f is $(n-1)$-connected.*

LEMMA 5.6 [7; Theorem 2.1]. *Let Ψ be the pair-map $(g_1,g_2): f_1 \to f_2$ in* (1.3) *such that A, B, X and Y are 1-connected, $\pi_q(g_1)=0$ for $0<q<m$ $(m>1)$, and $\pi_q(f_2)=0$ for $0<q<n$ $(n>1)$. Let* (A) *and* (B) *be the following statements:*

(A) $H_q(\Psi)=0$ *for* $q \leqq r$, (B) $\pi_q(\Psi)=0$ *for* $1<q\leqq r$.

Then if $1<r\leqq m+n-2$, (A) *implies* (B), *and if $1<r\leqq m+n-1$,* (B) *implies* (A).

Let Ψ be a weak fibration with fibre $f_{X,Y}$ as before, and we assume that A, B, X and Y are 1-connected, g_1 is m-connected $(m>1)$, f_2 is n-connected $(n>1)$, Y is $(r-1)$-connected $(r>1)$, and $\pi_q(\Psi)=0$ for $q\leqq l$ $(l>1)$.

Consider the following diagram

$$
(5.7) \qquad
\begin{tikzcd}
F_X * \Omega X \arrow[r, "f_{X,Y} * \Omega f_2"] \arrow[d, "j_0"] & F_Y * \Omega Y \arrow[d, "j_0'"] \\
C_{j_X} \arrow[r, "\tilde{f}"] \arrow[d, "r_X"] \arrow[dr, phantom, "\Downarrow \equiv"] & C_{j_Y} \arrow[d, "r_Y"] \\
X \arrow[r, "f_2"] & Y
\end{tikzcd} ,
$$

where j_0, j_0', r_X and r_Y are maps given in section 1 [2], and $\tilde{f} = f \cup Cf_{X,Y}$.

Then the upper diagram is homotopy commutative (c.f. [2; Proposition 1.3]) and the lower diagram is commutative.

PROPOSITION 5.8. *$f_{X,Y} * \Omega f_2$ is* Min.$(m+n, l+r-1)$*-connected.*

PROOF. Since $f_{X,Y} * \Omega f_2 = (1 * \Omega f_2) \circ (f_{X,Y} * 1)$, we shall prove that $1 * \Omega f_2$ and $f_{X,Y} * 1$ are $(m+n)$-connected and $(l+r-1)$-connected, respectively.

Now we introduce the homotopy commutative diagram

$$
\begin{tikzcd}
F_X * \Omega X \arrow[r, "1 * \Omega f_2"] \arrow[d, "w_X"] & F_X * \Omega Y \arrow[d, "w_Y"] & \\
\Sigma(F_X \# \Omega X) \arrow[r, "\Sigma(1 \# \Omega f_2)"] & \Sigma(F_X \# \Omega Y) \arrow[r] & \Sigma(F_X \# \Omega Y) \cup_{\Sigma(1 \# \Omega f_2)} C\Sigma(F_X \# \Omega X)
\end{tikzcd} ,
$$

where w's are maps defined in [9; p. 134] and these maps are homotopy equivalences by Proposition 1.2, and the lower row in the diagram is the extended cofibration. Then we have

$$
\begin{aligned}
\Sigma(F_X \# \Omega Y) \cup_{\Sigma(1 \# \Omega f_2)} C\Sigma(F_X \# \Omega X) &= \Sigma((F_X \# \Omega Y) \cup_{1 \# \Omega f_2} C(F_X \# \Omega X)) \\
&= \Sigma(F_X \# (\Omega Y \cup_{\Omega f_2} C\Omega X)) \quad \text{(c.f. [10])} \\
&\equiv F_X * (\Omega Y \cup_{\Omega f_2} C\Omega X) ,
\end{aligned}
$$

where $X \equiv Y$ implies that X and Y have the same homotopy type. Since $\Omega Y \cup_{\Omega f_2} C\Omega X$ is homology $(n-1)$-connected (see Lemma 5.5) and simply connected, we see that $\Omega Y \cup_{\Omega f_2} C\Omega X$ is $(n-1)$-connected, and also F_X is $(m-1)$-connected. Hence $F_X * (\Omega Y \cup_{\Omega f_2} C\Omega X)$ is $(m+n)$-connected. On the other hand, we get $H_q(1 * \Omega f_2) \cong H_q(\Sigma(1 \# \Omega f_2)) \cong H_q(F_X * (\Omega Y \cup_{\Omega f_2} C\Omega X))$ for all q. Hence

$1*\Omega f_2$ is homology $(m+n)$-connected, and the Whitehead theorem [5] we deduce that $1*\Omega f_2$ is $(m+n)$-connected. Similarly, $f_{X,Y}*1$ is $(l+r-1)$-connected. Therefore we have the desired result.

Now we introduce the commutative diagram

$$\begin{array}{ccc}
\pi_{q+1}(r_X) & \xrightarrow{\Xi^T_*} & \pi_{q+1}(r_Y) \\
\Big\downarrow \varepsilon^{-1}_{r_X} & & \Big\downarrow \varepsilon^{-1}_{r_Y} \\
\pi_q(E_{r_X}) & \xrightarrow{u_*} & \pi_q(E_{r_Y}) \\
\Big\uparrow w_{r_X} & & \Big\uparrow w_{r_Y} \\
\pi_q(F_X*\Omega X) & \xrightarrow{(f_{X,Y}*\Omega f_2)_*} & \pi_q(F_Y*\Omega Y)
\end{array},$$

where $\Xi^T=(\tilde{f},f_2)\colon r_X\to r_Y$, and u is determined by $\tilde{f}$ and f_2, and w's are maps given by section 1 [2]. Then $\varepsilon^{-1}_{r_X}$, $\varepsilon^{-1}_{r_Y}$, w_{r_X} and w_{r_Y} are isomorphisms, and $f_{X,Y}*\Omega f_2$ is Min.$(m+n,\ l+r-1)$-connected, Ξ^T is Min.$((m+n,\ l+r-1)+1)$-connected; hence so is $\Xi=(r_X,r_Y)\colon \tilde{f}\to f_2$. Since f_2 is n-connected and r_X is $(m-2)$-connected [2], and Min.$(m+n,\ l+r-1)+1<m+n+2$, we may apply Lemma 5.6 to the pair-map Ξ in (5.7), and we have $\Xi_\#=(r_X,r_Y)_\#\colon H_q(\tilde{f})\to H_q(f_2)$ is monomorphic for $q\leqq$ Min.$(m+n,\ l+r-1)$ and epimorphic for $q\leqq$ Min.$(m+n,\ l+r-1)+1$.

Now the homology excision homomorphism $\varepsilon'^H_{2,\Pi}\colon H_q(\Pi)\to H_q(f_2)$ defined by $\varepsilon'^H_{2,\Pi}(x,y,a,b)=(g_1a,g_2b)$ for $x\in C_{q-2}(F_X)$, $y\in C_{q-1}(F_Y)$, $a\in C_{q-1}(A)$, $b\in C_q(B)$. If we consider the extended weak cofibration Π with cofibre $\tilde{f}$, then $\bar{\varepsilon}^H_{2,\Pi}\colon H_q(\Pi)\to H_q(\tilde{f})$ is isomorphic for all q, and we have $\varepsilon'^H_{2,\Pi}=\Xi_\#\circ\bar{\varepsilon}^H_{2,\Pi}$. Thus the results obtained above is described as follows.

THEOREM 3. *Let Ψ be a weak fibration with fibre $f_{X,Y}$:*

$$\begin{array}{ccc}
F_X & \xrightarrow{f_{X,Y}} & F_Y \\
\Big\downarrow j_X & \Downarrow \Pi & \Big\downarrow j_Y \\
A & \xrightarrow{f_1} & B \\
\Big\downarrow g_1 & \Downarrow \Psi & \Big\downarrow g_2 \\
X & \xrightarrow{f_2} & Y
\end{array},$$

and we assume that

A, B, X and Y are 1-connected, g_1 is m-connected ($m > 1$), f_2 is n-connected ($n > 1$), Y is $(r-1)$-connected ($r > 1$), $\pi_q(\Psi) = 0$ for $q \leqq l$ ($l>1$).
Then the excision homomorphism

$$\mathcal{E}'^{H}_{2,\Pi} : H_q(\Pi) \longrightarrow H_q(f_2)$$

is isomorphic for $q \leqq \text{Min.}(m+n,\ l+r-1)$ and epimorphic for $q \leqq \text{Min.}(m+n, l+r-1) + 1$.

LEMMA 5.9 [6; Lemma 4.1]. *Let $f\colon X \to Y$ be a map, and if the induced homomorphism $f_{\#}\colon H_q(X) \to H_q(Y)$ is isomorphic for $q < N$ and epimorphic for $q = N$, then $f^*\colon \pi(Y, W) \to \pi(X, W)$ is 1-1 for $\pi_q(W) = 0$, $q \geqq N+1$ and onto for $\pi_q(W) = 0$, $q \geqq N$.*

COROLLARY 5.10. *Under the assumptions of Theorem 3, the excision correspondence*

$$\mathcal{E}'_{2,\Pi} : \pi_1(f_2, W) \longrightarrow \pi_2(\Pi, W)$$

is 1-1 for $\pi_q(W) = 0$, $q \geqq \text{Min.}(m+n, l+r-1)+2$ and onto for $\pi_q(W) = 0$, $q \geqq \text{Min.}(m+n, l+r-1) + 1$.

PROOF. We consider the following commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f_1"] \arrow[dd, "g_1"'] \arrow[dr, "i_{j_X}"] & & B \arrow[dd, "g_2"] \arrow[dr, "i_{j_Y}"] & & \\
& C_{j_X} \arrow[rr, "\tilde{f}"] \arrow[dl, "r_X"] & & C_{j_Y} \arrow[r, "i_{\tilde{f}}"] \arrow[dl, "r_Y"] & C_{\tilde{f}} \arrow[d, "\tilde{r}"] \\
X \arrow[rr, "f_2"] & & Y \arrow[rr, "i_{f_2}"] & & C_{f_2}
\end{tikzcd}$$

where $C_{\tilde{f}} = C_{j_Y} \cup_{\tilde{f}} CC_{j_X}$ and $C_{f_2} = Y \cup_{f_2} CX$, and $\tilde{f} = f_1 \cup Cf_{X,Y}$ and $\tilde{r} = r_Y \cup Cr_X$.
Then we obtain the commutative diagram

$$\begin{tikzcd}
H_q(\Pi) \arrow[r, "\bar{\mathcal{E}}^H_{2,\Pi}"] \arrow[d, equal] & H_q(\tilde{f}) \arrow[r, "\bar{\mathcal{E}}^H_{2,\tilde{f}}"] \arrow[d, "\Xi_{\#}"] & H_q(C_{\tilde{f}}) \arrow[d, "\tilde{r}_{\#}"] \\
H_q(\Pi) \arrow[r, "\mathcal{E}^H_{2,\Pi}"] & H_q(f_2) \arrow[r, "\bar{\mathcal{E}}^H_{2,f_2}"] & H_q(C_{f_2})
\end{tikzcd}.$$

Since $\bar{\mathcal{E}}^{H}_{2,\Pi}$, $\bar{\mathcal{E}}^{H}_{2,\tilde{f}}$ and $\bar{\mathcal{E}}^{H}_{2,f_2}$ are isomorphic we obtain $\tilde{r}_{\#}$ is monomorphic for $q \leqq \mathrm{Min.}(m+n, l+r-1)$ and epimorphic for $q \leqq \mathrm{Min.}(m+n, l+r-1)+1$. Hence by Lemma 5.9 we get $r^*: \pi(C_{f_2}, W) \to \pi(C_{\tilde{f}}, W)$ is 1-1 for $\pi_q(W)=0$, $q \geqq \mathrm{Min.}(m+n, l+r-1)+2$ and onto for $\pi_q(W)=0$, $q \geqq \mathrm{Min.}(m+n, l+r-1)+1$.

Next if we consider the following commutative diagram

$$\begin{tikzcd}
\pi_2(\Pi, W) \arrow[r, "\mathcal{E}'^{-1}_{\Pi}"] \arrow[d, equal] & \pi_1(\tilde{f}, W) \arrow[r, "\mathcal{E}'^{-1}_{\tilde{f}}"] & \pi(C_{\tilde{f}}, W) \\
\pi_2(\Pi, W) & \pi_1(f_2, W) \arrow[l, "\mathcal{E}'_{2,\Pi}"'] \arrow[u, "\Xi^*"'] \arrow[r, "\mathcal{E}'^{-1}_{f_2}"] & \pi(C_{f_2}, W) \arrow[u, "\tilde{r}^*"']
\end{tikzcd},$$

then $\mathcal{E}'^{-1}_{\Pi}$ $\mathcal{E}'^{-1}_{\tilde{f}}$ and $\mathcal{E}'^{-1}_{f_2}$ are 1-1 and onto (see remarks of section 2 and 3). Therefore we have the desired result.

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MATHEMATICAL INSTITUTE
AKITA UNIVERSITY
AKITA, JAPAN