Tôhoku Math. Journ.
23(1971), 13-15.

NOTE ON THE CLUSTER SETS OF MEROMORPHIC FUNCTIONS

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(Received May 16, 1970)

1. Hervé [1] and Matsumoto [2] have proved the interesting theorems on cluster sets of single-valued meromorphic functions in general domains.

In this note, we shall show that the same assertion as in these theorems holds under a somewhat weak condition.

Let E be a totally disconnected compact set in the z-plane, F the complementary domain of E with respect to the extended z-plane and $\{F_n\}$ $(n=0,1,\cdots)$ an exhaustion of F. We consider the graph: $0<u(z)<R$, $0<v(z)<2\pi$ associated with the exhaustion $\{F_n\}$ in the sense of Noshiro [3]. The niveau curve $\gamma_r: u(z)=r(0<r<R)$ consists of a finite number of analytic closed curves $\gamma_r^{(i)}$ $(i=1,2,\cdots,m(r))$. We put $\Lambda(r)=\max_{1\leq i\leq m(r)}\int_{\gamma_r^{(i)}}dv$. We denote by $\log\mu_n$ the harmonic modulus of $F_n-\overline{F}_{n-1}$ and we put $\tau_n=\sum_{k=1}^{n}\log\mu_k$.

In the following, we consider a set E such that F has an exhaustion satisfying the condition

$$\text{(A)} \qquad \limsup_{n\to\infty}\ \log\mu_n\int_0^{\tau_{n-1}}\frac{dr}{\Lambda(r)}=\infty.$$

It is known that if E is of logarithmic capacity zero, then there exists an exhaustion of F satisfying (A), and that if there exists an exhaustion of F satisfying (A), then E belongs to the class N_B^0 (cf. Matsumoto [2]). Matsumoto [2] gave an example of a Cantor set E which is of logarithmic capacity positive and whose complement has an exhaustion satisfying (A).

2. Let D be a domain in the z-plane, Γ its boundary, E a totally disconnected compact set on Γ and z_0 a point of E. Let $f(z)$ be a single-valued and mermorphic function in D such that $\Omega=C_D(f,z_0)-C_{\Gamma-E}(f,z_0)$ is not empty.

Hervé [1] gave a theorem which can be stated as follows.

If E is of logarithmic capacity zero and if each point of E belongs to some component, of Γ, consisting of a non-degenerate continuum, then Ω is an open set and $w=f(z)$ takes every value, with two possible exceptions, belonging to each component Ω_n of Ω, infinitely often in the intersection of any neighborhood of z_0 and D.

Matsumoto's theorem [2] can be formulated in the following form.

If there exists an exhaustion of the complementary domain F of E satisfying the condition (A) *and if E is contained in a single component of Γ, then the same assertion as in the above theorem holds.*

As an extension of above theorems, we can get the following.

THEOREM. *If there exists an exhaustion of the complementary domain F of E satisfying the condition* (A) *and if each point of E belongs to a component of Γ consisting of a non-degenerate continuum, then the same assertion as in Hervé's theorem holds.*

For the proof of Theorem we need only to follow an argument due to Hervé [1] (cf. Noshiro [3]). Here we briefly explain the essential part of the proof.

Suppose that a component Ω_n of Ω contains an exceptional value w_0 in the intersection of any neighborhood of z_0 and D. Let U be a simply-connected domain containing w_0 whose closure $\overline{U}$ lies in Ω_n. We can take a positive number r_0 such that $f(z)\neq w_0$ in $\overline{(K)}\cap D$ and such that $\overline{U}\cap\overline{C_D(f,\zeta)}=\phi$ for all ζ belonging to $\overline{(K)}\cap(\Gamma-E)$, where $\overline{(K)}$ denotes the closure of the open disc $(K)=\{z\,|\,|z-z_0|<r_0\}$. Inside U we draw two simple closed curves $\mathcal{L}$ and $\mathcal{L}'$ such that the interior of $\mathcal{L}$ contains w_0 and $\mathcal{L}'$ and such that w_0 lies outside $\mathcal{L}'$. We denote by $E(U)$ the set of all points in $\overline{(K)}\cap E$ such that $C_D(f,\zeta)\supset U$.

Suppose that L is a simple closed curve in (K) which does not pass through any point of E and whose interior contains at least one point ζ of $E(U)$.

Since $w_0\in C_D(f,\zeta)-C_{\Gamma-E}(f,\zeta)-R_D(f,\zeta)$ and $E\in N_B^0$, there exists a path Λ inside L, terminating at a point $\zeta'\in E\cap(L)$ such that w_0 is an asymptotic value of $w=f(z)$ along Λ (cf. Noshiro [4]). We may assume that the image of Λ by $w=f(z)$ lies completely in the ring-domain $(\mathcal{L},\mathcal{L}')$ bounded by $\mathcal{L}$ and $\mathcal{L}'$. We consider the open set of all points z in $D\cap(L)$ such that $f(z)\in(\mathcal{L},\mathcal{L}')$ and we denote by Δ its component containing the path Λ. Denote by Δ_r the common part of Δ and the exterior of γ_r, by $A(r)$ the area of the Riemannian image of Δ_r under the function $w=f(z)$ and by $L(r)$ the total length of the image of

$\Delta \cup \gamma_r$.

By Matsumoto's argument [2], we can get easily the following lemma under the assumption of our Theorem.

LEMMA 1.

$$\liminf_{r\to\infty} L(r)/A(r) = 0.$$

By this lemma, we can apply Ahlfors theory of covering surfaces with an aide of the function $u(z)$. Therefore, we get the following lemma which is similar to that of Hervé [1].

LEMMA 2. *The component Δ is infinitely connected.*

Using this lemma and the same argument as in the proof of Hervé's theorem, we easily see that our assertion of Theorem is ture.

REMARK. By modifying Matsumoto's example, we can show the existence of a domain D and a set E which satisfy all conditions of our Theorem, but do not satisfy a condition of Matsumoto's theorem and Hervé's one.

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