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POSITIVELY CURVED COMPLEX HYPERSURFACES IMMERSED IN A COMPLEX PROJECTIVE SPACE

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1. **Introduction.** Let $P_m(C)$ be a complex projective space of complex dimension m with the Fubini-Study metric of constant holomorphic sectional curvature 1. Recently, using topological methods in algebraic geometry, we have proved the following result.

PROPOSITION A ([2]). *Let M be a complete complex hypersurface imbedded in $P_{n+1}(C)$. If $n \geqq 2$ and if every sectional curvature of M with respect to the induced metric is positive, then M is complex analytically isometric to a hyperplane $P_n(C)$.*

S. Tanno has tried to generalize this result to *immersed* hypersurfaces and obtained the following.

PROPOSITION B ([4]). *Let M be a complete complex hypersurface immersed in $P_{n+1}(C)$. If $n \geqq 2$ and if every sectional curvature of M with respect to the induced metric is greater than $(1/4)\{1-(n+2)/(3n)\}$, then M is complex analytically isometric to a hyperplane $P_n(C)$.*

The purpose of this paper is to prove the following theorem which is a generalization of Proposition A and is also an improvement of proposition B in the case $n \geqq 4$.

THEOREM. *Let M be a complete complex hypersurface immersed in $P_{n+1}(C)$. If $n \geqq 4$ and if every sectional curvature of M with respect to the induced metric is positive, then M is complex analytically isometric to a hyperplane $P_n(C)$.*

2. **Proof of Theorem.** First we note that since every sectional curvature of M is positive, M is compact (cf., Proposition 3.1 in [2]).

Let J(resp. $\tilde{J}$) be the complex structure of M(resp. $P_{n+1}(C)$) and g (resp. $\tilde{g}$) be the Kaehler metric of M(resp. $P_{n+1}(C)$). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation with respect to g (resp. $\tilde{g}$). Then the second fundamantal form σ of the immersion is given by

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$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y .$$

Let R be the curvature tensor field of M. Then the equation of Gauss is

$$\begin{aligned} g(R(X, Y)Z, W) = \tilde{g}(\sigma(X, W), \sigma(Y, Z)) - \tilde{g}(\sigma(X, Z), \sigma(Y, W)) \\ + \frac{1}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ + g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) \\ + 2g(X, JY)g(JZ, W)\} . \end{aligned}$$

Let ξ, $\tilde{J}\xi$ be local fields of orthonormal vectors normal to M. If we set

$$g(AX, Y) = \tilde{g}(\sigma(X, Y), \xi) ,$$
$$g(A^*X, Y) = \tilde{g}(\sigma(X, Y), \tilde{J}\xi) ,$$

then A and A^* are local fields of symmetric linear transformations. We can easily see that $A^* = JA$ and $JA = -AJ$ so that, in particular,

$$\operatorname{tr} A = \operatorname{tr} A^* = 0 .$$

The equation of Gauss can be written in terms of A's as

(1) $$\begin{aligned} g(R(X, Y)Z, W) = g(AX, W)g(AY, Z) - g(AX, Z)g(AY, W) \\ + g(JAX, W)g(JAY, Z) - g(JAX, Z)g(JAY, W) \\ + \frac{1}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ + g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) \\ + 2g(X, JY)g(JZ, W)\} . \end{aligned}$$

Let ρ be the scalar curvature of M. Then we have

(2) $$\rho = n(n+1) - \|\sigma\|^2 ,$$

where $\|\sigma\|$ is the length of the second fundamental form of the immersion so that $\|\sigma\|^2 = 2\operatorname{tr} A^2$. We can see from (1) that the sectional curvature K of M determined by orthonormal vectors X and Y is given by

(3) $$\begin{aligned} K(X, Y) = \frac{1}{4}\{1 + 3g(X, JY)^2\} \\ + g(AX, X)g(AY, Y) - g(AX, Y)^2 \\ + g(JAX, X)g(JAY, Y) - g(JAX, Y)^2 . \end{aligned}$$

At each point x of M, we can choose an orthonormal basis $e_1, \cdots, e_n$, $Je_1, \cdots, Je_n$ of $T_x(M)$ with respect to which the matrix of A is of the form

$$\begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ 0 & & & & & -\lambda_n \end{pmatrix}$$

so that

$$\| \sigma \|^2 = 2 \operatorname{tr} A^2 = 4 \Sigma \lambda_\alpha^2 .$$

From (3) we have, for $\alpha \neq \beta$,

$$K\left(\frac{e_\alpha + e_\beta}{\sqrt{2}}, \frac{Je_\alpha - Je_\beta}{\sqrt{2}}\right) = \frac{1}{4} - \frac{\lambda_\alpha^2 + \lambda_\beta^2}{2} .$$

Since every sectional curvature of M is positive, we have

$$\lambda_\alpha^2 + \lambda_\beta^2 < \frac{1}{2} . \tag{4}$$

From (4) we have

$$\lambda_\alpha^4 + \lambda_\alpha^2 \lambda_\beta^2 \leqq \frac{1}{2} \lambda_\alpha^2$$

and hence

$$(n-1) \Sigma \lambda_\alpha^4 + \sum_{\alpha \neq \beta} \lambda_\alpha^2 \lambda_\beta^2 \leqq \frac{n-1}{2} \Sigma \lambda_\alpha^2 ,$$

or

$$(n-2) \Sigma \lambda_\alpha^4 + (\Sigma \lambda_\alpha^2)^2 \leqq \frac{n-1}{2} \Sigma \lambda_\alpha^2 .$$

Therefore we have

$$(n-2) \operatorname{tr} A^4 + \frac{1}{2} (\operatorname{tr} A^2)^2 \leqq \frac{n-1}{2} \operatorname{tr} A^2 ,$$

i.e.,

$$(n-2) \operatorname{tr} A^4 + \frac{1}{8} \| \sigma \|^4 \leqq \frac{n-1}{4} \| \sigma \|^2 . \tag{5}$$

On the other hand, we know that the second fundamental form σ satisfies the following differential equation ([2]).

$$\frac{1}{2} \Delta \| \sigma \|^2 = \| \nabla' \sigma \|^2 - 8 \operatorname{tr} A^4 - \frac{1}{2} \| \sigma \|^4 + \frac{n+2}{2} \| \sigma \|^2 , \tag{6}$$

where Δ denotes the Laplacian and ∇' the covariant differentiation with respect to the connection in (tangent bundle) $\oplus$ (normal bundle). From (5) we have

$$-8\operatorname{tr} A^4 - \frac{1}{2}\|\sigma\|^4 + \frac{n+2}{2}\|\sigma\|^2 \geqq \frac{n-4}{2(n-2)}\|\sigma\|^2(n - \|\sigma\|^2) .$$

Moreover we can see from (4) that $\Sigma\lambda_\alpha^2 < n/4$, i.e., $\|\sigma\|^2 < n$. Therefore we have

$$-8\ \operatorname{tr} A^4 - \frac{1}{2}\|\sigma\|^4 + \frac{n+2}{2}\|\sigma\|^2 \geqq 0 ,$$

which, together with (6), implies $\Delta\|\sigma\|^2 \geqq 0$. Hence, by a well-known theorem of E. Hopf, $\|\sigma\|^2$ is a constant so that the scalar curvature ρ is a constant. This, combined with Theorem 1 in [1], implies that M is complex analytically isometric to a complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature. Moreover by Theorem 3 in [3] M is imbedded as a totally geodesic hypersurface. This completes the proof of Theorem.

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