Tôhoku Math. Journ.
24 (1972), 319-329.

ON THE LAW OF THE ITERATED LOGARITHM FOR LACUNARY TRIGONOMETRIC SERIES

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

SHIGERU TAKAHASHI

(Received Jan. 18, 1971)

1. Introduction. Throughout this note we set

$$S_N(x) = \sum_{k=1}^{N} a_k \cos 2\pi(n_k x + \alpha_k) \quad \text{and} \quad A_N = \Bigl(2^{-1} \sum_{k=1}^{N} a_k^2\Bigr)^{1/2} ,$$

where $\{n_k\}$ is a sequence of positive integers and we assume that

$$A_N \to +\infty \; ; \qquad \text{as } N \to +\infty \; . \tag{1.1}$$

In [2] M. Weiss has proved the following

THEOREM. *If $\{n_k\}$ and $\{a_k\}$ satisfy the conditions*

$$n_{k+1}/n_k > 1 + c \, , \qquad \textit{for some } c > 0 \, , \tag{1.2}$$

and

$$a_N = o(\sqrt{A_N^2/\log\log A_N}) \, , \qquad \textit{as } N \to +\infty \, , \tag{1.3}$$

then we have, for any sequence of real numbers $\{\alpha_k\}$,

$$\overline{\lim_{N\to\infty}}\, (2A_N^2 \log\log A_N)^{-1/2}\, S_N(x) = 1 \, , \qquad \text{a.e.} \, .$$

That is, the same law of the iterated logarithm holds for

$$\{\cos 2\pi(n_k x + \alpha_k)\}$$

as for the sequence of normalized, uniformly bounded independent random variables with vanishing mean values.

The purpose of the present note is to weaken the *lacunarity* condition (1.2). But we could show only the inequality "$\overline{\lim} \leqq 1$". In fact we prove the following

THEOREM. *Let $\{n_k\}$ and $\{a_k\}$ satisfy the conditions*

$$n_{k+1}/n_k > 1 + c\, k^{-\alpha} \, , \qquad \textit{for some } c > 0 \textit{ and } 0 < \alpha \leqq 1/2 \, , \tag{1.4}$$

and

$$a_N = O(\sqrt{A_N^2/N^{2\alpha}(\log A_N)^{1+\varepsilon}}) \, , \qquad \textit{for some } \varepsilon > 0, \textit{ as } N \to +\infty \, . \tag{1.5}$$

Then we have, for any sequence of real numbers $\{\alpha_k\}$,

$$\varlimsup_{N\to\infty} (2A_N^2 \log\log A_N)^{-1/2} S_N(x) \leqq 1 \qquad \text{a.e.} \tag{1.6}$$

If $\alpha = 0$, then the condition (1.4) is (1.2). It seems to me that the condition (1.5) is more restrictive than (1.3) is due to the magnitude of $\| S_N(x)/A_N \|_p$, $p \geqq 2$. In fact, we have noticed that for any given (c, α) such that $c > 0$ and $0 < \alpha \leqq 1/2$, there exists a sequence $\{n_k\}$ satisfying (1.4) which is not a $\Lambda(2)$-set (cf. [1]).

2. **Some Lemmas.** From now on let $\{n_k\}$ and $\{a_k\}$ satisfy the conditions (1.4) and (1.5), respectively.

(i) Let us put

$$p(0) = 0,\ p(k) = \max\{m;\ n_m \leqq 2^k\} \quad \text{for} \quad k \geqq 1\,,$$

$$\Delta_k(x) = \sum_{m=p(k)+1}^{p(k+1)} a_m \cos 2\pi(n_m x + \alpha_m) \quad \text{and} \quad B_k = A_{p(k+1)}\,.^{*)}$$

If $p(k) + 1 < p(k+1)$, then from (1.4) we have

$$\begin{aligned} 2 > n_{p(k+1)}/n_{p(k)+1} &> \prod_{m=p(k)+1}^{p(k+1)-1} (1 + cm^{-\alpha}) \\ &> 1 + c\{p(k+1) - p(k) - 1\}p^{-\alpha}(k+1)\,. \end{aligned}$$

Therefore, we have

$$p(k+1) - p(k) = O(p^{\alpha}(k))\,, \qquad \text{as } k \to +\infty\,, \tag{2.1}$$

and hence

$$\begin{aligned} \| \Delta_k \|_\infty \leqq \sum_{m=p(k)+1}^{p(k+1)} | a_m | &\leqq \max_{m \leqq p(k+1)} | a_m | \{p(k+1) - p(k)\} \\ &= O\,(B_k(\log B_k)^{-(1+\varepsilon)/2})\,, \qquad \textit{as } k \to +\infty\,. \end{aligned} \tag{2.2}$$

LEMMA 1. *For any given* k, j, q *and* h *satisfying*

$$p(j) + 1 < h \leqq p(j+1) < p(k) + 1 < q \leqq p(k+1)\,,$$

the number of solutions (n_r, n_i) *of the equation*

$$n_q - n_r = n_h - n_i\,,$$

where $p(j) < i < h$ *and* $p(k) < r < q$, *is at most* $C\,2^{j-k}p^{\alpha}(k)$, *where* C *is a positive constant independent of* k, j, q *and* h.

PROOF. Let (n_r, n_i) be any solution, then we have

$$n_r = n_q - (n_h - n_i) > n_q - 2^j > n_q(1 - 2^{j-k}) \geqq n_q(1 + 2^{j-k+1})^{-1}\,.$$

*) For some k, $p(k)$ may be equal to $p(k+1)$. Then we put $\Delta_k(x) = 0$.

If m_1(or m_2) denotes the smallest (or the largest) index of n_r of the solutions (n_r, n_i), then (1.4) implies that

$$1 + 2^{j-k+1} \geqq n_q/n_{m_1} \geqq n_{m_2+1}/n_{m_1} > \prod_{m=m_1}^{m_2} (1 + \mathrm{c}m^{-\alpha})$$
$$> 1 + c\,(m_2 - m_1 + 1)\, p^{-\alpha}(k+1)\ .$$

Since $p(k+1)/p(k) \to 1$, as $k \to +\infty$, $m_2 - m_1 + 1 < C\,2^{j-k}p^{\alpha}(k)$, for some constant C. Further, for any given q, r and h, there exists at most one n_i satisfying the equation. Hence we can complete the proof of the lemma.

In the same way we can prove the following

LEMMA 2. *For any given k, j, q and h satisfying*

$$j \leqq k-2,\ p(j+1) < h \leqq p(j+2) \quad \textit{and} \quad p(k+1) < q \leqq p(k+2)\ ,$$

the number of solutions (n_r, n_i) of the equation

$$n_q - n_r = n_h - n_i\ ,$$

where $p(j) < i \leqq p(j+1)$ and $p(k) < r \leqq p(k+1)$, is at most $C\,2^{j-k}p^{\alpha}(k)$, where C is a positive constant independent of k, j, q and h.

(ii) Let $\{\rho_k\}$ be a non-decreasing sequence of positive integers such that $\rho_1 = 2$, $\rho_k \to +\infty$ and $\rho_k = O((\log B_k)^{\varepsilon/4})$, as $k \to +\infty$. Putting $\phi(k) = \sum_{m=1}^{k} \rho_m$, we can take a sequence of nonnegative integers $\{q(k)\}$ satisfying the following conditions:

$$\begin{cases} q(0) = 0 \text{ and for } k \geqq 1,\ \phi(2k-1) < q(k) \leqq \phi(2k) \\ \text{and } \|\Delta_{q(k)-1}\|_2^2 \leqq \rho_{2k}^{-1} \displaystyle\sum_{m=\phi(2k-1)}^{\phi(2k)-1} \|\Delta_m\|_2^2\ . \end{cases}$$

If we put

$$Q_k(x) = \sum_{m=q(k-1)}^{q(k)-2} \Delta_m(x) \text{ and } D_k = B_{q(k)-2} = \left(\sum_{m=1}^{q(k)-2} \|\Delta_m\|_2^2\right)^{1/2},$$

then we have, by (2.2),

$$\text{(2.3)} \qquad \|Q_k\|_\infty \leqq \sum_{m=q(k-1)}^{q(k)-2} \|\Delta_m\|_\infty \leqq 3\rho_{2k} \sup_{m<q(k)-1} \|\Delta_m\|_\infty$$
$$= O\,(\rho_{q(k)-2} D_k (\log D_k)^{-(1+\varepsilon)/2}) = O\,(D_k(\log D_k)^{-(2+\varepsilon)/4})\ ,\ ^{*)}$$

and

$$\text{(2.4)} \qquad D_k^2 - D_{k-1}^2 = \sum_{m=q(k-1)-1}^{q(k)-2} \|\Delta_m\|_2^2 = O\,(D_k^2(\log D_k)^{-1-\varepsilon/2})\ , \quad \text{as } k \to +\infty\ .$$

---

*) It is seen that $q(k) > \phi(2k-1) \geqq (2k-1)\rho_1 = 4k-2$. Hence $q(k)-2 \geqq 2k$.

Further, we have, from the definition of $q(k)$,

$$\sum_{m=1}^{k-1} \| \Delta_{q(m)-1} \|_2^2 = o(D_k^2) \,, \qquad \text{as } k \to +\infty \,. \tag{2.5}$$

(iii) LEMMA 3. *We have the following ralations:*

i. *For any* $N > M \geqq 0$,

$$\left\| \sum_{k=M}^{N} \{\Delta_k^2 - \| \Delta_k \|_2^2\} \right\|_2^2 \leqq C \sum_{K=M}^{N} \| \Delta_k \|_2^2 \, B_N^2 (\log B_N)^{-(1+\varepsilon)} \,,$$

where C *is a constant idependent of* N *and* M.

ii. $$\left\| \sum_{k=1}^{N-1} \{\Delta_{q(k)-1}^2 - \| \Delta_{q(k)-1} \|_2^2\} \right\|_2^2 = O(D_N^4 (\log D_N)^{-(1+\varepsilon)}) \,, \qquad \textit{as } N \to +\infty \,.$$

iii. $$\left\| \sum_{k=1}^{N} \{Q_k^2 - \| Q_k \|_2^2\} \right\|_2^2 = O(D_N^4 (\log D_N)^{-1-\varepsilon/2}) \,, \qquad \textit{as } N \to +\infty \,.$$

RROOF. For simplicity of writing the formula we may assume that $\alpha_k = 0$, $k = 1, 2, \cdots$, that is, we prove the lemma only for cosine series. The general case follows the same lines.

i. We write $\Delta_k^2 - \| \Delta_k \|_2^2 = U_k(x) + V_k(x)$, where

$$\begin{cases} U_k(x) = \displaystyle\sum_{q=p(k)+1}^{p(k+1)} a_q \sum_{r=p(k)+1}^{q} a_r \cos 2\pi(n_q + n_r)\, x \,, \\ V_k(x) = \displaystyle\sum_{q=p(k)+2}^{p(k+1)} a_q \sum_{r=p(k)+1}^{q-1} a_r \cos 2\pi(n_q - n_r)\, x \,. \end{cases}$$

Then (2.2) implies that

$$\| U_k \|_2 \leqq \sum_{q=p(k)+1}^{p(k+1)} | a_q | \, \| \Delta_k \|_2 = O\,(B_k \| \Delta_k \|_2 (\log B_k)^{-(1+\varepsilon)/2}) \,,$$
$$\| V_k \|_2 = O\,(B_k \| \Delta_k \|_2 (\log B_k)^{-(1+\varepsilon)/2}) \,, \qquad \text{as } k \to +\infty \,.$$

Since the sequence $\{U_k(x)\}$ is orthogonal on $(0, 1)$, we have

$$\left\| \sum_{k=M}^{N} U_k \right\|_2^2 = \sum_{k=M}^{N} \| U_k \|_2^2 \leqq C \sum_{k=M}^{N} \| \Delta_k \|_2^2 \, B_N^2 (\log B_N)^{-1-\varepsilon} \,.$$

Hence, for the proof of the first relation in the lemma it is sufficient to show that for some constant C,

$$\sum_{k=M+1}^{N} \sum_{j=M}^{k-1} \left| \int_0^1 V_k(x)\, V_j(x) dx \right| \leqq C \sum_{k=M}^{N} \| \Delta_k \|_2^2 \, B_N^2 (\log B_N)^{-1-\varepsilon} \,. \tag{2.6}$$

From Lemma 1 and (1.5), we obtain, for $N \geqq k > j$,

$$\left|\int_0^1 V_k(x)\,V_j(x)dx\right|$$
$$\leqq C2^{j-k}p^{\alpha}(k)\sum_{q=p(k)+1}^{p(k+1)}|a_q|\max_{p(k)<r<q}|a_r|\sum_{h=p(j)+1}^{p(j+1)}|a_h|\max_{p(j)<i<h}|a_i|$$
$$\leqq C'2^{j-k}p^{-\alpha}(j)B_N^2(\log B_N)^{-(1+\varepsilon)}\sum_{q=p(k)+1}^{p(k+1)}|a_q|\sum_{h=p(j)+1}^{p(j+1)}|a_h|\,, \qquad (C'>0)\,.^{*)}$$

Further, from (2.1) we have

$$\sum_{m=p(i)+1}^{p(i+1)}|a_m| \leqq 2\,||\varDelta_i||_2\,\{p(i+1)-p(i)\}^{1/2} = O\,(||\varDelta_i||_2\,p^{\alpha/2}(i))\,, \quad \text{as } i\to+\infty\,.$$

Thus, we have

$$\left|\int_0^1 V_k(x)\,V_j(x)dx\right| \leqq CB_N^2(\log B_N)^{-(1+\varepsilon)}\,||\varDelta_k||_2\,||\varDelta_j||_2\,2^{j-k}p^{\alpha/2}(k)p^{-\alpha/2}(j)\,.$$

Since $p(j+1)/p(j)\to 1$, as $j\to+\infty$, we have $\sum_{j=1}^{k-1}2^{j-k}p^{-\alpha}(j)\leqq C\,p^{-\alpha}(k)$, for all $k\geqq 1$. Hence we have

$$\sum_{k=M+1}^{N}\sum_{j=M}^{k-1}||\varDelta_k||_2\,||\varDelta_j||_2\,2^{j-k}p^{\alpha/2}(k)p^{-\alpha/2}(j)$$
$$\leqq \sum_{k=M+1}^{N}||\varDelta_k||_2\,p^{\alpha/2}(k)\left\{\sum_{j=M}^{k-1}2^{j-k}p^{-\alpha}(j)\right\}^{1/2}\left\{\sum_{j=M}^{k-1}2^{j-k}\,||\varDelta_j||_2^2\right\}^{1/2}$$
$$= C\sum_{k=M+1}^{N}||\varDelta_k||_2\left\{\sum_{j=M}^{k-1}2^{j-k}\,||\varDelta_j||_2^2\right\}^{1/2}$$
$$\leqq C\left\{\sum_{k=M+1}^{N}||\varDelta_k||_2^2\right\}^{1/2}\left\{\sum_{k=M+1}^{N}\sum_{j=M}^{k-1}2^{j-k}\,||\varDelta_j||_2^2\right\}^{1/2}\leqq C\sum_{k=M}^{N}||\varDelta_k||_2^2\,.$$

The last two relations proves the first part of the Lemma.

ii. We can prove the second part in the same way.

iii. We have

$$Q_m^2(x) - ||Q_m(x)||_2^2$$
$$= \sum_{k=q(m-1)}^{q(m)-2}\{\varDelta_k^2 - ||\varDelta_k||_2^2\} + 2\sum_{k=q(m-1)+2}^{q(m)-2}\varDelta_k\sum_{j=q(m-1)}^{k-2}\varDelta_j + 2\sum_{k=q(m-1)+1}^{q(m)-2}\varDelta_k\varDelta_{k-1}\,.$$

By the Minkowski inequality and the preceding relations, we have

$$\left\|\sum_{m=1}^{N}\sum_{k=q(m-1)}^{q(m)-2}\{\varDelta_k^2-||\varDelta_k||_2^2\}\right\|_2$$
$$\leqq \left\|\sum_{k=1}^{q(N)-2}\{\varDelta_k^2-||\varDelta_k||_2^2\}\right\|_2 + \left\|\sum_{m=1}^{N-1}\{\varDelta_{q(m)-1}^2-||\varDelta_{q(m)-1}||_2^2\}\right\|_2$$
$$= O\,(D_N^2(\log D_N)^{-(1+\varepsilon)/2})\,, \qquad \text{as } N\to+\infty\,.$$

*) We may assume that $p(j)\geqq p(1)>0$.

Since $\{\Delta_{3k+r} \sum_{j=q(m)}^{3k+r-2} \Delta_j\}$ is orthogonal for each r, we have, by (2.3),

$$\left\| \sum_{m=1}^{N} \sum_{k=q(m-1)+2}^{q(m)-2} \Delta_k \sum_{j=q(m-1)}^{k-2} \Delta_j \right\|_2^2 \leqq 3 \sum_{m=1}^{N} \sum_{k=q(m-1)+2}^{q(m)-2} \left\| \Delta_k \sum_{j=q(m-1)}^{k-2} \Delta_j \right\|_2^2$$
$$= O\Big(D_N^2(\log D_N)^{-1-\varepsilon/2} \sum_{k=1}^{q(N)-2} \| \Delta_k \|_2^2\Big) = O(D_N^4(\log D_N)^{-1-\varepsilon/2}) ,$$
$$\text{as } N \to +\infty .$$

Further, we have

$$\left\| \sum_{m=1}^{N} \sum_{k=q(m-1)+1}^{q(m)-2} \Delta_k \Delta_{k-1} \right\|_2^2$$
$$\leqq \sum_{k=1}^{q(N)-2} \| \Delta_k \Delta_{k-1} \|_2^2 + 2 \sum_{k=2}^{q(N)-2} \sum_{j=2}^{k-1} \left| \int_0^1 \Delta_k \Delta_{k-1} \Delta_j \Delta_{j-1} dx \right|$$
$$= O(D_N^4(\log D_N)^{-1-\varepsilon}) + 2 \sum_{k=2}^{q(N)-2} \sum_{j=1}^{k-1} \left| \int_0^1 \Delta_k \Delta_{k-1} \Delta_j \Delta_{j-1} dx \right| .$$

Using Lemma 2, the last term is estimated in the same way as that of (2.6) and we obtain

$$\sum_{k=2}^{q(N)-2} \sum_{j=1}^{k-1} \left| \int_0^1 \Delta_k \Delta_{k-1} \Delta_j \Delta_{j-1} dx \right| = O(D_N^4(\log D_N)^{-1-\varepsilon}) , \qquad \text{as } N \to +\infty .$$

Hence, we can prove the last part of the lemma.

3. Method of the proof of the Theorem. Let δ be an arbitrary positive number and let us take a positive number θ such that $0 < \theta - 1 < \delta^2$. For this θ, we put

$$M_k = \max\{m;\ D_m^2 \leqq \theta^k\} \quad \text{and} \quad m_k = \max\{m;\ B_m^2 \leqq \theta^k\} .$$

Then from (2.2) and (2.3) it is seen that there exists an integer K such that $k \geqq K$ implies that

$$D_{M_k}^2 \leqq B_{m_k}^2 \leqq \theta^k < B_{m_k+1}^2 \leqq D_{M_k+1}^2 < \theta^{k+1} . \tag{3.1}$$

If we prove that the following two relations

$$\overline{\lim_{k\to\infty}}(2\theta^k \log\log \theta^k)^{-1/2} \sum_{m=1}^{M_k} Q_m(x) \leqq 1 , \quad \text{a.e.} \tag{3.2}$$

and

$$\lim_{k\to\infty}(2\theta^k \log\log \theta^k)^{-1/2} \sum_{m=1}^{M_k-1} \Delta_{q(m)-1}(x) = 0 , \quad \text{a.e.} \tag{3.3}$$

hold, then we have, by (2.3),

$$\overline{\lim_{k\to\infty}}(2\theta^k \log\log \theta^k)^{-1/2} \sum_{m=1}^{m_k} \Delta_m(x) \leqq 1 , \quad \text{a.e.} .$$

Further, if we prove that

$$(3.4)\qquad \overline{\lim_{k\to\infty}} \sup_{m_k<m\leqq m_{k+1}} \{2(\theta^{k+1}-\theta^k)\log\log\theta^k\}^{-1/2}\sum_{j=m_k+1}^{m}\Delta_j(x)\leqq 4,\quad \text{a.e.},$$

then we have

$$\overline{\lim_{k\to\infty}} \sup_{m_k\leqq m<m_{k+1}} (2\theta^k\log\log\theta^k)^{-1/2}\sum_{j=1}^{m}\Delta_j(x)$$
$$\leqq 1+4\sqrt{\theta-1}\leqq 1+4\,\delta\,,\quad \text{a.e.}\,.$$

Since (3.1) and (2.2) imply that $B^2_{m_k}\sim\theta^k$, as $k\to+\infty$,*) and δ is arbitrary, we have

$$\overline{\lim_{k\to\infty}}(2B_k^2\log\log B_k)^{-1/2}\sum_{m=1}^{k}\Delta_m(x)\leqq 1\quad \text{a.e.},$$

and by (2.2), the last relation implies (1.6). Therefore, for the proof of the theorem it is sufficient to show that (3.2), (3.3) and (3.4) hold.

To this end we need the follwing two lemmas.

LEMMA 4. *We have, for* a.e. x,

$$\sum_{m=1}^{M_k}Q_m^2(x)\sim\theta^k\quad\text{and}\quad \sum_{m=m_k+1}^{m_{k+1}}\Delta_m^2(x)\sim(\theta^{k+1}-\theta^k)\,,\qquad \textit{as } k\to+\infty\,.$$

PROOF. Since $D^2_{M_k}\sim\theta^k$, as $k\to+\infty$, we have, by the last relation in Lemma 3,

$$\sum_{k=1}^{\infty}\left\|\theta^{-k}\sum_{m=1}^{M_k}\{Q_m^2-\|Q_m\|_2^2\}\right\|_2^2=O\Big(\sum_{k=1}^{\infty}k^{-(1+\varepsilon/2)}\Big)=O(1)\,.$$

Hence, we have, for a.e. x,

$$\lim_{k\to\infty}\theta^{-k}\sum_{m=1}^{M_k}\{Q_m^2(x)-\|Q_m\|_2^2\}=0\,.$$

On the other hand from (2.5) it is seen that

$$\sum_{m=1}^{M_k}\|Q_m\|_2^2=D^2_{M_k}-\sum_{m=1}^{M_k-1}\|\Delta_{q(m)-1}\|_2^2\sim D^2_{M_k}\sim\theta^k\,,\qquad \text{as } k\to+\infty\,.$$

Hence, we can prove the first part of the lemma. The remaining one can be proved in the same way.

LEMMA 5. *There exists a sequence* $\{\eta_k\}$ *satisfying the conditions;*

i. $\lim_{k\to\infty}\eta_k\,\theta^{-k}\sum_{m=1}^{M_k-1}\Delta^2_{q(m)-1}(x)=0\,,$ a.e. ,

ii. $\eta_k\to+\infty$ and $\eta_k=o(\sqrt{\log\log\theta^k})\,,$ *as* $k\to+\infty$.

*) For two sequences $\{a_k\}$ and $\{b_k\}$, $a_k\sim b_k$ means that $\lim_{k\to\infty}a_k/b_k=1$.

PROOF. We can easily prove this lemma by (2.5) and the second relation of Lemma 3.

4. Proof of the Theorem. In this paragraph we use frequently the following formula:

$$\exp(x - 2^{-1}x^2 - |x|^3) \leqq (1 + x)\,, \quad \text{for} \quad |x| < 1/2\,. \tag{4.1}$$

(i) Let η be an arbitrary positive number and let us put

$$\lambda_k = (2\theta^{-k} \log\log \theta^k)^{1/2} \quad \text{and} \quad y_k = (1 + \eta)\,\lambda_k^{-1} \log\log \theta^k\,.$$

Then we have, by (2.3), $\lambda_k \sup_{m \leqq M_k} \| Q_m \|_\infty = o(1)$, as $k \to +\infty$. Therefore, for sufficiently large k we have, by (4.1)

$$\exp\left\{\lambda_k \sum_{m=1}^{M_k} Q_m(x) - 2^{-1}\lambda_k^2 \sum_{m=1}^{M_k} Q_m^2(x) - \lambda_k^3 \sum_{m=1}^{M_k} | Q_m^3(x) |\right\}$$
$$\leqq \prod_{m=1}^{M_k} \{1 + \lambda_k Q_m(x)\}\,.$$

From the definition of $\{Q_m(x)\}$, the sequence of functions $\{Q_m(x)\}$ is multiplicatively orthogonal on $(0, 1)$, that is,

$$\int_0^1 \prod_{j=1}^{n} Q_{s_j}(x)dx = 0\,, \quad \text{for} \quad s_1 < s_2 < \cdots < s_n\,.$$

Hence we have $\int_0^1 \prod \{1 + \lambda_k Q_m(x)\}dx = 1$ and obtain

$$\int_0^1 \exp\left\{\lambda_k \sum_{m=1}^{M_k} Q_m(x) - 2^{-1}\lambda_k^2 \sum_{m=1}^{M_k} Q_m^2(x) - \lambda_k^3 \sum_{m=1}^{M_k} | Q_m^3(x) |\right\}dx \leqq 1\,.$$

Putting $F_k(x) = 2^{-1}\lambda_k \sum_{m=1}^{M_k} Q_m^2(x) + \lambda_k^2 \sum_{m=1}^{M_k} | Q_m^3(x) |$, we have, by the Tchebyschev inequality,

$$\left|\left\{x;\, x \in (0, 1),\, \sum_{m=1}^{M_k} Q_m(x) > F_k(x) + y_k\right\}\right|$$
$$\leqq e^{-\lambda_k y_k} = O(k^{-(1+\eta)})\,, \qquad \text{as } k \to +\infty\,,^{*)}$$

and hence

$$\sum_{k=1}^{\infty} \left|\left\{x;\, x \in (0, 1),\, \sum_{m=1}^{M_k} Q_m(x) > F_k(x) + y_k\right\}\right| < +\infty\,.$$

Therefore, for a.e. x there exists an integer $K(x)$ such that $k \geqq K(x)$ implies

$$\sum_{m=1}^{M_k} Q_m(x) \leqq F_k(x) + y_k\,.$$

---

*) For a measurable set E, $|E|$ denotes its Lebesgue measure.

On the other hand we have, by Lemma 4 and (2.3),

$$F_k(x) + y_k \sim (2 + \eta)(2^{-1}\theta^k \log\log \theta^k)^{1/2}\ , \quad \text{a.e.}\ .$$

Hence, we have

$$\overline{\lim_{k\to\infty}}(2\theta^k \log\log \theta^k)^{-1/2} \sum_{m=1}^{M_k} Q_m(x) \leqq (1 + \eta/2) \quad \text{a.e.}\ .$$

Since $\eta > 0$ is arbitrary, we can prove (3.2).

(ii) Using the sequence $\{\eta_k\}$ in Lemma 5, we put

$$\lambda_k = (\eta_k \theta^{-k} \log\log \theta^k)^{1/2} \quad \text{and} \quad y_k = 2\lambda_k^{-1} \log\log \theta^k\ .$$

Then we have, by (2.2) and ii in Lemma 5, $\lambda_k \sup_{m<M_k} \| \Delta_{q(m)-1} \|_\infty = o(1)$, as $k \to +\infty$. Using the same method as above, we have, for a.e. x,

$$\sum_{m=1}^{M_k-1} \Delta_{q(m)-1}(x) \leqq G_k(x) + y_k\ , \quad \text{for} \quad k \geqq K(x)\ ,$$

where $G_k(x) = 2^{-1} \lambda_k \sum_{m<M_k} \Delta^2_{q(m)-1}(x) + \lambda_k^2 \sum_{m<M_k} | \Delta^3_{q(m)-1}(x) |$. On the other hand from Lemma 5 and (2.2) it is seen that

$$G_k(x) + y_k = o((\theta^k \log\log \theta^k)^{1/2})\ , \quad \text{a.e.}\ .$$

Hence, we can prove (3.3).

(iii) Let us put $\mu_{o,k} = \mu_0 = m_k$, $\mu_{k,k} = \mu_k = m_{k+1}$ and

$$\mu_{j,k} = \mu_j = \max\{m;\ B_m^2 \leqq \theta^k + j(\theta^{k+1} - \theta^k)k^{-1}\}\ , \quad \text{for} \quad j = 1, 2, \cdots, k-1\ .$$

Since (2.2) implies that $\sup_{n<m_{k+1}} \| \Delta_n \|_2^2 = O(\theta^k \cdot k^{-(1+\varepsilon)})$, as $k \to +\infty$, we have $\theta^k + (j-1)(\theta^{k+1} - \theta^k)k^{-1} < B^2_{\mu_j} \leqq \theta^k + j(\theta^{k+1} - \theta^k)k^{-1}$, for $j = 1, 2, \cdots, k$, and $k \geqq K_0$. Hence, we have, for $j = 0, 1, \cdots, k-1$, and $k \geqq K_0$,

$$\sum_{n=\mu_j+1}^{\mu_{j+1}} \| \Delta_n \|_2^2 \leqq 2\,(\theta^{k+1} - \theta^k)\,k^{-1}\ . \tag{4.3}$$

On the other hand if $\Delta_j(x) \neq 0$, then the frequencies of terms of $\Delta_j(x)$ lie in the interval $[2^j + 1, 2^{j+1}]$. Therefore, by the theorems on trigonometric series (cf. (4.4) p. 231 and (4.24) p. 233 in [3]), we have, for some constants C_1 and C_2 independent of j and k,

$$\begin{aligned}\Big\| \sup_{\mu_j<m\leqq\mu_{j+1}} \sum_{n=\mu_j+1}^{m} \Delta_n \Big\|_4^4 &\leqq C_1 \Big\| \sum_{n=\mu_j+1}^{\mu_{j+1}} \Delta_n \Big\|_4^4 \leqq C_2 \Big\| \sum_{n=\mu_j+1}^{\mu_{j+1}} \Delta_n^2 \Big\|_2^2 \\ &\leqq 2\,C_2 \Big\| \sum_{n=\mu_j+1}^{\mu_{j+1}} \{\Delta_n^2 - \| \Delta_n \|_2^2\} \Big\|_2^2 + 2C_2 \Big(\sum_{n=\mu_j+1}^{\mu_{j+1}} \| \Delta_n \|_2^2 \Big)^2 .\end{aligned}$$

By Lemma 3 and (4.3), we have, for some C_3,

$$\left\| \sup_{\mu_j < m \leqq \mu_{j+1}} \sum_{n=\mu_j+1}^{m} \varDelta_n \right\|_4^4 \leqq C_3\, \theta^{2k}\, k^{-2}\,.$$

Hence we have

$$\sum_{k=1}^{\infty} \sum_{j=1}^{k} \left\| \{(\theta^{k+1} - \theta^k) \log\log \theta^k\}^{-1/2} \sup_{\mu_j < m \leqq \mu_{j+1}} \sum_{n=\mu_j+1}^{m} \varDelta_n \right\|_4^4 < +\infty\,,$$

and this proves that for a.e. x,

$$\lim_{k\to\infty} \{(\theta^{k+1} - \theta^k) \log\log \theta^k\}^{-1/2} \sup_{j \leqq k} \sup_{\mu_j < m \leqq \mu_{j+1}} \sum_{n=\mu_j+1}^{m} \varDelta_n(x) = 0\,. \tag{4.4}$$

(iv) If we put $\lambda_k = \sqrt{(\theta^{k+1} - \theta^k)^{-1} \log\log \theta^k}$ and $y_k = 3\lambda_k^{-1} \log\log \theta^k$, then we have $\lambda_k \sup_{m \leqq m_{k+1}} \| \varDelta_m \|_\infty = o(1)$, as $k \to +\infty$. Therefore, for sufficiently large k we have, by (4.1),

$$\begin{aligned}
&\exp\left\{\lambda_k \sum_{n=m_k+1}^{\mu_j} \varDelta_n(x) - 2\lambda_k^2 \sum_{n=m_k+1}^{m_{k+1}} \varDelta_n^2(x)\right\} \\
&\leqq \exp\left\{\lambda_k \sum_{n=m_k+1}^{\mu_j} \varDelta_n(x) - 2\lambda_k^2 \sum_{n=m_k+1}^{\mu_j} \varDelta_n^2(x)\right\} \\
&\leqq \exp\left\{\lambda_k \sum_{n=m_k+1}^{\mu_j} \varDelta_n(x) - \lambda_k^2 \sum_{n=m_k+1}^{\mu_j} \varDelta_n^2(x) - 4\lambda_k^3 \sum_{n=m_k+1}^{\mu_j} | \varDelta_n^3(x) |\right\} \\
&\leqq \left[\prod_{n=m_k+1}^{\mu_j} \{1 + 2\lambda_k \varDelta_n(x)\}\right]^{1/2}.
\end{aligned}$$

Since the both sequences $\{\varDelta_{2n}(x)\}$ and $\{\varDelta_{2n+1}(x)\}$ are multiplicatively orthogonal on (0, 1), we have

$$\begin{aligned}
&\int_0^1 \exp\left\{\lambda_k \sum_{n=m_k+1}^{\mu_j} \varDelta_n(x) - 2\lambda_k^2 \sum_{n=m_k+1}^{m_{k+1}} \varDelta_n^2(x)\right\} dx \\
&\leqq \int_0^1 \left[\prod_{n=m_k+1}^{\mu_j} \{1 + 2\lambda_k \varDelta_n(x)\}\right]^{1/2} dx \\
&\leqq \left[\int_0^1 \prod\nolimits_1 \{1 + 2\lambda_k \varDelta_{2n}(x)\} dx \int_0^1 \prod\nolimits_3 \{1 + 2\lambda_k \varDelta_{2n+1}(x)\} dx\right]^{1/2} = 1\,,
\end{aligned}$$

where $\prod_1$ (or $\prod_2$) is the product over all n such that $m_k < 2n \leqq \mu_j$ (or $m_k < 2n + 1 \leqq \mu_j$). Hence, we have

$$\begin{aligned}
&\left| \left\{x;\, x \in (0, 1);\, \sum_{n=m_k+1}^{\mu_j} \varDelta_n(x) > 2\lambda_k \sum_{n=m_k+1}^{m_{k+1}} \varDelta_n^2(x) + y_k\right\} \right| \\
&\leqq e^{-\lambda_k y_k} = O(k^{-3}),\ \text{for } j = 1, \cdots, k,\ \text{as } k \to +\infty\,,
\end{aligned}$$

and hence, we have

$$\sum_{k=1}^{\infty} \sum_{j=1}^{k} \left| \left\{x;\, x \in (0, 1);\, \sum_{n=m_k+1}^{\mu_j} \varDelta_n(x) > 2\lambda_k \sum_{n=m_k+1}^{m_{k+1}} \varDelta_n^2(x) + y_k\right\} \right| < +\infty\,.$$

This shows that for a.e. x, there exists an integer $K(x)$ such that

$$\sup_{1\leqq j\leqq k}\sum_{n=m_k+1}^{\mu_j}\varDelta_n(x)\leqq 2\lambda_k\sum_{n=m_k+1}^{m_{k+1}}\varDelta_n^2(x)+y_k\ ,\quad \text{if}\quad k\geqq K(x)\ .$$

On the other hand by Lemma 4 we have, for a.e. x,

$$2\lambda_k\sum_{n=m_k+1}^{m_{k+1}}\varDelta_n^2(x)+y_k\sim 5\sqrt{(\theta^{k+1}-\theta^k)\log\log\theta^k}\ ,\quad \text{as}\quad k\to+\infty\ .$$

Therefore, we have

$$\lim_{k\to\infty}\{(\theta^{k+1}-\theta^k)\log\log\theta^k\}^{-1/2}\sup_{j\leqq k}\sum_{n=m_k+1}^{\mu_j}\varDelta_n(x)\leqq 5\ ,\quad \text{a.e.}\ . \tag{4.5}$$

By (4.4) and (4.5), we can prove (3.4).

REFERENCES

[1] S. TAKAHASHI, On trigonometric Fourier coefficients. Tôhoku Math. Jour., 21 (1969), 405-418.

[2] M. WEISS, The law of the interated logarithm for lacunary trigonometric series, Trans. Amer. Math. Soc., 91 (1959), 444-469.

[3] A. ZYGMUND, Trigonometric Series, Vol. II, Cambridge University Press, 1959.

DEPARTMENT OF MATHEMATICS
KANAZAWA UNIVERSITY
KANAZAWA, JAPAN