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ON CONVEXITY THEOREMS FOR RIESZ MEANS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Let $\sum_{n=0}^{\infty} a_n$ be an infinite series, and let $\{\lambda_n\}$ be positive numbers tending to the infinity. We write $A_n = a_0 + a_1 + \cdots + a_n$, and if $x > 0$, $\lambda_n \leqq x < \lambda_{n+1}$, then $A_\lambda(x) \equiv A_n = a_0 + a_1 + \cdots + a_n = \sum_{\lambda_i \leqq x} a_i$, and for $k > 0$,

$$A_\lambda^k(x) = \frac{1}{\Gamma(k)}\int_0^x (x-t)^{k-1} A_\lambda(t)dt \ .$$

We define $A_\lambda^0(x) \equiv A_\lambda(x)$, and if $x < \lambda_0$, $A_\lambda^k(x) \equiv 0$ for every $k \geqq 0$.

Let us set $b_n = \lambda_n a_n$, $B_\lambda(x) = \sum_{\nu=0}^{n} \lambda_\nu a_\nu$, $\lambda_n \leqq x < \lambda_{n+1}$,

$$B_\lambda^k(x) = \frac{1}{\Gamma(k)}\int_0^x (x-t)^{k-1} t A_\lambda(t)dt \ , \qquad (k > 0) \ .$$

We then have [2]

$$B_\lambda^k(x) = x A_\lambda^k(x) - k A_\lambda^{k+1}(x) \ . \tag{1.1}$$

If we write $C_\lambda^k(x) = x^{-k} A_\lambda^k(x)$, then $C_\lambda^k(x)$ is called the Riesz mean of order k and type λ, while $A_\lambda^k(x)$ is called the Riesz sum of order k and type λ associated with the series $\sum a_n$.

Since no confusion will arise, we write simply $A^k(x)$ in place of $A_\lambda^k(x)$.

2. The author [6] proved the following theorem.

THEOREM A. *Let $V(x)$ and $W(x)$ be positive functions defined for $x > 0$, such that*

$$\begin{cases} \text{(i)} \quad x^\alpha W(x) \ \textit{is non-decreasing for some} \ \alpha, 0 \leqq \alpha < 1 \ , \\ \text{(ii)} \quad x^\beta V(x) \ \textit{is non-decreasing for some} \ \beta, \beta \geqq 0, \ \textit{and} \end{cases} \tag{2.1}$$

$$W(x)/V(x) = O(x^\delta) \qquad (\delta > 0) \qquad \textit{as} \ x \to \infty \ . \tag{2.2}$$

Then

$$A^\delta(x) = o[W(x)] \tag{2.3}$$

and

$$A(x) = O[V(x)] \tag{2.4}$$

together imply, for any γ *such that* $0<\gamma<\delta$,

$$A^{\gamma}(x)=o[(V(x))^{1-\gamma/\delta}(W(x))^{\gamma/\delta}] \quad as \quad x\rightarrow\infty . \tag{2.5}$$

In the case $\alpha=0$, $\beta=0$, Theorem A is reduced to M. Riesz's convexity theorem [5]. The following Theorem I shows an order-relation for $A^{\gamma}(x)$ with a hypothesis being different from $A(x)$ in Theorem A. The theorem is an extension of Theorem 3 in L. S. Bosanquet's paper [I], though the conditions are not exactly the same.

THEOREM I. *Let* $V(x)$ *and* $W(x)$ *be positive functions defined for* $x>0$, *such that*

$$\begin{cases} \text{(i)} & x^{\alpha}W(x) \textit{ is non-decreasing for some } \alpha,\ -1<\alpha<1, \\ \text{(ii)} & x^{\beta}V(x) \textit{ is non-decreasing for some } \beta, \textit{ and} \end{cases} \tag{2.6}$$

$$W(x)/V(x)=O(x^{\delta+\eta}) , \quad (\delta>0, \eta>0) . \tag{2.7}$$

Then

$$A(x)-A(x-t)=O[t^{\eta}V(x)] , \quad (\eta>0) \quad 0<t=O\{(W(x)/V(x))^{1/(\delta+\eta)}\} , \tag{2.8}$$

and

$$A^{\delta}(x)=o[W(x)] , \qquad \delta>0 \tag{2.9}$$

together imply, for any γ *such that* $0<\gamma<\delta$,

$$A^{\gamma}(x)=o[(V(x))^{1-(\gamma+\eta)/(\delta+\eta)}(W(x))^{(\gamma+\eta)/(\delta+\eta)}] \quad as \quad x\rightarrow\infty . \tag{2.10}$$

$V(x)$ and $W(x)$ above mentioned are quasi-monotonic functions for $\beta>0$ and $0<\alpha<1$.

We shall prove this theorem in section 4.

We have also a one-sided convexity theorem, as follows.

THEOREM II. *Let* $V(x)$ *and* $W(x)$ *be positive functions defined for* $x>0$. *If* (2.6) *and* (2.7) *hold, and*

$$A(x)-A(x-t)>-Kt^{\eta}V(x) , \qquad (\eta>0) \tag{2.11}$$

where $0<t=O\{(W(x)/V(x))^{1/(\delta+\eta)}\}$, *and*

$$A^{\delta}(x)=o[W(x)] , \qquad (\delta>0) \tag{2.12}$$

where $W(x')/W(x)<H$ *for* $0<x'-x=O\{(W(x)/V(x))^{1/(\delta+\eta)}\}$, *then we have, for* $0<\gamma<\delta$,

$$A^{\gamma}(x)=o[(V(x))^{1-(\gamma+\eta)/(\delta+\eta)}(W(x))^{(\gamma+\eta)/(\delta+\eta)}] \quad as \quad x\rightarrow\infty . \tag{2.13}$$

We shall prove this theorem in section 5.

3. Lemmas. The following lemmas are required for the proof of the above theorems:

LEMMA A. *Let $\varphi(x)$ be a positive, non-decreasing fuction of $x > 0$, and let $0 < \xi < x$, $0 < l < 1$, $k \geqq 0$. Then $A^{k+l}(x) = o[\varphi(x)]$ implies,*

$$(3.1)\qquad g(\xi, x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\Gamma(l)}\int_0^\xi (x-t)^{l-1}A^k(t)dt = o[\varphi(x)]\ .$$

This is given in [2].

LEMMA B. *If $k > 0$, $l > 0$, then*

$$(3.2)\qquad A^{k+l}(x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\Gamma(l)}\int_0^x (x-t)^{l-1}A^k(t)dt\ .$$

This is given in [4].

LEMMA C. *If $\zeta > 0$, m is a positive integer, $\gamma \geqq 0$ and $0 \leqq \beta < 1$, then*

$$(3.3)\qquad \zeta^{m+\beta}A^\gamma(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+m+1)}\Delta_\zeta^{m+\beta}A^{\gamma+m}(x) - \Delta_\zeta^\beta\left[\int_x^{x+\zeta}dt_1\int_{t_1}^{t_1+\zeta}dt_2\cdots\int_{t_{m-1}}^{t_{m-1}+\zeta}[A^\gamma(t_m) - A^\gamma(x)]dt_m\right.\ ,$$

and

$$(3.4)\qquad \zeta^{m+\beta}A^\gamma(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+m+1)}\Delta_{-\zeta}^{m+\beta}A^{\gamma+m}(x) + \Delta_{-\zeta}^\beta\left[\int_{x-\zeta}^{x}dt_1\int_{t_1-\zeta}^{t_1}dt_2\cdots\int_{t_{m-1}-\zeta}^{t_{m-1}}[A^\gamma(x) - A^\gamma(t_m)]dt_m\right.\ .$$

See [2] for finite differences.

4. Proof of Theorem 1.

(1) Proof for the case: $-1 < \alpha \leqq 0$ and $\beta \leqq 0$. Let us put, for any $\varepsilon > 0$, $\zeta = [\varepsilon W(x)/V(x)]^{1/(\delta+\eta)}$.

Then we have some ε such that $x - (p+1)\zeta > 0$ by (2.7), and let $\delta = p + a$, where $0 < a < 1$ and p is a non-negative integer. Then we have, by (3.4)

$$(4.1)\qquad \zeta^{p+a}A(x) = \frac{\Delta_{-\zeta}^{p+a}A^p(x)}{\Gamma(p+1)} + \Delta_{-\zeta}^a\left[\int_{x-\zeta}^x dt_1\int_{t_1-\zeta}^{t_1}dt_2\cdots\int_{t_{p-1}-\zeta}^{t_{p-1}}\{A(x) - A(t)\}dt_p\right]$$
$$= J_1 + J_2\ ,\quad \text{say.}$$

By (2.9) and Lemma A, we have

$$(4.2)\qquad J_1 = \frac{1}{\Gamma(p+1)}\Delta_{-\zeta}^{p+a}A^p(x) = \frac{a}{\Gamma(p+1)}\int_{x-\zeta}^x (x-t)^{a-1}\Delta_{-\zeta}^p A^p(t)dt$$

$$= \frac{a}{\Gamma(p+1)}\int_{x-\zeta}^{x}(x-t)^{a-1}\sum_{m=0}^{p}(-1)^m\binom{p}{m}A^p(t-m\zeta)dt$$
$$= \frac{a}{\Gamma(p+1)}\sum_{m=0}^{p}(-1)^m\binom{p}{m}\left\{\int_0^{x-m\zeta}(x-m\zeta-u)^{a-1}A^p(u)du\right.$$
$$\left.-\int_0^{x-(m+1)\zeta}(x-m\zeta-u)^{a-1}A^p(u)du\right\}$$
$$= o[W(x)] \text{ for sufficiently large } x\,.$$

By (2.8) and (2.6) (ii) we obtain

$$(4.3)\quad J_2 = \Delta^a_{-\zeta}\left[\int_{x-\zeta}^{x}dt_1\int_{t_1-\zeta}^{t_1}dt_2\cdots\int_{t_{p-1}-\zeta}^{t_{p-1}}\{A(x)-A(t_p)\}dt_p\right.$$
$$= \Delta^a_{-\zeta}\left[\int_0^{\zeta}dt_1\int_0^{\zeta}dt_2\cdots\int_0^{\zeta}\{A(x)-A(x-t_1-t_2-\cdots-t_p)\}dt_p\right]$$
$$= a\int_{x-\zeta}^{x}(x-u)^{a-1}\left[\int_0^{\zeta}dt_1\int_0^{\zeta}dt_2\cdots\int_0^{\zeta}\{A(u)-A(u-t_1-t_2-\cdots-t_p)\}dt_p\right]du$$
$$= O\left[\int_{x-\zeta}^{x}(x-u)^{a-1}\left[\int_0^{\zeta}dt_1\int_0^{\zeta}dt_2\cdots\int_0^{\zeta}(t_1+t_2+\cdots+t_p)^{\eta}V(u)dt_p\right]du\right.$$
$$= O\left[\zeta^{\eta+p}\int_{x-\zeta}^{x}(x-u)^{a-1}V(u)du\right]$$
$$= O\left[\zeta^{\delta+\eta}V(x)\right].$$

Hence

$$(4.4)\qquad A(x) = O[(1/\zeta^{\delta})W(x)] + [\zeta^{\eta}V(x)]$$
$$= O[(V(x))^{1-(\eta/(\delta+\eta))}(W(x))^{\eta/(\delta+\eta)}] \quad \text{as} \quad x\to\infty\,.$$

The one of the two hypotheses of Theorem A is satisfied with

$$(V(x))^{1-(\eta(\delta+\eta))}(W(x))^{\eta/(\delta+\eta)}\,,$$

with the other hypothesis $W(x)$ unchanged, instead of $V(x)$. Hence, using Theorem A, we obtain

$$(4.5)\quad A^{\gamma}(x) = o[\{(V(x))^{1-(\eta/(\delta+\eta))}(W(x))^{\eta/(\delta+\eta)}\}^{1-\gamma/\delta}\{W(x)\}^{\gamma/\delta}]$$
$$= o[(V(x))^{1-(\gamma+\eta)/(\delta+\eta)}(W(x))^{(\gamma+\eta)/(\delta+\eta)}],\ (0<\gamma<\delta) \quad \text{as} \quad x\to\infty\,.$$

Now, if δ is an integer then we can prove the case (1) by the similar method.

(II) Proof of the case: $0<\alpha<1$ and $\beta\leqq 0$. First assume Theorem I with α replaced by $\alpha-1$ (with β unchanged). Then, since $0<\alpha<1$, it follows from (2.9) and (2.6) (i) that

$$(4.6) \qquad A^{\delta+1}(x) = \int_0^x A^\delta(t)dt = o\left[\int_0^x W(t)dt\right] = o\left[\int_0^x t^\alpha t^{-\alpha} W(t)dt\right]$$
$$= o\left[x^\alpha W(x)\int_0^x t^{-\alpha}dt\right] = o[xW(x)] \ .$$

By (1.1) and (4.6), we obtain

$$(4.7) \qquad B^\delta(x) = o[xW(x)] \ .$$

By (2.8), we have, for $0 < t = O[W(x)/V(x)]^{1/(\delta+\eta)}$,

$$(4.8) \quad B(x) - B(x-t) = (x-t)[A(x) - A(x-t)] + \int_{x-t}^x [A(x) - A(u)]du$$
$$= O[t^\eta x V(x)] \ .$$

Thus, the hypotheses of Theorem I are satisfied with $B(x) - B(x-t)$ instead of $A(x) - A(x-t)$, with $B^\delta(x)$ instead of $A^\delta(x)$ and with $\alpha - 1$ instead of α, respectivly. We have, from the case assumed,

$$(4.9) \quad B^\gamma(x) = o[(xV(x))^{1-(\gamma+\eta)/(\delta+\eta)}(xW(x))^{(\gamma+\eta)/(\delta+\eta)}]$$
$$= o[x(V(x))^{1-(\gamma+\eta)/(\delta+\eta)}(W(x))^{(\gamma+\eta)/(\delta+\eta)}] \ , \qquad (0 < \gamma < \delta) \ .$$

Next suppose that $\gamma > 0$ and $\delta - 1 < \gamma < \delta$, then we obtain

$$(4.10) \ A^{\gamma+1}(x) = \frac{\Gamma(\gamma+2)}{\Gamma(\delta+1)\Gamma(\gamma-\delta+1)}\int_0^x (x-t)^{\gamma-\delta}A^\delta(t)dt$$
$$= o\left[\int_0^x (x-t)^{\gamma-\delta}t^{-\alpha}t^\alpha W(t)dt\right] = o\left[x^\alpha W(x)\int_0^x (x-t)^{\gamma-\delta}t^{-\alpha}dt\right]$$
$$= o\left[x^{\gamma-\delta+1}W(x)\int_0^1 (1-u)^{\gamma-\delta}u^{-\alpha}du\right] = o[x^{\gamma-\delta+1}W(x)] \ ,$$

by (2.6) (i) and (2.9). From (4.9), (4.10) and (2.7), we get

$$(4.11) \ A^\gamma(x) = (1/x)[B^\gamma(x) + \gamma A^{\gamma+1}(x)]$$
$$= o[(V(x))^{1-(\gamma+\eta)/(\delta+\eta)}(W(x))^{(\gamma+\eta)/(\delta+\eta)} + x^{\gamma-\delta}W(x)]$$
$$= o[(V(x))^{1-(\gamma+\eta)/(\delta+\eta)}(W(x))^{(\gamma+\eta)/(\delta+\eta)}\{1 + x^{\gamma-\delta}(W(x)/V(x))^{1-(\gamma+\eta)/(\delta+\eta)}\}]$$
$$= o[(V(x))^{1-(\gamma+\eta)/(\delta+\eta)}(W(x))^{(\gamma+\eta)/(\delta+\eta)}] \quad \text{as} \quad x \to \infty \ .$$

If $0 < \delta \leqq 1$, the result may be proved. And if $\delta > 1$, suppose now $0 < \gamma < \delta - 1$ and assume the result with γ replaced by $\gamma + 1$. Then it follows that

$$(4.12) \qquad A^\gamma(x) = (1/x)\{B^\gamma(x) + \gamma A^{\gamma+1}(x)\}$$
$$= o[(1/x)\{x(V(x))^{1-(\gamma+\eta)/(\delta+\eta)}(W(x))^{(\gamma+\eta)/(\delta+\eta)}$$
$$+ (V(x))^{1-(\gamma+1+\eta)/(\delta+\eta)}(W(x))^{(\gamma+1+\eta)/(\delta+\eta)}\}]$$
$$= o[(V(x))^{1-(\gamma+\eta)/(\delta+\eta)}(W(x))^{(\gamma+\eta)/(\delta+\eta)}] \quad \text{as} \quad x \to \infty \ .$$

and the result is proved by induction on γ.

(III) Proof of the case. $-1 < \alpha < 1$ and $\beta > 0$. First assume Theorem I with β replaced by $\beta - 1$. Then, since $-1 < \alpha < 1$, it follows from (2.9) and (2.6) (i) that

$$(4.13)\quad A^{\delta+1}(x) = \int_0^x A^\delta(t)dt = o\left[\int_0^x W(t)dt\right]$$
$$= o\left[\int_0^x t^\alpha t^{-\alpha} W(t)dt\right] = o\left[x^\alpha W(x)\int_0^x t^{-\alpha}dt\right] = o[xW(x)] .$$

By (1.1) and (4.13), we get

$$(4.14)\quad B^\delta(x) = o[xW(x)] ,$$

By (2.8), we have, for $0 < t = O[W(x)/V(x)]^{1/(\delta+\eta)}$,

$$(4.15)\quad B(x) - B(x-t) = (x-t)[A(x) - A(x-t)] + \int_{x-t}^x [A(x) - A(u)]du$$
$$= O[xt^\eta V(x)] + O\left[\int_0^t v^\eta V(x)dv\right] = O[t^\eta x V(x)] .$$

Thus, the hypotheses of Theorem I are satisfied with $B(x) - B(x-t)$ instead of $A(x) - A(x-t)$, with $B^\delta(x)$ instead of $A^\delta(x)$ and with $\beta - 1$ instead of β.

The rest of the proof is essentially the same as for case (II), then the result is proved by induction on β.

5. Proof of Theorem II. First suppose that $-1 < \alpha \leqq 0$, $\beta \leqq 0$ and let us put, for any $\varepsilon > 0$, $\zeta = [\varepsilon W(x)/V(x)]^{1/(\delta+\eta)}$. Then we have some ε such that $x - (p+1)\zeta > 0$ by (2.7), and let $\delta = p + a$, where $0 < a < 1$ and p is a non-negative integer.

Then we obtain, by (3.3),

$$(5.1)\quad \zeta^{p+a}A(x) = (1/\Gamma(p+1))\Delta_\zeta^{p+a}A^p(x)$$
$$- \Delta_\zeta^a\left[\int_x^{x+\zeta}dt_1\int_{t_1}^{t_1+\zeta}dt_2\cdots\int_{t_{p-1}}^{t_{p-1}+\zeta}[A(t_p) - A(x)]dt_p\right]$$
$$= I_1 + I_2 ,\quad \text{say.}$$

By (2.12) and Lemma A, we have

$$(5.2)\quad I_1 = \Delta_\zeta^{p+a}A^p(x) = a\int_x^{x+\zeta}(x+\zeta-t)^{a-1}\Delta_\zeta^p A^p(t)dt$$
$$= a\sum_{m=0}^p(-1)^m\binom{p}{m}\int_x^{x+\zeta}(x+\zeta-t)^{a-1}A^p(t+(p-m)\zeta)dt$$
$$= a\sum_{m=0}^p(-1)^m\binom{p}{m}\int_{x+(p-m)\zeta}^{x+(p+1-m)\zeta}(x+(p+1-m)\zeta-u)^{a-1}A^p(u)du$$
$$= o[W\{(x+(p+1-m)\zeta\}] = o[W(x)] .$$

By (2.11) we have, for $x - (p+1)\zeta > 0$,

$$I_2 = -\Delta_\zeta^a\left[\int_x^{x+\zeta} dt_1 \cdots \int_{t_{p-1}}^{t_{p-1}+\zeta} [A(t_p) - A(x)]dt_p\right] \tag{5.3}$$

$$= -a\int_0^\zeta (\zeta - u)^{a-1}du\left[\int_0^\zeta dt_1 \cdots \int_0^\zeta \{A(u + t_1 + \cdots + t_p) - A(u)\}dt_p\right]$$

$$< aK(p\zeta)^\eta \zeta^p \int_0^\zeta (\zeta - u)^{a-1} V(u + p\zeta)du$$

$$< Kp^\eta \zeta^{p+a+\eta} V(x) .$$

Then, we have, by (5.1), (5.2) and (5.3),

$$A(x) < (\varepsilon/\zeta^\delta) W(x) + K'\zeta^\eta V(x) \tag{5.4}$$

$$= (1 + K')\varepsilon^{\eta/(\delta+\eta)}(V(x))^{1-(\eta/(\delta+\eta))}(W(x))^{\eta/(\delta+\eta)}$$

Next, by (3.4) and (2.11) we obtain

$$A(x) = (1/\zeta^\delta)\Big[(1/\Gamma(p+1))\Delta_{-\zeta}^{p+a} A^p(x) \tag{5.5}$$

$$+ \Delta_{-\zeta}^a\left\{\int_{x-\zeta}^x dt_1 \int_{t_1-\zeta}^{t_1} dt_2 \cdots \int_{t_{p-1}-\zeta}^{t_{p-1}} \{A(x) - A(t_p)\}dt_p\right\}\Big]$$

$$> -\varepsilon\zeta^{-\delta} W(x) - K''\zeta^\eta V(x)$$

$$> -(1 + K'')\varepsilon^{\eta/(\delta+\eta)}(V(x))^{1-(\eta/(\delta+\eta))}(W(x))^{\eta/(\delta+\eta)} .$$

Thus, by (5.4) and (5.5), we have

$$|A(x)| = O[(V(x))^{1-\eta/(\delta+\eta)}(W(x))^{\eta/(\delta+\eta)}] \quad \text{as} \quad x \to \infty . \tag{5.6}$$

The rest of the proof is similar to that of Theorem I.

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