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ON A TAUBERIAN THEOREM FOR SEQUENCES WITH GAPS AND ON FOURIER SERIES WITH GAPS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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Many statements about sequences $x = (x_k)$ with gaps, that is sequences x with the property that $x_k = 0$ for all k belonging to a certain index set $I \subset Z^+ = \{1, 2, \cdots\}$ are of one of the following two types:

1. All sequences which belong to a certain space A and which fulfill a certain gap condition, belong also to some space $B \subsetneqq A$.

2. All sequences x which belong to a certain space A have the following property: There exists a sequence $y = (y_n) \in B \subsetneqq A$ such that for a certain index set $I = \{n_k\} \subset Z^+$

$$y_n = \begin{cases} x_k & \text{if } n = n_k \\ 0 & \text{if } n \neq n_k \, . \end{cases}$$

All statements of type 1 can be expressed in the form

$$\omega_I \cap A = \omega_I \cap B$$

where ω_I is the space of all sequences $x = (x_k)$ for which $x_k = 0$ if $k \notin I$.

All statements of type 2 can be expressed in the form

$$A + \omega_{I'} = B + \omega_{I'}$$

where $I' = Z^+ \backslash I$ and for any sets E_1 and E_2 of sequences the sum $E_1 + E_2$ of the two sets is defined by

$$E_1 + E_2 = \{x = (x_k) : x_k = u_k + v_k \text{ where } (u_k) \in E_1, (y_k) \in E_2\} \, .$$

It is the purpose of this note to demonstrate how the theory of sums and of intersections of FK-spaces can be used to simplify the proofs of known theorems on lacunary sequences and to obtain new results.

The main results of this paper (except for a few basic theorems) can be summarized as follows: Let $\{n_j\}_{j=1}^{\infty}$ be an increasing sequence of positive integers and let ω be the linear space of all complex valued sequences $x = (x_k)_{k=1}^{\infty}$. Let $\omega_I = \{x \in \omega : x_k = 0 \text{ if } k \neq n_j, j = 1, 2, \cdots\}$, let E be an FK-space (see section 1), $E_N = \{x \in E : x$ has sectional convergence $(AK)\}$, $E_{1N} = \{x \in E : x$ has Cesàro-sectional convergence, i.e. $(x_1, (1 - n^{-1})x_2, (1 -$

$2n^{-1})x_3 \cdots, (1-(n-1)n^{-1})x_n, 0, 0, \cdots) \to x, (n \to \infty)$ in $E\}$, $cs = \{x \in \omega: \sum_{k=1}^{\infty} x_k$ exists$\}$, $\sigma s = \{x \in \omega: \lim_{n\to\infty} \sum_{k=1}^{n} (1-(k-1)n^{-1})x_k$ exists$\}$. Then the following is true: If $\sigma s \cap \omega_I = cs \cap \omega_I$ then $E_{1N} \cap \omega_I = E_N \cap \omega_I$. (ii) Let $n_{j+1} \geqq qn_j$ where $q > 1$ is fixed. If $\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikt} \sim f(t) \in L^1(T)$, then there exists a continuous function g on T with Fourier series of power series type, i.e. such that $g(t) \sim \sum_{k=0}^{\infty} \hat{g}(k)e^{ikt}$ and such that $\hat{g}(n_j) = \hat{f}(n_j)$ for $j = 1, 2, \cdots$. If $\mu \in M(T)$, $\mu(t) \sim \sum_{k=-\infty}^{\infty} \hat{\mu}(k)e^{ikt}$ where $\hat{\mu}(k) = 0$ if $0 < k \neq n_j$, then there exists a continuous function h on T such that $h(t) \sim \sum_{k=-\infty}^{\infty} \hat{h}(k)e^{ikt}$ where $\hat{h}(k) = 0$ if $0 < k \neq n_j$ and $\hat{h}(n_j) = \hat{\mu}(n_j)$ for $j = 1, 2, \cdots$.

This summary is identical with the abstract [8]. I am grateful to Professor J. Fournier for bringing to my attention that statement (ii) can be derived also from statements in his paper [4].

Theorem 4.4 is stronger than statement (ii). Since its formulation is a little complicated I have preferred to state the simpler result (ii) in the introduction.

1. **Definitions.** Let ω be the linear space of all complex sequences $x = (x_x)_{k=1}^{\infty}$. Any vector subspace E of ω will be called a sequence space.

A sequence space E with a locally convex topology τ is called a K-space, if the inclusion map $(E, \tau) \to \omega$ is continuous when ω is endowed with the topology of coordinatewise convergence. If in addition (E, τ) is complete and metrizable, then (E, τ) is called an FK-space. The basic properties of FK-spaces can be found in papers of Zeller [17], [18] and in books of Goffman and Pedrick [10], Wilansky [15] or Zeller and Beekmann [19]. ω is an FK-space whose topology is given by the semi norms $p_k (k = 1, 2, \cdots)$ where $p_k(x) = |x_k|$ $(x \in \omega)$. An FK-space whose topology is given by a norm is called a BK-space.

If $I \subset Z^+$ then $\omega_I = \{x \in \omega: x_k = 0$ if $k \notin I\}$ is a closed subspace of ω, hence itself an FK-space under the topology induced by ω.

If $x \in \omega$, then the sequences $P_n x$ defined by

$$(P_n x)_k = \begin{cases} x_x & \text{if } k = 1, 2, \cdots, n \\ 0 & \text{if } k = n+1, n+2, \cdots \end{cases}$$

are called the n-th sections of x, and the sequences $P_n^1 x$ defined by

$$(P_n^1 x)_k = \begin{cases} (1-(k-1)/n)x_k & \text{if } k = 1, 2, \cdots, n \\ 0 & \text{if } k = n+1, n+2, \cdots \end{cases}$$

are called the n-th Cesàro-sections (of order one) of x.

If E is a K-space, $x \in E$, $P_n x \in E$ for every $n = 1, 2, \cdots$ and if $P_n x \to x$ (resp. $P_n^1 x \to x$) in E as $n \to \infty$, then x is said to have sectional convergence or AK (resp. Cesàro-sectional convergence or σK). For a K-space

E we define the spaces

$$E_N = \{x \in E: x \text{ has } AK\}. \ E_{1N} = \{x \in E: x \text{ has } \sigma K\}$$
$$E_{AB} = \{x \in \omega: \{P_n x\}_{n=1}^{\infty} \text{ is a bounded subset of } E\}$$
$$E_{\sigma B} = \{x \in \omega: \{P_n^1 x\}_{n=1}^{\infty} \text{ is a bounded subset of } E\}.$$

E has AK (resp. σK) if $E = E_N$ (resp. $E = E_{1N}$). E has AB (resp. σB) if $E \subset E_{AB}$ (resp. $E \subset E_{\sigma B}$). It should be noted that it is not necessarily true that $E_{AB} \subset E$ or $E_{\sigma B} \subset E$.

If (p_j) are the semi norms defining the topology of an FK-space E, then E_N and E_{AB} are FK-spaces with the topology given by the semi norms $g_j (j = 1, 2, \cdots)$ where $g_j(x) = \sup_n p_j(P_n x)$ [7]. Correspondingly E_{1N} and $E_{\sigma B}$ are FK-spaces with the topology given by the semi norms $g_j (j = 1, 2, \cdots)$ where $g_j(x) = \sup_n p_j(P_n^1 x)$.

The following BK-spaces with the indicated norms will be important in the sequel: For $1 \leqq p < \infty$

$$l^p = \left\{x \in \omega: \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} = \|x\|_{l^p} < \infty\right\},$$
$$l^\infty = \left\{x \in \omega: \sup_k |x_k| = \|x\|_{l^\infty} < \infty\right\},$$

c_0 is the closed subspace of l^∞ consisting of x with $\lim_{k\to\infty} x_k = 0$,

$$cs = \left\{x \in \omega: \sum_{k=1}^{\infty} x_k \text{ exists}\right\}, \ \|x\|_{cs} = \sup_n \left|\sum_{k=1}^{n} x_k\right|,$$
$$\sigma s = \left\{x \in \omega: (C, 1) - \sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=1}^{n}\left(1 - \frac{k-1}{n}\right)x_k \text{ exists}\right\},$$
$$\|x\|_{\sigma s} = \sup_n \left|\sum_{k=1}^{n}\left(1 - \frac{k-1}{n}\right)x_k\right|,$$
$$bv = \left\{x \in \omega: \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty\right\},$$
$$\|x\|_{bv} = \sum_{k=1}^{\infty} |x_k - x_{k+1}| + \lim_{k\to\infty} |x_k|,$$
$$bv_0 = bv \cap c_0,$$
$$q = \left\{x \in l^\infty: \sum_{k=1}^{\infty} (k+1)|\Delta^2 x_k| < \infty, \text{ where } \Delta^2 x_k = x_k - 2x_{k+1} + x_{k+2}\right\},$$
$$\|x\|_q = \sum_{k=1}^{\infty} (k+1)|\Delta^2 x_k| + \sup_k |x_k|,$$
$$q_0 = q \cap c_0.$$

In addition we use the following notations:

$$\phi = \{x \in \omega: x_k = 0 \text{ except for finitely many } k \in Z^+\}.$$

If $x, y \in \omega$ then $xy = (x_k y_k)$.
If $E \subset \omega$, then for any real number α

$$d^\alpha E = \{x \in \omega: (k^{-\alpha} x_k) \in E\} \text{ and } \int^\alpha E = d^{-\alpha} E .$$

For any pair of sets $A \subset \omega$ and $B \subset \omega$ the space $(A \to B)$ of multipliers from A to B is defined by $(A \to B) = \{x \in \omega: xa \in B \text{ for every } a \in A\}$.

In particular we define the β-and σ-(dual Köthe-) spaces A^β respectively A^σ of A by $A^\beta = (A \to cs)$, $A^\sigma = (A \to \sigma s)$.

The set $A_\perp$ orthogonal to A, is defined by $A_\perp = (A \to \{0\})$ where $\{0\}$ is the set consisting of the null element in ω.

If E is any K-space then E' shall denote the space of linear continuous functionals on E.

A sequence space E is called solid (or normal) if $E = (l^\infty \to E)$.

2. Basic Theorems

LEMMA 2.1. *Suppose E is an FK-space.*

(i) *If E has AK, then E' can be identified with the quotient space $E^\beta/E_\perp$.*

(ii) *If E has σK, then E' can be identified with the quotient space $E^\sigma/E_\perp$.*

PROOF. (i) Suppose E is an FK-space with AK. Then $\varphi \in E'$ if and only if there exists a sequence $y \in E^\beta$ such that

$$\varphi(x) = \sum_{k=1}^{\infty} x_k y_k \tag{1}$$

for every $x \in E$ ([18], Satz 2.1 and Satz 3.4).

Evidently

$$\varphi(x) = \sum_{k=1}^{\infty} x_k (y_k + z_k)$$

for every $z \in E_\perp$ and $E_\perp$ is the kernel of the homomorphism $h: E^\beta \to E'$, where $h(y) = \varphi$ and φ is given by (1). Hence the mapping $g: E^\beta/E_\perp \to E'$ where $g(y + E_\perp) = \varphi$ is an isomorphism. Thus E' can be identified with $E^\beta/E_\perp$ by means of the isomorphism g.

(ii) If E has σK a similar proof can be given using in this case the fact that $\varphi \in E'$ if and only if there exists a sequence $y \in E^\sigma$ such that

$$\varphi(x) = (C. 1) - \sum_{k=1}^{\infty} x_k y_k$$

for every $x \in E$ ([7], Theorem 5).

THEOREM 2.2. *Let E be an FK-space, $\phi \subset E$. $I \subset Z^+$ and $I' = Z^+ \backslash I$.*

(i) *If E has AK, then* $(\omega_I \cap E)^\beta = \omega_{I'} + E^\beta$.

(ii) *If E has σK, then* $(\omega_I \cap E)^\sigma = \omega_{I'} + E^\sigma$.

PROOF. (i) Evidently $\omega_{I'} + E^\beta \subset (\omega_I \cap E)^\beta$. For the proof of the opposite inclusion we use Lemma 2.1. Since $\omega_I \cap E$ being the intersection of two FK-spaces with AK is also an FK-space with AK and since $\phi \subset E$ implies $(\omega_I \cap E)_\perp = \omega_{I'}$, we can write

$$(\omega_I \cap E)' = (\omega_I \cap E)^\beta / \omega_{I'} \tag{1}$$

where the equality here and in similar cases later means identification of the two spaces as explained in the proof of Lemma 2.1.

Now

$$(\omega_I \cap E)' = (\omega_I)'|_{\omega_I \cap E} + E'|_{\omega_I \cap E}$$

where $A'|B$ denotes the restriction to B of the functionals on A ([17], p. 472. Satz 4.7 b)). Again by Lemma 2.1 we obtain

$$(\omega_I)' | \omega_I \cap E = (\omega_I)^\beta / \omega_{I'}$$

and

$$E' | \omega_I \cap E = E^\beta / \omega_{I'} \,.$$

Hence

$$(\omega_I \cap E)' = (\omega_I)^\beta / \omega_{I'} + E^\beta / \omega_{I'} \,. \tag{2}$$

(1) and (2) imply

$$(\omega_I \cap E)^\beta / \omega_{I'} = (\omega_I)^\beta / \omega_{I'} + B^\beta / \omega_{I'} \,.$$

Hence

$$(\omega_I \cap E)^\beta \subset (\omega_I)^\beta + \omega_{I'} + E^\beta = \omega_{I'} + E^\beta$$

where the last equation follows from the obvious equation $(\omega_I)^\beta = \omega_{I'} + \phi$ and $\phi \subset E^\beta$. This together with the opposite inclusion proves the equation in (i).

(ii) This can be proved correspondingly using part (ii) of Lemma 2.1.

LEMMA 2.3. *Let A and B be FK-spaces.*

(i) *If A has AK then* $(A \to B) = (A \to B_N)$.

(ii) *If A has σK, then* $(A \to B) = (A \to B_{1N})$.

PROOF. (i) Evidently $(A \to B_N) \subset (A \to B)$ since $B_N \subset B$. Conversely suppose $x \in (A \to B)$. The mappings $T_x \colon A \to B$ where $T_x a = xa$ for any $a \in A$, are linear and continuous [17]. Hence if $\{p_j\}$ are the semi norms defining the topology in B, then $q_j = p_j \cdot T_x$ is a continuous semi norm on A. Hence for each $j = 1, 2, \cdots$

$$q_j(P_n a - P_m a) = p_j[T_x(P_n a - P_m a)] = p_j[P_n(xa) - P_m(xa)] \to 0$$

if $n > m \to \infty$. This shows that $xa \in B_N$. Thus $x \in (A \to B_N)$.

(ii) This can be proved correspondingly using now P_n^1 instead of P_n.

LEMMA 2.4. *If $I \subset Z^+$ and $\phi \cap \omega_I \subset E \subset d^n l^\infty$ for some $n \in Z^+$, then $(\omega_I \to E) = \omega_{I'} + \phi$.*

PROOF. Let $E \subset d^n l^\infty$ for some $n \in Z^+$. This implies $\int^{n+2} E \subset cs$. Hence

$$\begin{aligned}(\omega_I \to E) &= \left(\int^{n+2} \omega_I \to \int^{n+2} E\right) \subset \left(\int^{n+2} \omega_I \to cs\right) = \left(\int^{n+2} \omega_I\right)^\beta \\ &= d^{n+2}(\omega_{I'} + \phi) = \omega_{I'} + \phi .\end{aligned}$$

Conversely if $\phi \cap \omega_I \subset E$ then $(\omega_{I'} + \phi) \subset (\omega_I \to E)$ since $\omega_I \cdot \omega_{I'} = \{0\}$ and $\phi \cdot \omega_I \subset \phi \cap \omega_I \subset E$.

2.5. Let A and B be FK-spaces whose topologies are defined by the collection of semi norms $\{p_j\}_{j=1}^\infty$ and $\{q_j\}_{j=1}^\infty$ respectively. Then $A + B$ is an FK-space in the topology defined by the semi norms $r_{j,k}(j, k \in Z^+)$ where

$$r_{j,k}(x) = \inf \{p_j(a) + q_k(b): a \in A, b \in B, x = a + b\}$$

(see [14] or [16]).

LEMMA. (i) *If A and B have AB, then $(A + B)_N = A_N + B_N$.*

(ii) *If A and B have σB, then $(A + B)_{1N} = A_{1N} + B_{1N}$.*

PROOE. (i) Clearly $A_N + B_N \subset (A + B)_N$. The opposite inclusion follows from the following equation of Garling ([5]; p. 1007): If E is an FK-space then $E_N = bv_0 \cdot E_{AB} = bv_0 \cdot E_N$. Since $A + B$ has AB if A and B have AB it follows that $(A+B)_N = bv_0 \cdot (A+B) \subset bv_0 \cdot A + bv_0 \cdot B = A_N + B_N$.

(ii) This can be proved correspondingly using the obvious inclusion $A_{1N} + B_{1N} \subset (A + B)_{1N}$ and the equation of Buntinas ([3], p. 197): If E is an FK-space then $E_{1N} = q_0 \cdot E_{\sigma B} = q_0 \cdot E_{1N}$.

LEMMA 2.6. *Let A, B, D be subsets of ω and $B = ((B \to D) \to D)$. Then $(A \to B) = [(B \to D) \to (A \to D)] = ([(A \to D) \to D] \to B)$.*

PROOF. Suppose $x \in (A \to B)$, $a \in A$. Then $xa \in B$. Hence if $h \in (B \to D)$, then $xah = (xh)a \in D$ implies $x \in [(B \to D) \to (A \to D)]$. If $B = ((B \to D) \to D)$, then by the same reasoning

$$\begin{aligned}[(B \to D) \to (A \to D)] &\subset \{[(A \to D) \to D] \to [(B \to D) \to D]\} \\ &= ([(A \to D) \to D] \to B) \subset (A \to B) ,\end{aligned}$$

where the last inclusion follows from $A \subset [(A \to D) \to D]$ ([7]; p. 139).

3. Lacunary sequence spaces with AK or with σK

THEOREM 3.1. *Suppose $I \subset Z^+$ such that*

$$\omega_I \cap \sigma s = \omega_I \cap cs \tag{1}$$

and suppose E is an FK-space with the property $\phi \cap \omega_I \subset E \subset d^n l^\infty$ for some $n \in Z^+$. Then

$$\omega_I \cap E_{1N} = \omega_I \cap E_N .$$

PROOF. σs is an FK-space with σK[12]. Hence by Theorem 2.2

$$(\omega_I \cap \sigma s)^\sigma = \omega_{I'} + (\sigma s)^\sigma .$$

Now $(\sigma s)^\sigma = q$, [2]. Hence

$$(\omega_I \cap \sigma s)^\sigma = \omega_{I'} + q . \tag{2}$$

Since $\omega_I \cap cs$ is an FK-space with AK and since $(\sigma s)_N = cs$ we have by Lemma 2.3 (i) and by Theorem 2.2

$$(\omega_I \cap cs)^\sigma = (\omega_I \cap cs)^\beta = \omega_{I'} + (cs)^\beta . \tag{3}$$

Now $(cs)^\beta = bv$ [2]. Hence (1), (2) and (3) imply

$$\omega_{I'} + q = \omega_{I'} + bv .$$

Hence $([\omega_{I'} + q] \to E_{1N}) = ([\omega_{I'} + bv] \to E_{1N})$, which clearly implies

$$(\omega_{I'} \to E_{1N}) \cap (q \to E_{1N}) = (\omega_{I'} \to E_{1N}) \cap (bv \to E_{1N}) . \tag{4}$$

Evidently $bv = bv_0 + [1]$, where [1] is the linear space spanned by the sequence $x = (x_k)$ with $x_k = 1$ for all k. Therefore

$$(bv \to E_{1N}) = (bv_0 + [1] \to E_{1N}) = (bv_0 \to (E_{1N})_N \cap E_{1N} \tag{5}$$

where the last equation follows from Lemma 2.3 since bv_0 has AK. Now $(E_{1N})_N = E_N$ since $E_{1N} \subset E$ implies $(E_{1N})_N \subset E_N = (E_N)_N \subset (E_{1N})_N$ where the equation follows from ([5], p. 1016, Cor. 2). Hence

$$(bv_0 \to (E_{1N})_N) = (bv_0 \to E_N) . \tag{6}$$

Now

$$(bv_0 \to E_N) = E_{AB} \tag{7}$$

because $bv_0 \cdot E_{AB} \subset E_N$ ([5], p. 1006, Th. 4) implies $E_{AB} \subset (bv_0 \to E_N)$ and conversely $(bv_0 \to E_N) \subset (bv \to E_{AB}) \subset E_{AB}$ ([5], p. 1002, Prop. 3, $(e) \Rightarrow (f)$).

Therefore (5), (6) and (7) imply

$$(bv \to E_{1N}) = E_{AB} \cap E_{1N} . \tag{8}$$

Furthermore

$$(q \to E_{1N}) = E_{1N} \quad ([3],\ \text{Th. } 3.11). \tag{9}$$

Thus (4), (8), (9) and Lemma 2.4 imply

$$(\omega_I + \phi) \cap E_{AB} \cap E_{IN} = (\omega_I + \phi) \cap E_{1N} \,.$$

Intersection of the sets on both sides of this equation with ω_I yields

$$\omega_I \cap E_{AB} \cap E_{1N} = \omega_I \cap E_{1N} \,.$$

Thus

$$\omega_I \cap E_{1N} \subset [\omega_I \cap E_{AB} \cap E_{1N}]_{1N} = \omega_I \cap (E_{AB})_{1N} \cap E_{1N}$$

since $(E_{1N})_{1N} = E_{1N}$ ([3], Th. 3.11 and Prop. 3.6). But $(E_{AB})_{1N} = q_0 \cdot E_{AB}$ ([3], Th. 3.11) and $q_0 \cdot E_{AB} \subset bv_0 \cdot E_{AB} = E_N$ ([5], p. 1006, Th. 4). Hence $\omega_I \cap E_{1N} \subset \omega_I \cap E_N$. Since the opposite inclusion is obvious, the theorem is proved.

EXAMPLE 3.2. Suppose $I = \Lambda = \{\lambda_k \in Z^+ \colon \lambda_{k+1} > q\lambda_k$ for some $q > 1$ and for all $k = 1, 2, \cdots\}$. For any sequence space E the intersection $\omega_\Lambda \cap E$ is the space of sequences in E which have Hadamard gaps. It is well known that $\omega_\Lambda \cap \sigma s = \omega_\Lambda \cap cs$ ([20], vol. I, p. 70). Hence the last theorem is applicable with $I = \Lambda$.

REMARK 3.3. The last proof contains several equations which are of independent interest. For example

$$\omega_{I'} + q = \omega_{I'} + bv$$

and

$$(bv \to E_{1N}) = E_{AB} \cap E_{1N} \,.$$

However the proof can be shortened considerably and the condition $\phi \cap \omega_I \subset E \subset d^n l^\infty$ can be omitted if one uses the concept of weak sectional convergence ($=SAK$; see [18]).

Let E be an FK-space. An element $x \in E$ has SAK, if for every linear continuous functional $\varphi \in E'$

$$\lim_{n \to \infty} \varphi(P_n x) = \varphi(x) \,.$$

Let

$$E_{SAK} = \{x \in E \colon x \text{ has } SAK\} \,.$$

In the proof of Theorem 3.4 we use the following important fact proved by Zeller ([18], Satz 3.4):

If E is an FK-space and $E = E_{SAK}$, then $E = E_N$.

THEOREM 3.4. *Suppose* $I \subset Z^+$ *such that*

$$\omega_I \cap \sigma s = \omega_I \cap cs. \tag{1}$$

Then for any FK-space E *we have*

$$\omega_I \cap E_{1N} = \omega_I \cap E_N . \tag{2}$$

PROOF. Suppose $\varphi \in (\omega_I \cap E_{1N})'$. Since $\omega_I \cap E_{1N}$ is an FK-space with σK, we have for $x \in \omega_I \cap E_{1N}$

$$\varphi(x) = (C, 1) - \sum_{k=1}^{\infty} x_k \varphi(e^k)$$

where e^k is the sequence $(u_j^k)_{j=1}^{\infty}$ with $u_j^k = 1$ if $j = k$ and $u_j^k = 0$ if $j \neq k$. Hence $\{x_k\varphi(e^k)\}_{k=1}^{\infty} \in \sigma s \cap \omega_I$ which by (1) implies $\{x_k\varphi(e^k)\} \in cs \cap \omega_I$. Hence $\lim_{n\to\infty} \varphi(P_n x) = \sum_{k=1}^{\infty} x_k \varphi(e^k) = \varphi(x)$ exists for every $x \in \omega_I \cap E_{1N}$ and for every $\varphi \in (\omega_I \cap E_{1N})'$. Thus by ([18], Satz 3.4)

$$(\omega_I \cap E_{1N}) = (\omega_I \cap E_{1N})_{SAK} = (\omega_I \cap E_{1N})_N = \omega_I \cap (E_{1N})_N = \omega_I \cap E_N$$

where we used again the fact that $(E_{1N})_N = E_N$ (see proof of Teeorem 3.1 just before equation (6)). Thus (2) follows.

4. Lacunary Fourier Series

4.1. In this section the technique developed in sections 2 and 3 will be applied to spaces of sequences of Fourier coefficients. As usual 1) $L^p(1 \leqq p < \infty)$, 2) L^∞, 3) C, 4) M will denote respectively the space of 2π-periodic, complex valued measurable functions f or measures μ on the real line for which 1) $\left(\int_0^{2\pi} |f|^p dt\right)^{1/p} = ||f||_p < \infty$, 2) ess. $\sup_{0\leqq t\leqq 2\pi} |f(t))| = ||f||_\infty < \infty$, 3) f is continuous, $||f||_C = \max_{0\leqq t\leqq 2\pi} |f(t)|$, 4) μ is a regular Borel measure, $||\mu|| = \int_0^{2\pi} |d\mu|$. If E is any of the above spaces, then the space of sequences of Fourier coefficients of even (resp. odd) functions or measures in E will be denoted by $\hat{E}_c$ (resp. $\hat{E}_s$). Thus

$$\hat{E}_c = \left\{x: \sum_{k=1}^{\infty} x_k \cos kt \sim g(t), \text{ where } g \in E\right\},$$

and

$$\hat{E}_s = \left\{x: \sum_{k=1}^{\infty} x_k \sin kt \sim h(t), \text{ where } h \in E\right\}.$$

For convenience we have assumed that $\int_0^{2\pi} g dt = 0$ $\left(\text{resp. } \int_0^{2\pi} dg = 0 \text{ if } g = \mu\right)$. This will affect our results only in an obvious way.

The spaces $\hat{E}_c$ and $\hat{E}_s$ are evidently BK-spaces with the norms $||x||_{\hat{E}_c} =$

$\|g\|_E$ and $\|x\|_{\hat{E}_s} = \|h\|_E$.

4.2. Our starting point is the following set of theorems of Paley and Banach for sequences with Hadamard gaps and a theorem of Helson (which had its origin in a theorem of Paley).

Let Λ be an index set defining Hadamard gaps (3.2); then the following statements are true:

a) (Paley [13] or [20], vol. II, p. 133)

$$\hat{L}_c \cap \hat{L}_s + \omega_{\Lambda'} = l^2 + \omega_{\Lambda'} .$$

b) (Banach [1] or [20], vol. II, p. 131 (7.1))

$$l^2 + \omega_{\Lambda'} = \hat{C}_c + \omega_{\Lambda'} = \hat{C}_s + \omega_{\Lambda'} .$$

c) (Banach [1] or [20], vol. II, p. 131 (7.1))

$$\hat{M}_c \cap d(\hat{C}_s) + \omega_{\Lambda'} = \hat{M}_s \cap d(\hat{C}_c) + \omega_{\Lambda'} = l^\infty + \omega_{\Lambda'} .$$

d) (Helson [11])

$$(\hat{C}_c \cap \hat{C}_s \to l^1) = l^2 .$$

4.3. As immediate consequences of these statements we list the following:

e) $$\hat{L}_c \cap \hat{L}_s + \omega_{\Lambda'} = \hat{C}_c + \omega_{\Lambda'} = \hat{C}_s + \omega_{\Lambda'} .$$

PROOF. This follows from a) and b).

f) $$l^\infty \cdot (\hat{L}_c \cap \hat{L}_s + \omega_{\Lambda'}) = (\hat{C}_c + \omega_{\Lambda'})_N = (\hat{C}_s + \omega_{\Lambda'})_N .$$

PROOF. The spaces in e) as sums of FK-spaces are FK-spaces (2.5). Since each of them is equal to the FK-space $l^2 + \omega_{\Lambda'}$ by a), which is a solid (= normal) FK-space with AK, f) follows.

g) $$l^\infty \cdot (\hat{M}_c \cap \omega_\Lambda) = l^\infty \cdot (\hat{M}_s \cap \omega_\Lambda) = (l^\infty \to (\hat{C}_c + \hat{C}_s)_N \cap \omega_\Lambda) .$$

PROOF. By taking the σ-dual spaces E^σ of the spaces E in e) and observing that $\hat{L}_c^\sigma = \widehat{L_c^\infty}$, $\hat{L}_s^\sigma = \widehat{L_s^\infty}$ ([6], p. 369) hence $(\hat{L}_c \cap \hat{L}_s)^\sigma = \widehat{L_c^\infty} + \widehat{L_s^\infty}$ ([9], Th. 4) and $\hat{C}_c^\sigma = \hat{M}_c$, $\hat{C}_s^\sigma = \hat{M}_s$ ([6], p. 369) we obtain

$$(\widehat{L_c^\infty} + \widehat{L_s^\infty}) \cap (\omega_\Lambda + \phi) = \hat{M}_c \cap (\omega_\Lambda + \phi) = \hat{M}_s \cap (\omega_\Lambda + \phi) .$$

Intersection with ω_Λ gives

$$(\widehat{L_c^\infty} + \widehat{L_s^\infty}) \cap \omega_\Lambda = \hat{M}_c \cap \omega_\Lambda = \hat{M}_s \cap \omega_\Lambda .$$

Since the spaces in the last line are evidently equal to the solid FK-space $l^2 \cap \omega_\Lambda$ which has AK (since $\widehat{L_c^\infty} + \widehat{L_s^\infty} \subset l^2 \subset \hat{M}_c$ and $l^2 \subset \hat{M}_s$), we obtain

$$(\widehat{L_c^\infty} + \widehat{L_s^\infty}) \cap \omega_A = (\widehat{L_c^\infty} + \widehat{L_s^\infty})_N \cap \omega_A = (l^\infty \to (\hat{C}_c + \hat{C}_s)_N \cap \omega_A)$$
$$= l^\infty \cdot (\hat{M}_c \cap \omega_A) = l^\infty \cdot (\hat{M}_s \cap \omega_A) .$$

Here we have used the relation $(\widehat{L_c^\infty} + \widehat{L_s^\infty})_N \subset (\widehat{L_c^\infty} + \widehat{L_s^\infty})_{1N} = \hat{C}_C + \hat{C}_s$. For the last equation see Lemma 2.5 and observe that $(\widehat{L_c^\infty})_{1N} = \hat{C}_c$, $(\widehat{L_s^\infty})_{1N} = \hat{C}_s$ and hence $(\widehat{L_c^\infty} + \widehat{L_s^\infty})_N \subset (\hat{C}_c + \hat{C}_s)_N$ ([18], Satz. 2.3).

We now present the main theorem of this section.

THEOREM 4.4. *If* $A = \{\lambda_k \in Z^+ : \lambda_{k+1} > q\lambda_k$ *for some* $q > 1$ *and* $k = 1, 2, \cdots\}$ *then*

a) $l^\infty \cdot [(\hat{M}_c + \hat{M}_s) \cap \omega_A] = (l^\infty \to (\hat{C}_c + \hat{C}_s)_N \cap \omega_A)$

b) $l^\infty \cdot (\hat{L}_c \cap \hat{L}_s + \omega_{A'}) = (l^\infty \to [\hat{C}_c \cap \hat{C}_s + \omega_{A'}]_N)$.

PROOF. a) by 4.2 d)

$$(1) \qquad (\hat{C}_c \cap \hat{C}_s \to l^1) = l^2 .$$

Also

(2) $(\hat{C}_c \cap \hat{C}_s \to l^1) = (l^\sigma \to (\hat{C}_c \cap \hat{C}_s)^\sigma)$ by Lemma 2.6 with $D = \sigma s$ since $(l^1 \to \sigma s)^\sigma = (l^\infty)^\sigma = l^1$ and $(\hat{C}_c \cap \hat{C}_s)^\sigma = \hat{C}_c^\sigma + \hat{C}_s^\sigma = \hat{M}_c + \hat{M}_s$ where the first equation follows from ([9], Th. 4) and the second from ([6], p. 369). Hence by (1) and (2) we have

$$(3) \qquad l^2 = (l^\infty \to \hat{M}_c + \hat{M}_s) .$$

By 4.2 c) we have

$$(4) \qquad l^\infty + \omega_{A'} = \hat{M}_c + \omega_{A'} .$$

This implies

$$(5) \qquad ((l^\infty + \omega_{A'}) \to (\hat{M}_c + \hat{M}_s)) = ((\hat{M}_c + \omega_{A'}) \to (\hat{M}_c + \hat{M}_s)) .$$

The left term is equal to $(l^\infty \to \hat{M}_c + \hat{M}_s) \cap (\omega_{A'} \to \hat{M}_c + \hat{M}_s)$ where $(\omega_{A'} \to \hat{M}_c + \hat{M}_s) = \omega_A + \phi$ by Lemma 2.4.

The right term in (5) is equal to

$(\hat{M}_c \to \hat{M}_c + \hat{M}_s) \cap (\omega_A + \phi)$ (again by Lemma 2.4).

Now $(\hat{M}_c \to \hat{M}_c + \hat{M}_s) = \hat{M}_c + \hat{M}_s$ since (1) $\in \hat{M}_c$ and $\hat{M}_c \circ (\hat{M}_c + \hat{M}_s) \subset \hat{M}_c \cdot \hat{M}_c + \hat{M}_c \cdot \hat{M}_s \subset \hat{M}_c + \hat{M}_s$. Thus by (5)

$$(6) \qquad (l^\infty \to \hat{M}_c + \hat{M}_s) \cap (\omega_A + \phi) = (\hat{M}_c + \hat{M}_s) \cap (\omega_A + \phi) .$$

(6) and (3) imply

$$l^2 \cap (\omega_A + \phi) = (\hat{M}_c + \hat{M}_s) \cap (\omega_A + \phi) .$$

By intersection with ω_A we obtain

$$(7) \qquad l^2 \cap \omega_A = (\hat{M}_c + \hat{M}_s) \cap \omega_A .$$

It follows from 4.3, g) that $l^2 \cap \omega_A = (l^\infty \to (C_c + C_s)_N \cap \omega_A)$. This implies with (7)

$$(8) \qquad (l^\infty \to (C_c + C_s)_N \cap \omega_A) = l^\infty \cdot [(\hat{M}_c + \hat{M}_s) \cap \omega_A] .$$

Thus a) is proved.

b) The equation in b) can be considered as the dual equation of the following equation

$$(9) \qquad (\hat{C}_c + \hat{C}_s) \cap \omega_A = (\hat{M}_c + \hat{M}_s) \cap \omega_A$$

which is immediate by (8).

Indeed (9) implies, by Theorem 2.2 (since $\hat{C}_c + \hat{C}_s$ as sum of two FK-space with with σK is also an FK-space σK), that

$$\begin{aligned}[(\hat{C}_c + \hat{C}_s) \cap \omega_A]^\sigma &= (\hat{C}_c^\sigma \cap \hat{C}_s^\sigma) + \omega_{A'} = (\hat{M}_c \cap \hat{M}_s) + \omega_{A'} \\ &= [(\hat{L}_c + \hat{L}_s) \cap \omega_A]^\sigma = \widehat{L_c^\infty} \cap \widehat{L_s^\infty} + \omega_{A'} .\end{aligned}$$

Since the space represented by this equation is equal to $l^2 + \omega_{A'}$ which is a solid FK-space with AK, we also have

$$\begin{aligned}l^\infty \cdot (\hat{M}_c \cap \hat{M}_s + \omega_{A'}) &= (\widehat{L_c^\infty} \cap \widehat{L_s^\infty} + \omega_{A'})_{1N} = (\hat{C}_c \cap \hat{C}_s + \omega_{A'})_N \\ &= (l^\infty \to [\hat{C}_c \cap \hat{C}_s + \omega_{A'}]_N) .\end{aligned}$$

Thus (b) follows.

REMARKS. 4.5. 1. Statement 4.4 b) improves the theorem of Paley stated in 4.2 a) and the theorem of Banach in 4.2 b), as well as the statements 4.3 e) and 4.3 f).

2. Statement 4.4 a) can also easily be derived from 4.4 b). Thus both statements are equivalent (since we derived 4.4 b) from 4.4 a)).

3. To our knowledge theorem 4.4 contains the best results concerning Hadamard gaps of Fourier series in the sense that it states the equalities $E \cap \omega_A = l^2 \cap \omega_A$ and $E_1 + \omega_{A'} = l^2 + \omega_{A'}$ for the widest range of spaces E and E_1.

4. In view of Theorem 4.4 we note that $l^\infty \cdot (\hat{M}_c + \hat{M}_s) = l^\infty$, but $l^\infty \cdot (\hat{L}_c \cap \hat{L}_s) = c_0 \cdot (\hat{L}_c \cap \hat{L}_s) \subsetneqq c_0$. It would be interesting to know what $c_0 \cdot (\hat{L}_c \cap \hat{L}_s)$ is.

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