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FIXED POINT FREE INVOLUTIONS ON HOMOTOPY SPHERES

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1. Introduction. The original motivation for this work was the following idea. Suppose one could prove that if $T:S^n \to S^n$ is a PL involution without fixed points, then there exists a PL sphere $S^{n-1} \subset S^n$ such that $TS^{n-1} = S^{n-1}$. This would constitute the induction step in a proof that each fixed point free PL involution on S^n is conjugate, in the group of homeomorphisms of S^n, to the antipodal map, $\mathfrak{A}$. In the smooth case, one might try to argue in a similar way, but with due regard for the groups θ_n, [8].

This idea does not work. There are obstructions to finding a PL sphere S^{n-1} in S^n such that $TS^{n-1} = S^{n-1}$ when n is odd. When $n = 4k - 1$, there is a symmetric bilinear form whose index is determined by the pair (S^n, T) and which is the obstruction in this dimension. When $n = 4k + 1$, the bilinear form is skew-symmetric with an associated quadratic form ψ_0 (over Z_2). The obstruction in this case is the Arf invariant of ψ_0. Similar obstructions are encountered in trying to answer the following question: Suppose S_0^{n-1} and S_1^{n-1} are two spheres in (S^n, T) such that $TS_i^{n-1} = S_i^{n-1}$, $i = 0, 1$. Then are the involutions $(S_0^{n-1}, T|S_0^{n-1})$ and $(S_1^{n-1}, T|S_1^{n-1})$ equivalent? Here we call two involutions (S_1, T_1) and (S_2, T_2) PL or smoothly equivalent if there is an equivariant homeomorphism, PL or smooth, of (S_1, T_1) onto (S_2, T_2). This paper supplies the proofs of the theorems announced in [5]. Since the work described here was completed, (in 1965) much progress has been made in classifying fixed point free involutions, smooth or PL, on homotopy spheres. It does not seem appropriate to list here all of these works, especially since the thesis [9] of Santiago Lopez de Medrano contains a complete bibliography of this subject. Let it suffice to say that many of the obvious questions raised here have now been answered. See, in particular [9] and [21]. We shall work in the smooth category, but all of the results hold in the PL category.

2. Characteristic submanifolds. Let $T: W \to W$ be a smooth, fixed point free involution of the smooth manifold W. Denote the orbit space

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(quotient space) by W/T. Then $\pi\colon W \to W/T$ is a principal Z_2-bundle, and is classified by a map $g\colon W/T \to P^N$ (real projective N-space) for N large, [16]. Choose a smooth map $h\colon W/T \to P^N$ homotopic to g and transverse regular [19] on P^{N-1}. Then h lifts to an equivariant map $\tilde{h}\colon (W, T) \to (S^N, \mathfrak{A})$ which is transverse regular on $S^{N-1} \subset S^N$. Let E and F be the upper and lower hemispheres of S^N. Then if $A = \tilde{h}^{-1}E$ and $B = \tilde{h}^{-1}F$, we have $TA = B$ and $A \cap B = M = \tilde{h}^{-1}(S^{N-1})$ is a T-invariant submanifold of W of codimension one.

DEFINITION. If M is a smooth submanifold of codimension one in W such that $W = A \cup B$ where $A \cap B = M$ and $TA = B$, then M is a *characteristic submanifold* of (W, T).

REMARK. It is easily seen that an equivariant map of M into S^{N-1} may be extended to a map of (A, M) into (E, S^{N-1}) with $A - M$ mapping into $E - S^{N-1}$, and then to an equivariant map of W into S^N which is transverse regular on S^{N-1}. Hence all characteristic submanifolds arise by the above construction, starting with the classifying map.

Let us specialize now to the case $W = \Sigma^n$, a smooth homotopy sphere of dimension n. The following lemma is the first step in making a characteristic submanifold $M \subset \Sigma^n$ as highly connected as possible.

LEMMA 2.1. *If $T\colon \Sigma^n \to \Sigma^n$ is a smooth, fixed point free involution and $n > 1$, then Σ^n contains a connected characteristic submanifold.*

PROOF. Let M be a characteristic submanifold. Then M/T carries the unique non-zero element of $H_{n-1}(\Sigma^n/T; Z_2)$, dual to the 1-dimensional cohomology class f^*x, where x generates $H^1(P^N)$ and $f\colon \Sigma/T \to P^N$ is the classifying map. Hence a (unique, by the ring structure of $H^*(\Sigma/T; Z_2)$) component of M/T carries this element, which implies that the double covering, M_0, of this component in Σ is connected. It is clear that the involution interchanges the two components of $\Sigma - M_0$, so that M_0 is a characteristic submanifold.

LEMMA 2.2. *If $n > 3$, then there exists a 1-connected characteristic manifold $M_1^n \subset \Sigma^{n+1}$.*

PROOF. Let M be a connected characteristic manifold. Let $g_1, \cdots, g_k, Tg_1, \cdots, Tg_k$ be a (possibly redundant) set of representatives of generators for $\pi_1(M)$, consisting of disjoint, simple closed, smooth curves. The proof of lemma 3.1 of [4] may now be applied to this situation, with a little care taken to see that the modified manifold remains T-invariant. This care amounts to choosing, for each disc d_i with boundary $\partial d_i = g_i$, the disc Td_i as the disc whose boundary is Tg_i. We may suppose d_i and Td_i

are disjoint since $n + 1 \geqq 5$.

Lemmas 2.1 and 2.2 are the first two steps in the process of transforming M, by equivariant handle exchanges into an $[(n-1)/2]$-connected characteristic manifold. The induction step is the following.

LEMMA 2.3. *Let $n \geqq 5$. If $\Sigma^{n+1} = A_{k-1} \cup B_{k-1}$, where $TA_{k-1} = B_{k-1}$, and $A_{k-1} \cap B_{k-1} = M^n_{k-1}$ is a smooth, $(k-1)$-connected n-submanifold, then, for $k \leqq [(n-1)/2]$, $\Sigma^{n+1} = A_k \cup B_k$, where $TA_k = B_k$, and $A_k \cap B_k = M^n_k$ is a smooth, k-connected, n-submanifold of Σ^{n+1}.*

PROOF. We shall indicate the proof only for the case n odd and $k = (n-1)/2$, since this case contains all of the ideas for the other cases. Let A_{k-1}, B_{k-1}, M_{k-1} be as above. Then from the Mayer-Vietoris sequence,

$$\cdots \longrightarrow H_{k+1}(\Sigma^{n+1}) \longrightarrow H_k(M_{k-1}) \longrightarrow H_k(A_{k-1}) \oplus H_k(B_{k-1}) \longrightarrow H_k(\Sigma^{n+1})$$

is exact, and hence, since $n + 1 > k + 1$, and $k > 0$,

$$0 \longrightarrow H_k(M_{k-1}) \longrightarrow H_k(A_{k-1}) \oplus H_k(B_{k-1}) \longrightarrow 0$$

is exact. Let $\alpha: M_{k-1} \longrightarrow A_{k-1}$ and $\beta: M_{k-1} \longrightarrow B_{k-1}$ be the inclusions. Then

$$H_k(M_{k-1}) = \ker(\alpha_* \oplus \ker \beta_*, \quad \text{and} \quad T_* \ker \alpha_* = \ker \beta_* .$$

Further, $\beta_*: \ker \alpha_* \longrightarrow H_k(B_{k-1})$ and $\alpha_*: \ker \beta_* \longrightarrow H_k(A_{k-1})$ are isomorphisms. Let x be a non-zero element of $\ker \alpha_*$. By the Hurewicz isomorphism theorem and results of Whitney (see for example [11], lemmas 6.11 and 6.12) x is represented by an imbedded sphere $iS^k \subset M_{k-1}$. Since $2k < n$, a general position argument allows us to assume $iS^k \cap TiS^k = \varnothing$, or one may argue as follows: the imbedding of S^k into M_{k-1}, followed by the projection $\pi: M_{k-1} \longrightarrow M_{k-1}/T$ (the quotient space) may be approximated by a homotopic map $g: S^k \longrightarrow M_{k-1}/T$ which imbeds S^k in M_{k-1}/T, [11]. Then $\pi^{-1}(gS^k) = iS^k \cup TiS^k$ is a disjoint union of spheres, where iS^k represents x, by the covering homotopy theorem.

Since $x \in \ker \alpha_*$, iS^k bounds a singular disc $iD^{k+1} \subset A_{k-1}$, which we may suppose is smooth and meets M_{k-1} orthogonally, only at points of iS^k. By an argument due to Milnor [12], we may suppose that D^{k+1} is actually imbedded in A_{k-1}. Supposing that M_{k-1} is totally geodesic [13] in a neighborhood of S^k, we may imbed $D^{k+1} \times D^{n-k}$ in A_{k-1}, using the exponential map from a suitable neighborhood of the zero cross-section of the normal bundle of D^{k+1} in A_{k-1} so that, if i denotes this imbedding,

1) $i(D^{k+1} \times D^{n-k}) \cap M_{k-1} = S^k \times D^{n-k}$,

2) $B_{k-1} \cup i(D^{k+1} \times D^{n-k})$ is a smooth manifold with boundary except for a corner along $i(S^k \times \partial D^{n-k})$.

We may further suppose, since $iS^k \cap TiS^k = \varnothing$, that $i(D^{k+1} \times D^{n-k}) \cap$

$Ti(D^{k+1} \times D^{n-k}) = \emptyset$. Let $i(D^{k+1} \times D^{n-k}) = N$, and let $\mathring{N}$ be the interior of N relative to A_{k-1}. Now let $A'_{k-1} = A_{k-1} \cup TN - \mathring{N}$, $B'_{k-1} = B_{k-1} \cup N - T\mathring{N}$. Now smooth the corners equivariantly on A'_{k-1} and B'_{k-1} to get A''_{k-1} and B''_{k-1}.

The above constructions may be carried out for $k \leqq [(n-1)/2]$. We now shall assume n is odd and $k = (n-1)/2$, and show that the above process for obtaining A''_{k-1} and B''_{k-1} from A_{k-1} and B_{k-1}, called equivariant handle exchange, can be repeated until the resulting characteristic submanifold, M_k, is k-connected. The argument is simpler for $k < (n-1)/2$. Before completing the proof of lemma 2.3, we need an additional lemma.

LEMMA 2.4. *If $H_k(M_{k-1}) = F_r \oplus \mathfrak{T}$, where F_r is free abelian of rank r and $\mathfrak{T}$ is a finite group, then M_{k-1} can be replaced by M'_{k-1} by equivariant $(k+1)$-handle exchange, so that $H_k(M'_{k-1}) \cong F_{r-2} \oplus \mathfrak{T}$.*

PROOF. Let x, as above, be a basis element for an infinite cyclic direct summand of $\ker \alpha_* \subset H_k(M_{k-1})$. Then since the sequence

$$\longrightarrow H_{k+1}(M_{k-1}) \xrightarrow{\alpha_*} H_{k+1}(A_{k-1}) \longrightarrow H_{k+1}(A_{k-1}, M_{k-1})$$
$$\xrightarrow{\partial} H_k(M_{k-1}) \xrightarrow{\alpha_*} H_k(A_{k-1})$$

is exact, and α_* is an epimorphism, there exists a unique $y \in H_{k+1}(A_{k-1}, M_{k-1})$ with $\partial y = x$, and y generates an infinite cyclic summand in $H_{k+1}(A_{k-1}, M_{k-1})$. Then by Poincaré duality, there exists $u \in H_{k+1}(A_{k-1})$ such that $y \cdot u = 1$. Hence $S'^k = \partial D^{n-k}$ is homologically trivial in $A - \mathring{N}$. The appropriate Mayer-Vietoris sequences give the exact sequences:

$$0 \longrightarrow H_k(S^k) \longrightarrow H_k(B_{k-1}) \longrightarrow H_k(B_{k-1} \cup N) \longrightarrow 0$$

and

$$\longrightarrow H_k(S'^k) \xrightarrow{0} H_k(A - \mathring{N}) \longrightarrow H_k(A) \longrightarrow 0 .$$

Hence $H_k(B_{k-1}) \cong H_k(B_{k-1} \cup N) \oplus Z$ and $H_k(A_{k-1}) \cong H_k(A_{k-1} - \mathring{N})$, where Z is generated by $\beta_* x$. Notice that $\partial T_* y = T_* x$, where $T_* x$, represented by TS^k, generates an infinite cyclic summand of $H_k(A_{k-1} - \mathring{N})$, and $T_* y \cdot T_* u = \pm 1$. Hence TS'^k is homologically trivial in $B_{k-1} - T\mathring{N}$, hence also in $B_{k-1} \cup N - T\mathring{N}$. Again, from the appropriate Mayer-Vietoris sequences, we have exact sequences

$$0 \longrightarrow H_k(TS^k) \longrightarrow H_k(A_{k-1} - \mathring{N}) \longrightarrow H_k(A_{k-1} - \mathring{N} \cup TN) \longrightarrow 0$$

and

$$\longrightarrow H_k(TS'^k) \xrightarrow{0} H_k(B_{k-1} \cup N - T\mathring{N}) \longrightarrow H_k(B_{k-1} \cup N) \longrightarrow 0 .$$

Hence $H_k(A_{k-1}) \cong H_k(A_{k-1} - \mathring{N}) \cong H_k(A_{k-1} - \mathring{N} \cup TN) \oplus Z$, where Z is generated by $\alpha_* T_* x$, and

$$H_k(B_{k-1} \cup N - T\mathring{N}) \oplus Z \cong H_k(B_{k-1} \cup N) \oplus Z \cong H_k(B_{k-1}) .$$

This completes the proof of Lemma 2.4.

By a finite number of applications of Lemma 2.4, we may reduce $H_k(M_{k-1})$ to a torsion group.

REMARK. Notice that the process used in Lemma 2.4 does not change $\mathfrak{T}$, a fact which will be of importance later.

In order to kill the torsion elements in $H_k(M_{k-1})$, we need to use linking numbers. Let $L(x, y) \in Q/Z$ be defined as in [15], page 524. (See also [8].) Since for $x \in \ker \alpha_*$ or $\ker \beta_* \subset H_k(M)$, $L(x, x) = 0$, the application here is somewhat different from that in [8]. There are two cases to consider.

(a) If $L(x, T_* x) = 0$, where $x \in \ker \alpha_* \subset H_k(M_{k-1})$, which we now may suppose is finite, then with S^k, S'^k, N and $\mathring{N}$ as before, we see that TS^k represents an element in $H_k(A_{k-1} - \mathring{N})$ whose order equals the order of $\beta_* x \in H_k(B_{k-1})$, represented by S^k. Now

$$\begin{aligned} H_k(A_{k-1} - \mathring{N}; Q) &\cong H^k(A_{k-1} - \mathring{N}; Q) \cong H_{k+2}(A_{k-1} - \mathring{N}, \partial(A_{k-1} - \mathring{N}); Q) \\ &\cong H_{k+2}(\Sigma, B_{k-1} \cup N; Q) \cong H_{k+1}(B_{k-1} \cup N; Q) . \end{aligned}$$

Since $H_{k+1}(B_{k-1}; Q) = 0$, (by Poincaré duality, if $H_k(M_{k-1})$ is finite, then since M_{k-1} is $(k-1)$-connected, $H_{k+1}(M_{k-1}) = 0$) we see that $H_{k+1}(B_{k-1} \cup N; Q) \cong Q$. Hence $H_k(A_{k-1} - \mathring{N})$ is the direct sum of a finite group and an infinite cyclic group. From the Mayer-Vietoris sequence of $(A_{k-1}, A_{k-1} - \mathring{N}, N)$ we have the exact sequence

$$\rightarrow H_k(S'^k) \rightarrow H_k(A_{k-1} - \mathring{N}) \rightarrow H_k(A_{k-1}) \rightarrow 0 .$$

Since $H_k(A_{k-1})$ is a finite group, and $H_k(A_{k-1} - \mathring{N})$ has an infinite cyclic summand, we see that the sequence becomes

$$0 \rightarrow Z \rightarrow Z + G' \rightarrow G \rightarrow 0 ,$$

where G' and G are finite groups. It is now easy to verify that G' has no more elements than does G. Hence the torsion subgroup of $H_k(A_{k-1} - \mathring{N})$ is no larger than that of $H_k(A_{k-1})$. The Mayer-Vietoris sequence for $(A_{k-1} - \mathring{N} \cup TN, A_{k-1} - \mathring{N}, TN)$ now gives us the exact sequence

$$\rightarrow H_k(TS^k) \rightarrow H_k(A_{k-1} - \mathring{N}) \rightarrow H_k(A_{k-1} - \mathring{N} \cup TN) \rightarrow 0 ,$$

where the generator of $H_k(TS^k)$ maps to an element of finite order in $H_k(A_{k-1} - \mathring{N})$. Therefore, the torsion subgroup of $H_k(A_{k-1} - \mathring{N} \cup TN)$ is smaller than that of $H_k(A_{k-1} - \mathring{N})$, and thus smaller than that of $H_k(A_{k-1})$. The same is obviously true of $H_k(B_{k-1} \cup N - T\mathring{N})$ and $H_k(B_{k-1})$. Now, by an equivariant handle exchange, we may kill the free parts of $H_k(B_{k-1} \cup N - T\mathring{N})$ and $H_k(A_{k-1} - \mathring{N} \cup TN)$ without changing the torsion parts, by Lemma 2.4. (See Remark following Lemma 2.4.) Thus in case there exists $x \in \ker \alpha_*$ such that $L(x, T_*x) = 0$, the number of elements in $H_k(A_{k-1})$ and $H_k(B_{k-1})$ may be reduced by equivariant $(k+1)$-handle exchange.

(b) Now suppose $L(x, Tx) \neq 0$, where x, S^k, S'^k, N, and $\mathring{N}$ have the same meaning as above. Let l be the order of x in $H_k(M_{k-1})$. Suppose $L(x, Tx) = l'/l$, where $0 < l' < l$. Since $\partial\colon H_{k+1}(A_{k-1}, M_{k-1}) \to H_k(M_{k-1})$ is an isomorphism of $H_{k+1}(A_{k-1}, M_{k-1})$ onto $\ker \alpha_*$ there exists a unique y, of order l, in $H_{k+1}(A_{k-1}, M_{k-1})$ such that $\partial y = x$.

Let $L_M (= L$, above) denote the pairing, by linking from $\ker \alpha_* \times \ker \beta_* \subset H_k(M_{k-1}) \times H_k(M_{k-1})$ to Q/Z, and let L_A denote the pairing, by linking, from $H_{k+1}(A_{k-1}, M_{k-1}) \times H_k(A_{k-1})$ to Q/Z. These functions are related as follows: $L_A(u, \alpha_* v) = L_M(\partial u, v)$. In particular, if D^{k+1} is a disc in A_{k-1} with boundary S^k, and brackets $\{\ \}$ denote homology class, then $L_M(x, Tx) = l'/l$ implies that $L_A(\{D^{k+1}\}, \{TS^k\}) = l'/l$.

Hence we see that $l'S'^k$ is homologous to lTS^k in $A_{k-1} - \mathring{N}$, and that $H_k(A_{k-1} - \mathring{N})$ has one infinite cyclic summand. From the Mayer-Vietoris sequences, we may extract the exact sequences

$$0 \to H_k(S'^k) \to H_k(A_{k-1} - \mathring{N}) \to H_k(A_{k-1}) \to 0$$

and

$$0 \to H_k(TS^k) \to H_k(A_{k-1} - \mathring{N}) \to H_k(A_{k-1} - \mathring{N} \cup TN) \to 0 .$$

Let ε' denote the image of the generator of $H_k(S'^k)$ in $H_k(A_{k-1} - \mathring{N})$, and let ε denote the image of the generator of $H_k(TS^k)$ in $H_k(A_{k-1} - \mathring{N})$. Then the relation $l'S'^k \sim lTS^k$ in $A_{k-1} - \mathring{N}$ implies that $l'\varepsilon' = l\varepsilon$. It is now an easy exercise with indices of subgroups to show that $H_k(A_{k-1} - \mathring{N} \cup TN)$ has fewer elements than does $H_k(A_{k-1})$. Hence we see that by equivariant handle exchange, we may reduce the size of $H_k(A_{k-1})$, whether or not $L(x, Tx) = 0$. By iterating these processes, we may finally obtain a k-connected characteristic submanifold $M_k = A_k \cap B_k$, where $A_k \cup B_k = \Sigma^{2k+1}$, and $TA_k = B_k$. This proves Lemma 2.3. Lemmas 2.1–2.4 imply

THEOREM 2.5. *If $n \geqq 5$ is odd, then (Σ^{n+1}, T) desuspends to $(S^n, T/S^n)$ for some T-invariant $S^n \subset \Sigma^{n+1}$.*

3. The signature of (Σ^{4k+3}, T). In order to define this invariant and not merely its absolute value, we must make some choices. First, fix an orientation for Σ^{4k+3} and an orientation for each standard sphere

$$S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n \subset \cdots ,$$

on which the antipodal map, $\mathfrak{A}$, acts. This will now determine an orientation for any characteristic manifold $M \subset \Sigma$ as follows. Let $f: (\Sigma, T) \rightarrow (S^{4k+3}, \mathfrak{A})$ be a smooth, equivariant map, transverse regular on S^{4k+2}, such that $f^{-1}(S^{4k+2}) = M$. Furthermore, require f to be orientation preserving. Then there is a unique orientation of $\tau(M)$, the tangent bundle of M, such that a) $\tau(M) \oplus \nu(M) \cong \tau(\Sigma)\,|\,M$, where $\tau(\Sigma)$ is the oriented tangent bundle of Σ, and b) $\tau(S^{4k+2}) \oplus f_*\nu(M) \cong \tau(S^{4k+3})\,|\,S^{4k+2}$, where $\tau(S^{4k+3})$ is the oriented tangent bundle of S^{4k+3}, and $\nu(M)$ is the normal bundle of M, oriented so that b) holds. Intersection numbers are now defined as follows. If $x \in H_{2k+1}(M)$, let $\bar{x} \in H^{2k+1}(M)$ denote the Poincaré dual of x (that is, $\bar{x} \cap \mu = x$, where μ is the generator of $H_{4k+2}(M)$ determined by the orientation of $\tau(M)$). Then if $x, y \in H_{2k+1}(M)$, the intersection number, $x \cdot y$, of x and y is defined by $x \cdot y = \langle \bar{x} \cup \bar{y}, \mu \rangle$ where $\langle\, , \rangle$ denotes Kronecker product. In terms of this bilinear form, a related bilinear form, B, is defined which determines the signature $\sigma(\Sigma^{4k+3}, T)$.

DEFINITION. If $x, y \in H_{2k+1}(M)$, let $B(x, y) = x \cdot T_* y$.

LEMMA 3.1. *B is a symmetric, unimodular bilinear form on the quotient of* $\ker \alpha_*$ *modulo its torsion subgroup.*

PROOF. The intersection form, $x \cdot y$, annihilates torsion elements, and by Poincaré duality, has determinant ± 1 on $H_{2k+1}(M)/\mathfrak{T}(H_{2k+1}(M))$, ($\mathfrak{T}(G)$ denotes the torsion subgroup of G). If $x, y \in \ker \alpha_*$, then $\bar{x}$ and $\bar{y}$ belong to the image $i^* H^{2k+1}(A)$. Let $\bar{x} = i^* u$, $\bar{y} = i^* v$. Then

$$x \cdot y = \langle i^* u \cup i^* v, \mu \rangle = \langle u \cup v, i_* \mu \rangle = \langle u \cup v, 0 \rangle = 0 .$$

Similarly $x \cdot y = 0$ if $x, y \in \ker \beta_*$.

Hence the form $x \cdot y$, when restricted to $\ker \alpha_* \times \ker \beta_*$ (modulo torsion) has determinant ± 1. Since $T_* \ker \alpha_* = \ker \beta_*$, B has determinant ± 1 when restricted to $\ker \alpha_*$ modulo its torsion subgroup. B is symmetric since

$$\begin{aligned} B(x, y) &= \langle \bar{x} \cup T^* \bar{y}, \mu \rangle = -\langle T^* \bar{y} \cup \bar{x}, \mu \rangle = -\langle T^*(\bar{y} \cup T^* \bar{x}), \mu \rangle \\ &= -\langle \bar{y} \cup T^* \bar{x}, T_* \mu \rangle = \langle \bar{y} \cup T^* \bar{x}, \mu \rangle = B(y, x) . \end{aligned}$$

Note that $T_* \mu = -\mu$ since T preserves orientation in Σ^{4k+3}, and hence reverses orientation in M, since T interchanges the components of $\Sigma - M$.

DEFINITION. The *signature*, $\sigma(\Sigma^{4k+3}, T)$, of the fixed point free, smooth involution T on the smooth homotopy sphere Σ^{4k+3} is the index of the symmetric bilinear form B defined on $\ker \alpha_*$ (modulo torsion) $\subset H_{2k+1}(M)/\mathfrak{T}(H_{2k+1}(M))$, where M is some characteristic submanifold of (Σ^{4k+3}, T).

Clearly, for this definition to have any meaning, we must prove

$\sigma(\Sigma^{4k+3}, T)$ is independent of the choice of characteristic submanifold, M.

LEMMA. 3.2. *Let M_0 and M_1 be two characteristic submanifolds in (Σ^{4k+3}, T). Let M_0 and M_1 determine $\sigma_0(\Sigma^{4k+3}, T)$ and $\sigma_1(\Sigma^{4k+3}, T)$, respectively. Then $\sigma_0 = \sigma_1$.*

PROOF. M_0 and M_1 determine equivariant, smooth maps $f_0, f_1: (\Sigma^{4k+3}, T) \to (S^N, \mathfrak{A})$ with N large, transverse regular on S^{N-1}, and with $f_i^{-1}(S^{N-1}) = M_i$, $i = 0, 1$. Since f_0/T and f_1/T classify the same Z_2-bundle, they are homotopic, by a map $F: (\Sigma^{4k+3} \times I)/T \times 1 \to P^N$. We may suppose that F is smooth, and transverse regular on P^{N-1}. F then is covered by an equivariant homotopy $\tilde{F}: (\Sigma^{4k+3} \times I, T \times 1) \to (S^N, \mathfrak{A})$ such that $\tilde{F}^{-1}(S^{N-1})$ is a characteristic submanifold W of $(\Sigma^{4k+3} \times I, T \times 1)$ with $\partial W = M_1 - M_0$. (We identify M_i with $M_i \times i \subset \Sigma \times I$.) We now have $\Sigma \times I = U \cup V$, with $U \cap V = W, \partial U = A_0 \cup W \cup A_1, \partial V = B_0 \cup W \cup B_1, \Sigma \times 0 = A_0 \cup B_0$, $\Sigma \times 1 = A_1 \cup B_1, A_0 \cap B_0 = M_0, A_1 \cap B_1 = M_1$.

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_{2k+1}(M_0) \arrow[r, "(\alpha_{0*}, \beta_{0*})"] \arrow[d, "m_{0*}"] & H_{2k+1}(A_0) \oplus H_{2k+1}(B_0) \arrow[r] \arrow[d, "(u_{0*} \oplus v_{0*})"] & 0 \\
0 \arrow[r] & H_{2k+1}(W) \arrow[r, "(u_*, v_*)"] & H_{2k+1}(U) \oplus H_{2k+1}(V) \arrow[r] & 0 \\
0 \arrow[r] & H_{2k+1}(M_1) \arrow[r, "(\alpha_{1*}, \beta_{1*})"] \arrow[u, "m_{1*}"] & H_{2k+1}(A_1) \oplus H_{2k+1}(B_1) \arrow[r] \arrow[u, "(u_{1*} \oplus v_{1*})"] & 0
\end{tikzcd}$$

where the maps are all induced by inclusions and the rows are taken from Mayer-Vietoris exact sequences. Let $j: \partial W \to W$ be the inclusion (so $j = m_0 \oplus m_1$), and let $x, y \in \ker j_*$. Then $\bar{x}, \bar{y}$ (Poincaré duals) belong to $j^* H^{2k+1}(W)$, so

$$B(x, y) = \langle \bar{x} \cup T^*\bar{y}, \mu \rangle = \langle j^*(\bar{x}' \cup T^*\bar{y}'), \mu \rangle = \langle \bar{x}' \cup T^*\bar{y}', j_*\mu \rangle = 0 ,$$

where $\bar{x}', \bar{y}' \in H^{2k+1}(W)$ are chosen so that $j^*\bar{x}' = \bar{x}, j^*\bar{y}' = \bar{y}$. Hence $\ker j_*$ is contained in its annihilator with respect to B. R. Thom, [20], has shown that the rank of $\ker j_*$ is half the rank of $H_{2k+1}(\partial W)$. What remains for us to prove is that $\ker j_* \cap (\ker \alpha_{0*} \oplus \ker \alpha_{1*})$ has half the rank of $\ker \alpha_{0*} \oplus \ker \alpha_{1*}$, since this will show that the subspace

$$\ker j_* \cap (\ker \alpha_{0*} \oplus \ker \alpha_{1*})$$

is its own annihilator will respect to B, which implies that B has zero index on $\ker \alpha_{0*} \oplus \ker \alpha_{1*}$, or that $\sigma_0 = \sigma_1$.

Since rank $(\ker j_*) =$ rank $(\ker \alpha_{0*} \oplus \ker \alpha_{1*}) = (1/2)$ rank $(H_{2k+1}(\partial W))$, it is enough to show that

$$\ker j_* = \ker j_* \cap (\ker \alpha_{0*} + \ker \alpha_{1*}) \oplus \ker j_* \cap (\ker \beta_{0*} \oplus \ker \beta_{1*}) ,$$

since T_* interchanges these two summands, and hence they have the same rank. But this follows from the commutativity of the above diagram. This proves the lemma.

THEOREM 3.3. *If $k > 0$, (Σ^{4k+3}, T) can be desuspended to $(S^{4k+2}, T|S^{4k+2})$ for some T-invariant $S^{4k+2} \subset \Sigma^{4k+3}$ if and only if $\sigma(\Sigma^{4k+3}, T) = 0$.*

PROOF. If (Σ^{4k+3}, T) contains a T-invariant S^{4k+2}, then since S^{4k+2} must be a characteristic submanifold, we must have $\sigma(\Sigma^{4k+3}, T) = 0$. Now suppose $\sigma(\Sigma^{4k+3}, T) = 0$. By lemma 2.3, there is a $2k$-connected characteristic manifold, M^{4k+2}. By the Hurewicz isomorphism theorem, and by a theorem of Whitney, (see [12]), we may represent a basis for $\ker \alpha_*$ in $H_{2k+1}(M)$ by disjoint, imbedded spheres $S_1, \cdots, S_m$. Furthermore, we may suppose that each intersection $S_i \cap TS_i$ is transverse. We now see that B takes on only even values along the diagonal, as follows. If p is a point of $S_i \cap TS_i$, then so is Tp. Since the intersection points occur in pairs, and each pair contributes either zero or ± 2 to the intersection number, it follows that $B(x, x)$ is even for each x in $\ker \alpha_*$. (Actually, for the characteristic manifold of dimension $4k + 2$, each pair of intersection points of $S \cap TS$ contributes ± 2, while for dimension $4k$, each pair contributes 0 to the intersection number $x \cdot Tx$. This is easy to prove using frames of vectors tangent to S and TS, where S is an imbedded sphere representing the homology class x.)

REMARK. It is a fact, which we do not use here, that $B(x, x)$ is even whether or not the characteristic manifold M^{4k+2} is $2k$-connected. The proof of this seems to need the same cohomology operation used later to define the Arf invariant. From lemma 4.5, we have

$$B_2(x, x) = \psi_0(x + x) + \psi_0(x) + \psi_0(x) \equiv 0 \bmod 2 .$$

Before completing the proof of theorem 3.3, we need two lemmas.

LEMMA 3.4. *If $\sigma(\Sigma^{4k+3}, T) = 0$, then there is a basis for $H_{2k+1}(M)$, where M is a $2k$-connected characteristic submanifold,*

$$x_1, \cdots, x_n, y_1, \cdots, y_n \in \ker \alpha_*, T_*x_1, \cdots, T_*x_n, T_*y_1, \cdots, T_*y_n \in \ker \beta_* ,$$

such that $B(x_i, x_j) = B(y_i, y_j) = 0$, $B(x_i, y_j) = \delta_{ij}$.

PROOF. Since the bilinear form B on $\ker \alpha_*$ has index zero and determinant ± 1, and $B(x, x)$ is even for every $x \in \ker \alpha_*$ this is Lemma 9 of Milnor, [12].

LEMMA 3.5. *If S^k is a sphere smoothly imbedded in M^{2k}, a $(k - 1)$-*

connected characteristic submanifold in Σ^{2k+1}, *if* S^k *represents an element of* $\ker \alpha_*$, *and if* $k > 2$, *then given a neighborhood* N *of* S^k *in* M^{2k}, *there exists a disc* D^{2k+1} *in* A *such that* $B \cup D$ *is a smooth submanifold with boundary in* Σ, *and* $D \cap M$ *is a tubular neighborhood of* S^k *contained in* N.

PROOF. Since $x \in \ker \alpha_*$, it follows from the Hurewicz theorem that S^k is the boundary of a singular disc in A. Using the collaring of ∂A, we may suppose there are no singularities near the boundary. Then we may apply Irwin's embedding theorem [7] to get a PL embedding $f: D^{k+1} \to A$ (in a PL structure on A compatible with its smooth structure). If D' is a regular neighborhood of D^{k+1} in A, then $B \cup D'$ may be smoothed [6] to yield $B \cup D$.

NOTE. D may not be a tubular neighborhood of a smooth $(k+1)$-cell in A whose boundary is S^k. In particular, when $k = 3$, S^3 may be knotted in ∂D^7, and not bound any non-singular smooth disc in D.

Now to continue with the proof of Theorem 3.3. Let $S_1, \cdots, S_n$ represent $x_1, \cdots, x_n$. Then since $B(x_1, x_1) = 0$, we may use Whitney's method [22] to remove intersections between S_1 and TS_1. If S_1 and TS_1 intersect at points $p_1, \cdots, p_s, Tp_1, \cdots, Tp_s$, then since the intersection number at p_1 is the same as that at Tp_1, there must be another point, say p_2, such that the intersections at p_1 and p_2 have opposite signs. Choosing the arcs and two-cells of Whitney's method equivariantly, we then simultaneously remove the intersections at p_1, p_2 and Tp_1, Tp_2, without introducing any new intersections with representatives of other basis elements. We may therefore suppose that representing the basis $x_1, \cdots, x_n, y_1, \cdots, y_n$ for $\ker \alpha_*$, we have disjoint spheres $S_1, \cdots, S_n, S'_1, \cdots, S'_n$ such that $S_i \cap TS_j = \varnothing$, $S'_i \cap TS'_j = \varnothing$, and $S_i \cap TS'_i$ consists of a single point where the intersection is transverse. The basis for $T_* \ker \alpha_* = \ker \beta_*$ now has these same properties. Now apply lemma 3.5 to the sphere S_1 to obtain a disc $D \subset A$. We may suppose $D \cap M$ is disjoint from representatives of all other basis elements of $H_{2k+1}(M)$. Then the appropriate Mayer-Vietoris sequence becomes

$$0 \longrightarrow Z \xrightarrow{h} H_{2k+1}(B) \longrightarrow H_{2k+1}(B \cup D) \longrightarrow 0$$

where $h(1) = x_1$. Hence $H_{2k+1}(B \cup D)$ is free with basis $x_2, \cdots, x_n, y_1, \cdots, y_n$. By Alexander duality, $H_{2k+1}(\overline{A - D})$ is free with basis $Tx_1, \cdots, Tx_n, Ty_2, \cdots, Ty_n$. Now apply the above argument with B replaced by $\overline{A - D}$, and with D replaced by TD, to obtain $\overline{B \cup D - TD} = B'$, $\overline{A \cup TD - D} = A'$, where $B' = TA'$, $A' \cap B' = M'$ is a $2k$-connected characteristic submanifold with $\operatorname{rank} H_{2k+1}(M') < \operatorname{rank} H_{2k+1}(M)$. We may clearly continue this process until rank $H_{2k+1}(M^{(i)}) = 0$.

REMARK. Even when $\sigma(\Sigma^{4k+3}, T) \neq 0$, the above process may be used to reduce rank $H_{2k+1}(M)$ to a minimum value determined by the signature.

4. **A cohomology operation.** Let K be a simplicial complex with $T\colon K \to K$ a simplicial, fixed point free involution. Then K/T, (the orbit space), is a simplicial complex and we may partially order the vertices so that the vertices of each simplex are linearly ordered. This ordering may then be lifted to an ordering of the vertices of K by defining $v < v'$ if and only if $\pi v < \pi v'$, where $\pi\colon K \to K/T$ is the projection. Clearly then, $v < v'$ if and only if $Tv < Tv'$. With such an ordering we will have $T(\sigma \cup_i \tau) = T\sigma \cup_i T\tau$, where $\cup_i$ is Streenrod's cup-sub-i product, [17].

Let $x \in Z^k(K; Z_2)$. Then there exists an element $v^k \in C^k(K; Z_2)$ such that

$$k) \qquad x \cup_k T^\# x = (1 + T^\#)v^k \,.$$

To see this, observe that $(1 + T^\#)(x \cup_k T^\# x) = x \cup_k T^\# x + T^\# x \cup_k x = 0$ since for k-dimensional simplexes σ and τ, $\sigma \cup_k \tau = \tau \cup_k \sigma$. ($\sigma \cup_k \tau = \sigma$ if $\sigma = \tau$, and $\sigma \cup_k \tau = 0$ otherwise.)

Hence a cochain v^k exists satisfying k). Now suppose cochains v^k, $v^{k+1}, \cdots, v^{k+i}$ are defined, with v^k satisfying k), v^{k+j} satisfying

$$k + j) \qquad x \cup_{k-j} T^\# x + \delta v^{k+j-1} = (1 + T^\#)v^{k+j} \quad \text{for all}$$

j, $1 \leqq j \leqq i$ and some i, $1 \leqq i \leqq k$. Then if $i < k$, let v^{k+i+1} be a cochain satisfying

$$k + i + 1) \qquad x_{k-i-1} T^\# x + \delta v^{k+i} = (1 + T^\#)v^{k+i+1} \,.$$

Such a cochain, v^{k+i+1}, exists if and only if $(1 + T^\#)(x \cup_{k-i-1} T^\# x + \delta v^{k+i}) = 0$. To see that this is true, recall Steenrod's coboundary formula. From [17], with Z_2 coefficients, we have

$$\delta(u \cup_i v) = u \cup_{i-1} v + v \cup_{i-1} u + \delta u \cup_i v + u \cup_i \delta v \,.$$

Hence

$$\begin{aligned}
&(1 + T^\#)(x \cup_{k-i-1} T^\# x + \delta v^{k+i}) \\
&\quad = x \cup_{k-i-1} T^\# x + T^\# x \cup_{k-i-1} x + (1 + T^\#)\delta v^{k+i} \\
&\quad = \delta(x \cup_{k-i} T^\# x) + \delta(1 + T^\#)v^{k+i} \\
&\quad = \delta(x \cup_{k-i} T^\# x + (1 + T^\#)v^{k+i}) \\
&\quad = \delta(x \cup_{k-i} T^\# x + \delta v^{k+i-1} + (1 + T^\#)v^{k+i}) \\
&\quad = 0 \,.
\end{aligned}$$

Therefore, a cochain v^{k+i+1} exists satisfying $(k + j)$ with $j = i + 1$, and therefore by induction, cochains v^{k+j} exist satisfying $(k + j)$ for all j, $0 \leqq$

$j \leqq k$.

LEMMA 4.1. *v^i is uniquely determined modulo*

$$\delta C^{i-1}(K; Z_2) + (1 + T^\#)C_i(K; Z_2) ,$$

for all i, $k \leqq i \leqq 2k$.

PROOF. Suppose v^k and $\bar{v}^k$ are two solutions of (k). Then

$$(1 + T^\#)(v^k + \bar{v}^k) = 0 , \quad \text{so} \quad v^k + \bar{v}^k \in (1 + T^\#)C^k(K; Z_2) .$$

Now suppose $v^k, v^{k+1}, \cdots, v^i$ and $\bar{v}^k, \bar{v}^{k+1}, \cdots, \bar{v}^i$ are two sets of solutions to $(k), (k+1), \cdots, (i)$, and that $v^j + \bar{v}^j \in \delta C^{j-1}(K; Z_2) + (1 + T^\#)C^j(K; Z_2)$, for $j < i$. Then from (i), we have

$$x \cup_{2k-i} T^\# x + \delta \bar{v}^{i-1} = (1 + T^\#)\bar{v}^i$$

and

$$x \cup_{2k-1} T^\# x + \delta v^{i-1} = (1 + T^\#)v^i .$$

Hence $(1 + T^\#)(v^i + \bar{v}^i) = \delta(v^{i-1} + \bar{v}^{i-1})$. By the induction hypothesis,

$$v^{i-1} + \bar{v}^{i-1} \in \delta C^{i-2}(K; Z_2) + (1 + T^\#)C^{i-1}(K; Z_2) .$$

Therefore

$$(1 + T^\#)(v^i + \bar{v}_i) \in (1 + T^\#)\delta C^{i-1}(K; Z_2) ,$$

so that

$$v^i + \bar{v}^i \in \delta C^{i-1}(K; Z_2) + (1 + T^\#)C^i(K; Z_2) .$$

We will now assume that K has a triangulation fine enough so that for each simplex $\sigma \in K$, $\mathrm{St}\,\sigma$ and $\mathrm{St}\,T\sigma$ have no common faces. Since $T: K \to K$ is fixed point free, this can be accomplished by subdividing K if necessary.

LEMMA 4.2. *Modulo $\delta C^{2k-1} + (1 + T^\#)C^{2k}$, v^{2k} depends only on the cohomology class of x.*

PROOF. Suppose $v^k, \cdots, v^{2k}$ satisfy equations $(k), (k+1), \cdots, (2k)$, while $\bar{v}^k, \cdots, \bar{v}^{2k}$ satisfy these equations with x replaced by $x + \delta\sigma$ and v^j replaced by $\bar{v}^j$, where by abuse of notation, σ denotes the elementary cochain which is one on the simplex σ and zero elsewhere. One choice of $\bar{v}^j$ is to let $\bar{v}^j = v^j + (T^\# x \cup_{2k-j} \delta\sigma)$ for $k \leqq j \leqq 2k$. This is shown by induction, supposing $\bar{v}^{j-1} = v^{j-1} + (T^\# x \cup_{2k-j+1} \delta\sigma)$, and applying Steenrod's coboundary formula again. Then by Lemma 4.1, $\bar{v}^{2k} = v^{2k} + (T^\# x \cup \delta\sigma)$ modulo $\delta C^{2k-1} +$ $(1 + T^\#)C^{2k}$, for any choice of $\bar{v}^{2k}$. Now since $\delta(T^\# x \cup \sigma) = T^\# x \cup \delta\sigma$, $\bar{v}^{2k} = v^{2k}$ modulo $\delta C^{2k-1} + (1 + T^\#)C^{2k}$.

Let $H_T^*(K; Z_2)$ denote the equivariant cohomology ring of (K, T), $H_*^T(K; Z_2)$ the invariant homology group.

LEMMA 4.3. *$(1 + T^\#)V^{2k}$ is a cocycle, and the cohomology class of $(1 + T^\#)v^{2k}$ in $H_T^{2k}(K; Z_2)$ is uniquely determined by the cohomology class of x in $H^k(K; Z_2)$.*

PROOF. Since $(1 + T^\#)v^{2k} = x \cup T^\# x + \delta v^{2k-1}$, it is clear that $\delta(1 + T^\#)v^{2k} = 0$. Uniqueness follows immediately from the previous two lemmas.

DEFINITION. Let $\psi\colon H^k(K; Z_2) \to H_T^{2k}(K; Z_2)$ be defined by $\psi(\{x\}) = \{(1 + T^\#)v^{2k}\}$, where x is a k-cocycle and v^{2k} is determined by x as above.

LEMMA 4.4. *Let $f\colon (K, T) \to (K', T')$ be an equivariant simplicial map. Then $f^*\psi = \psi f^*$.*

PROOF. We may partially order the vertices of K and K' so that T and T' preserve these orderings, and so that f is order preserving. This is done by ordering the vertices of K'/T' first, then ordering the vertices of K/T so that $f/T\colon K/T \to K'/T'$ is order-preserving, (see [17], page 294) and then lifting the ordering to K and K'. Then Theorem 3.1 of [17] applies, so that equations (k), $(k + 1)$, $\cdots$, $(2k)$ (of the beginning of this section) for $x' \in Z^*(K'; Z_2)$ are transformed by $f^\#$ into equations (k), $\cdots$, $(2k)$ for $f^\# x' \in Z^k(K; Z_2)$, with v^{k+j} chosen as $f^\# v'^{k+j}$ for $0 \leqq j \leqq k$. Hence $\psi f^*\{x\} = \{(1 + T^\#)f^\# v'^{2k}\} = \{f^\#(1 + T^\#)v'^{2k}\} = f^*\psi\{x'\}$.

Given $x \in H_T^p(K; Z_2)$ and $y \in H_p^T(K; Z_2)$. Define $\langle x, y\rangle_T$ as follows: choose an equivariant cocycle u representing x and invariant cycle v representing y. Then $v = (1 + T_\#)w$ for some p-chain w. Now let $\langle x, y\rangle_T = u(w)$. It is an elementary exercise to show that this is a well defined non-singular pairing of $H_T^p(K; Z_2)$ and $H_p^T(K; Z_2)$ to Z_2.

DEFINITION: If M^{2n} is a $2n$-dimensional, connected, smooth or P.L. manifold with a fixed point free, smooth or PL involution T, then let $\psi_0\colon H_n(M; Z_2) \to Z_2$ be defined by $\psi_0(x) = \langle \psi(\bar{x}), [M]\rangle_T$, where $\bar{x}$ is the Poincaré dual of $x \in H_n(M; Z_2)$ and $[M]$ denotes the generator of $H_{2n}^T(M; Z_2)$.

REMARK. We have not shown that $\psi\colon H_T^k(M; Z_2) \to H_T^{2k}(M; Z_2)$ is independent of the ordering and the triangulation chosen for M. This follows from the fact that ψ may be defined as above, using the coboundary formula, in singular cohomology. We understand that I. Berstein has a definition for the operation ψ, not using cup-sub-i products, analogous to the definition of Sq^i found in [18].

LEMMA 4.5. *Let M be a $2k$-dimensional manifold, with $x, y \in H_k(M; Z_2)$. Let $B_2(x, y) = x \cdot T_* y$. Then $\psi_0(x + y) = \psi_0(x) + \psi_0(y) + B_2(x, y)$.*

PROOF. Let s and t be cocycles representing Poincaré duals of x and y respectively. As in equations $(k), \cdots, (2k)$, let $v^{k+j}(s)$ satisfy

$$s \cup_{k-j} T^{\#}s + \delta v^{k+j-1}(s) = (1 + T^{\#})v^{k+j}(s)$$

and let $v^{k+j}(t)$ satisfy $t \cup_{k-j} T^{\#}t + \delta v^{k+j-1}(t) = (1 + T^{\#})v^{k+j}(t)$. We may choose $v^{k+j}(s + t)$ satisfying

$$(s + t) \cup_{k-j} T^{\#}(s + t) + \delta v^{k+j-1}(s + t) = (1 + T^{\#})v^{k+j}(s + t)$$

so that

$$v^{k+j}(s + t) = v^{k+j}(s) + v^{k+j}(t) + s \cup_{k-j} T^{\#}t \, .$$

It then follows from the coboundary formula that $v^{k+j+1}(s + t) = v^{k+j+1}(s) + v^{k+j+1}(t) + t \cup_{k-j-1} T^{\#}s$ will satisfy the equation $(s + t) \cup_{k-j-1} T^{\#}(s + t) + \delta v^{k+j}(s + t) = (1 + T^{\#})v^{k+j+1}(s + t)$. Hence $v^{2k}(s + t) = v^{2k}(s) + v^{2k}(t) +$ either $t \cup T^{\#}s$ or $s \cup T^{\#}t$ depending on the parity of k. But at the cohomology level, in either case, we will have

$$\psi(u + v) = \psi(u) + \psi(v) + (1 + T^{\#})(u \cup T^{\#}v) \, ,$$

where u and v denote the cohomology classes of s and t, respectively. Then from the definition of ψ_0, we have

$$\begin{aligned} \psi_0(x + y) &= \langle \psi(u + v), [M] \rangle_T \\ &= \langle \psi(u) + \psi(v) + (1 + T^*)(u \cup T^*v), [M] \rangle_T \\ &= \langle \psi(u), [M] \rangle_T + \langle \psi(v), [M] \rangle_T + \langle (1 + T^*)(u \cup T^*v), [M] \rangle_T \\ &= \psi_0(x) + \psi_0(y) + \langle u \cup T^*v, [M] \rangle \\ &= \psi_0(x) + \psi_0(y) + B_2(x, y) \, . \end{aligned}$$

LEMMA 4.6. *If S^k is a smooth sphere in the smooth manifold M^{2k} so that S^k and TS^k have only transverse intersections, and if $x \in H_{2k}(M; Z_2)$ is the homology class represented by S^k, then $\psi_0(x)$ is the number of pairs of points (p, Tp) in $S^k \cap TS^k$ reduced modulo 2.*

PROOF. Let the intersection points be $(p_1, Tp_1), \cdots, (p_n, Tp_n)$. Choose disjoint neighborhoods U_i, TU_i containing p_i, Tp_i respectively, $1 \leqq i \leqq n$. We may then choose the cochains $v^k, v^{k+1}, \cdots, v^{2k}$ as in equations $(k), \cdots, (2k)$ so that $v^{k+j} = \sum_{i=1}^{n} v_i^{k+j}$ where v_i^{k+j} is carried by U_i. Then if u is a cocycle dual to x, $u \cup T^{\#}u$ will be of the form $\Sigma_1^n (w_i + w_i')$ where w_i and w_i' each represent the generator of $H^{2k}(M^{2k}; Z_2)$, w_i is carried by U_i, and w_i' is carried by TU_i. Then $u \cup T^{\#}u + \delta v^{2k-1}$ differs from $u \cup T^{\#}u$ only in that w_i is replaced by $w_i + \delta v_i^{2k-1}$, so that $T^{\#}(w_i + \delta v_i^{2k-1}) = w_i'$. Hence $u \cup T^{\#}u + \delta v^{2k-1} = (1 + T^{\#})\Sigma_1^n w_i'$. Therefore $\psi(\{u\}) = (1 + T^{\#})\Sigma_1^n \{w_i'\}$, and $\psi_0(x) = \langle (1 + T^{\#}) \Sigma_1^n \{w_i'\}, [M] \rangle_T = \langle \Sigma_1^n \{w_1^n\}, [M] \rangle \equiv n \pmod 2$.

5. The Arf invariant of (Σ^{4k+1}, T). Let M^{4k} be a characteristic submanifold in Σ^{4k+1}, with $\Sigma^{4k+1} = A \cup TA$, $M^{4k} = \partial A = A \cap TA$. Let $\alpha\colon M \to A$ be the inclusion map, and $\ker \alpha_*$ the kernel of $\alpha_*\colon H_{2k}(M; Z_2) \to H_{2k}(A; Z_2)$. The bilinear form $B_2\colon \ker i_* \times \ker i_* \to Z_2$ is symmetric, and unimodular since $\ker \alpha_*$ is its own annihilator with respect to the usual intersection form: $H_{2k}(M; Z_2) \times H_{2k}(M; Z_2) \to Z_2$. Since we have seen that $B_2(x, x) = 0$ for all $x \in \ker \alpha_*$, it follows as in [12] that B_2 admits a symplectic basis, say $x_1, \cdots, x_n, y_1, \cdots, y_n$, as basis for $\ker i_*$ such that $B_2(x_i, x_j) = B_2(y_i, y_j) = 0$, $B_2(x_i, y_j) = \delta_{ij}$. Then the Arf invariant $c(\Sigma, T)$ is defined by

$$c(\Sigma, T) = \textstyle\sum_{i=1}^{n} \psi_0(x_i)\psi_0(y_i) .$$

As with the signature, we must show that $c(\Sigma^{4k+1}, T)$ does not depend on the choice of characteristic submanifold. That $c(\Sigma^{4k+1}, T)$ does not depend on the choice of symplectic basis is shown in [2]. Provisionally, let $c(\Sigma, T, M)$ be the Arf invariant determined by the characteristic submanifold $M \subset \Sigma$. Exactly as in the signature case, (lemma 3.2), we have the involution $T \times 1$ on $\Sigma \times I$, with $\Sigma \times I = U \cup V$, $U \cap V = W = \partial U = \partial V$, $TU = V$, and $\partial W = M_0 \cup M_1$, where M_i is a characteristic submanifold of $(\Sigma \times i, T \times 1 \mid \Sigma \times i)$, $i = 0, 1$. We must show that $c(\Sigma, T, M_0) = c(\Sigma, T, M_1)$. (Here we identify $M_i \subset \Sigma$ with $M_i \subset \Sigma \times i$.)

LEMMA 5.1. $c(\Sigma, T, M_0) = c(\Sigma, T, M_1)$.

PROOF. Let $M = M_0 \cup M_1$, and let $A = \partial U \cap (\Sigma \times \partial I)$. Then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & H_{2k}(M) \arrow[r, "(\alpha,\beta)"] \arrow[d, "j"] & H_{2k}(A) \oplus H_{2k}(TA) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H_{2k}(W) \arrow[r, "(u,v)"] & H_{2k}(U) \oplus H_{2k}(V) \arrow[r] & 0 . \end{tikzcd}$$

The rows are taken from Mayer-Vietoris sequences, and all maps are induced by inclusions.

What we wish to prove is clearly equivalent to $c(\Sigma \times \partial I, T \times 1, M) = 0$. As in lemma 3.2, we have $\ker j = \ker j \cap \ker \alpha \oplus \ker j \cap \ker \beta$, and the rank of $\ker j \cap \ker \alpha =$ half the rank of $\ker \alpha$. Since $x \cdot y = 0$ for all $x, y \in \ker j$, and $T_*(\ker j) = \ker j$, we have $B_2(x, y) = 0$ for all $x, y \in \ker j \cap \ker \alpha$. Hence we may pick a symplectic basis $x_1, \cdots, x_n, y_1, \cdots, y_n$ (with respect to B_2) for $\ker \alpha$ with $x_1, \cdots, x_n \in \ker j \cap \ker \alpha$. Let $x \in \ker j \cap \ker \alpha$, and let $\bar{x}$ denote the Poincaré dual of x. Then $\bar{x} = j^* w$ for some $w \in H^{2k}(W; Z_2)$. Then

$$\psi_0(x) = \langle \psi(\bar{x}), [M] \rangle_T = \langle \psi(j^* w), [M] \rangle_T = \langle j^* \psi(w), [M] \rangle_T = \langle \psi(w), j_*[M] \rangle_T .$$

But since $[M]$ is the boundary of the T-invariant element $[W]$, we have $j_*[M] = 0$ as an element of $H_{4k}^T(W; Z_2)$, and so $\langle \psi(w), j_*[M]\rangle_T = \psi_0(x) = 0$. Now since $\psi_0(x_i) = 0$ for $1 \leqq i \leqq n$, we have

$$c(\Sigma \times \partial I, T \times 1, M) = \sum_{i=1}^{n} \psi_0(x_i)\psi_0(y_i) = 0 .$$

THEOREM 5.2. *If $n = 0 \pmod 4$ and $n > 4$, then (T, Σ^{n+1}) can be desuspended to $(T|S^n, S^n)$ if and only if $c(T, \Sigma^{n+1}) = 0$.*

PROOF. Clearly, if (T, Σ^{n+1}) can be desuspended to $(T|S^n, S^n)$ then $c(T, \Sigma^{n+1}) = 0$, since in this case, S^n is a characteristic submanifold. Conversely, if $c(T, \Sigma^{n+1}) = 0$, then choosing first a $(2k - 1)$-connected characteristic submanifold $M^{4k} \subset \Sigma^{4k+1}$, where $n = 4k$, we may then find, as in Kervaire-Milnor [8] a symplectic basis $x_1, \cdots, x_r, y_1 \cdots, y_r$ for $\ker \alpha_*$, where $\psi_0(x_i) = 0$ for $1 \leqq i \leqq r$, and the x_i are respresented by disjoint imbedded spheres, S_i^{2k}. Since $\psi_0(x_i) = 0$ upon making the intersections of S_i^{2k} with TS_i^{2k} all transverse, [11], we will have an even number of pairs of intersection points $(p_1, Tp_1), \cdots, (p_{2t}, Tp_{2t})$. These may all be eliminated by the method of Whitney [22]. Since the intersection numbers at p_i and Tp_i differ in sign, then at either p_1 and p_2 or p_1 and Tp_2 they differ in sign. With no loss of generality, suppose these numbers differ at p_1 and p_2. Then Whitney's method gives an isotopy of S_i^{2k} whose end result is a sphere $\bar{S}_i^{2k}$ whose intersection with TS_i^{2k} consists of Tp_1, Tp_2, and the pairs $(p_3, Tp_3), \cdots (p_{2t}, Tp_{2t})$ Then apply this isotopy composed with T to TS_i^{2k} to get a sphere $T\bar{S}_i^{2k}$ such that $\bar{S}_i^{2k} \cap T\bar{S}_i^{2k}$ consists of transverse intersections at $(p_3, Tp_3), \cdots, (p_{2t}, Tp_{2t})$. One needs to take care that the arcs and 2-disc of Whitney's process do not intersect their T-images, but since $n > 4$, this presents no problem.

We now may represent the element $x_1, \cdots, x_r$ by imbedded spheres $S_1, \cdots, S_r$ such that $TS_i \cap S_i = \varnothing$. Just as in the signature case, we may now perform equivariant handle exchange between A and TA, using the "core spheres" $S_1, \cdots, S_r$ and $TS_1, \cdots, TS_r$, thus killing $H_{2k}(M)$. This proves the theorem.

6. Uniqueness of the desuspension.

Suppose $(S_0^n, T|S_0^n)$ and $(S_1^n, T|S_1^n)$ are two desuspensions of (Σ^{n+1}, T). Then are $(S_0^n, T|S_0^n)$ and $(S_1^n, T|S_1^n)$ equivalent? That is does there exist an equivariant diffeomorphism of S_0^n onto S_1^n? In trying to answer this question, we again encounter the familiar obstructions in half of the cases, and obtain an affirmative answer in the other half.

Suppose we are given (Σ^{n+1}, T) and desuspensions $(S_0^n, T|S_0^n)$ and $(S_1^n, T|S_1^n)$.

DEFINITION. $(S_0^n, T|S_0^n)$ and $(S_1^n, T|S_1^n)$ are *equivariantly concordant in* $(\Sigma^{n+1} \times I, T \times 1)$ if for some fixed point free involution τ on $S^n \times I$, the inclusions $S_0^n \times 0 \subset \Sigma^{n+1} \times I$, $S_1^n \times 1 \subset \Sigma^{n+1} \times I$ extend to an equivariant imbedding of $(S^n \times I, \tau)$ into $(\Sigma^{n+1} \times I, T \times 1)$. (We are identifying $S_i^n \times i$ with $S^n \times i$, $i = 0, 1$.)

THEOREM 6.1. *If $n \geqq 4$ is even, and (Σ^{n+1}, T) desuspends to $(S_0^n, T|S_0^n)$ and to $(S_1^n, T|S_1^n)$, then $(S_0^n, T|S_0^n)$ and $(S_1^n, T|S_1^n)$ are equivariantly concordant in $(\Sigma^{n+1} \times I, T \times 1)$.*

PROOF. As in the proof of lemma 3.2, there exists a characteristic submanifold W of $(\Sigma^{n+1} \times I, T \times 1)$ such that $\partial W = S_0^n \times 0 \cup S_1^n \times 1$. Since W has odd dimension $\geqq 5$, and since ∂W has homology only in dimensions 0 and n, the proof of theorem 2.5 shows that by equivariant handle exchange we may replace W by an $[(n+1)/2]$-connected manifold $W' \subset \Sigma \times I$ with $\partial W' = \partial W$. Hence W' is an equivariant h-cobordism between $(S_0^n, T|S_0^n)$ and $(S_1^n, T|S_1^n)$. Since $Wh(Z_2) = 0$, [14], W' is equivariantly diffeomorphic to $(S_0^n \times I, (T|S_0^n) \times 1)$ [10], [3]. This proves the theorem.

REMARK. Note that $(S_0^n, T|S_0^n)$ and $(S_1^n, T|S_1^n)$ are equivariantly diffeomorphic if they are equivariantly concordant, and $n \geqq 5$.

Now suppose n is odd. Then (Σ^{n+1}, T) always desuspends, for $n \geqq 5$. Let $(S_0^n, T|S_0^n)$ and $(S_1^n, T|S_1^n)$ be two desuspensions, which we identify with $S_i^n \subset \Sigma^{n+1} \times i$, for $i = 0, 1$. As above, there is a characteristic submanifold $W \subset (\Sigma^{n+1} \times I, T \times 1)$ with $\partial W = S_0^n \cup S_1^n$. The obstructions to obtaining an equivariant concordance in $(\Sigma^{n+1} \times I, T \times 1)$ between $(S_0^n, T|S_0^n)$ and $(S_1^n, T|S_1^n)$ will be the obstructions to killing the middle dimensional homology of W by equivariant handle exchange. (We may suppose, with no loss of generality, that W is $(n-1)/2$-connected.)

CASE 1. $n = 4k - 1$. Then T preserves orientation in S_0^n, and hence $(T \times 1)|W$ preserves orientation in W. Since W is $4k$-dimensional, the bilinear form $B(x, y) = x \cdot T_* y$ defined on $H_{2k}(W)$, modulo its torsion subgroup, will be symmetric. ($B(x, y) = \langle \bar{x} \cup T^*\bar{y}, [W] \rangle = \langle T^*\bar{y} \cup \bar{x}, [W] \rangle = \langle \bar{y} \cup T^*\bar{x}, T_*[W] \rangle = \langle \bar{y} \cup T^*\bar{x}, [W] \rangle = B(y, x)$, where $x, y \in H_{2k}(W)$, and $\bar{x}$, $\bar{y}$ are Poincaré duals of x and y.) Let $\Sigma^{n+1} \times I = U \cup V$ with $U \cap V = W$, $TU = V$. Let $u: W \to U$, $v: W \to V$ be the inclusions. We have, from the Mayer-Vietoris sequence the exact sequence

$$0 \to H_{2k}(W) \xrightarrow{(u_*, v_*)} H_{2k}(U) \oplus H_{2k}(V) \longrightarrow 0 .$$

DEFINITION. Let $\sigma(\Sigma^{n+1}, T, S_0^n, S_1^n)$ be the signature of the bilinear form B restricted to the kernel of u_*.

NOTE. If W_0 and W_1 are two different characteristic submanifolds in $\Sigma^{n+1} \times I$, with $\partial W_0 = \partial W_1 = S_0^n \cup S_1^n$, then as in the proof of lemma 3.2, we may use a characteristic manifold $X \subset (\Sigma^{n+1} \times I \times I, T \times 1 \times 1)$ with $\partial X = W_0 \cup W_1 \cup S_0^n \times 0 \times I \cup S_1^n \times 1 \times I$ to show that W_0 and W_1 determine the same signature, so that $\sigma(\Sigma^{n+1}, T, S_0^n, S_1^n)$ is well defined.

CASE 2. $n = 4k + 1$. Then T preserves orientation in S_0^n, and hence $(T \times 1) | W$ preserves orientation in W. But now W is $(4k + 2)$-dimensional, so the form B is skew-symmetric. We are in the Arf invariant case. We choose a symplectic basis $x_1, \cdots, x_n, y_1, \cdots, y_n$ for the kernel of u_*, and define $c(\Sigma^{n+1}, T, S_0^n, S_1^n) = \sum_{i=1}^n \psi_0(x_i)\psi_0(y_i)$.

Again, as in case 1, this invariant is independent of the choice of W, and the proof of this parallels Lemma 5.1 so closely that we do not give it.

Given (Σ^{2n}, T), with $n > 2$, by Theorem 2.5, we can always desuspend. Suppose then that we have two desuspensions, S_0^{2n-1}, and S_1^{2n-1}.

THEOREM 6.2. *If $k > 1$, S_0^{4k-1} and S_1^{4k-1} are concordant in $(\Sigma^{4k} \times I, T \times 1)$ if and only if $\sigma(\Sigma^{4k}, T, S_0^{4k-1}, S_1^{4k-1}) = 0$. In particular, if $\sigma = 0$, then $(S_0^{4k-1}, T | S_0^{4k-1})$ and $(S_1^{4k-1}, T | S_1^{4k-1})$ are equivariantly diffeomorphic.*

PROOF. In $(\Sigma^{4k} \times I, T \times 1)$ we find a characteristic submanifold W whose boundary is $-S_0^{4k-1} \times 0 + S_1^{4k-1} \times 1$. The proof of Theorem 3.3 now applies to show that by equivariant handle exchange $\Sigma^{4k} \times I$ we may change W into an equivariant h-cobordism between S_0^{4k-1} and S_1^{4k-1}. Since $Wh(Z_2) = 0$, [14], this yields a diffeomorphism between S_0^{4k-1}/T and S_1^{4k-1}/T, by the s-cobordism theorem, [10].

THEOREM 6.3. *If $k \geqq 1$, S_0^{4k+1} and S_1^{4k+1} are equivariantly concordant in $(\Sigma^{4k+2} \times I, T \times 1)$ if and only if $c(\Sigma^{4k+2}, T, S_0^{4k+1}, S_1^{4k+1}) = 0$.*

PROOF. The proof is the same as that of Theorem 6.2, except the reference to Theorem 3.3 is replaced by Theorem 5.2.

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