Tôhoku Math. Journ.
25 (1973), 197-199.

ANALYTIC SUBGROUPS OF $GL(n, \boldsymbol{R})$

Dedicated to Professor Shigeo Sasaki on his 60th birthday

MORIKUNI GOTO

(Received July 1, 1972; Revised October 30, 1972)

1. Let L be a Lie group, let G be a connected Lie group, and let f be a continuous one-one homomorphism from G into L. Then the analytic structure of G is determined by the image $f(G)$. The set $f(G)$ with the Lie group structure of G is called an *analytic subgroup* of L. (See Chevalley [2].) $f(G)$ is not necessarily closed in L. Obviously, we can find continuous monomorphisms from $\boldsymbol{R}^m$ into a toral group of dimension greater than m. The purpose of this short note is to prove the following theorem, by which we can say, roughly speaking, that the above example exhausts all non-closed analytic subgroups in the case when L is the general linear group $GL(n, \boldsymbol{R})$.

For a subset M of $GL(n, \boldsymbol{R})$, $\overline{M}$ will denote the closure of M. The identity element of a group in question will always be denoted by e.

THEOREM. *Let G be a connected Lie group, and let f be a continuous one-one homomorphism from G into $GL(n, \boldsymbol{R})$. Then we can find a closed subgroup V, which is isomorphic with $\boldsymbol{R}^k$ for suitable $k = 0, 1, 2, \cdots$, and a connected closed normal subgroup N, such that G is a semi-direct product: $G = VN$, $V \cap N = e$. Here V and N may be selected so that $\overline{f(V)}$ is a toral group, $f(N)$ is closed, and $\overline{f(G)}$ is a local semi-direct product of $\overline{f(V)}$ and $f(N)$: $\overline{f(G)} = \overline{f(V)}f(N)$, and $\overline{f(V)} \cap f(N)$ is finite. Moreover, in this case $\overline{f(G)}$ is diffeomorphic with the direct product $\overline{f(V)} \times N$.*

2. Following a method in Borel [1], we shall first prove a lemma.

LEMMA. *Let L be a Lie group, and N a connected closed normal subgroup of L. Let T be a toral subgroup of L. If $L = TN$ and $T \cap N$ is finite, then the space of L is diffeomorphic with the product space $T \times N$.*

PROOF. First we note that it is enough to show the lemma for $\dim T = 1$. Then $L/N = T$ is a circle. On the other hand, a principal fibre bundle with a circle as its base space and with a connected Lie group as its fibre is always trivial. (A special case of Corollary 18.6 in Steenrod [5].) q.e.d.

3. Proof of Theorem. The commutator subgroup $f(G)'$ of $f(G)$ is a closed subgroup of $GL(n, \boldsymbol{R})$, see e.g. Goto [3]. Notice that *this is the only place where the property of $GL(n, \boldsymbol{R})$ is used in the proof.* Also we know that $f(G)'$ coincides with the commutator subgroup of $\overline{f(G)}$, see Goto loc. cit.. Let us pick up a maximal analytic subgroup N of G, containing the commutator subgroup G' of G, such that $f(N)$ is closed in $GL(n, \boldsymbol{R})$. Then $f(N)$ is a closed normal subgroup of $\overline{f(G)}$ and the factor group $\overline{f(G)}/f(N)$ is abelian. Since the abelian Lie group $\overline{f(G)}/f(N)$ contains a dense analytic subgroup $f(G)/f(N)$ which contains no closed analytic subgroup except $\{e\}$, we can conclude that $\overline{f(G)}/f(N)$ is a toral group and G/N is isomorphic with $\boldsymbol{R}^k$ for a suitable k.

Let K be a maximal compact subgroup of $\overline{f(G)}$. By Iwasawa [4], $Kf(N)/f(N)$ is a maximal compact subgroup of $\overline{f(G)}/f(N)$, which is compact. Hence we have that $Kf(N) = \overline{f(G)}$. Let K' denote the semisimple part of K, and T_1 the identity component of the center of K. Then we have a local direct product decomposition: $K = K'T_1$ and $K' \cap T_1$ is finite. Since $K' \subset f(G)' \subset f(N)$, we have that $\overline{f(G)} = T_1 f(N)$. Next, let T_2 denote the identity component of $T_1 \cap f(N)$. Then we can find a toral subgroup T with $T_1 = TT_2$ and $T \cap T_2 = \{e\}$. Thus we have that $\overline{f(G)} = Tf(N)$ and $T \cap f(N)$ is finite. That is, $\overline{f(G)} = Tf(N)$ is a local direct product decomposition. By the above lemma, $\overline{f(G)}$ is diffeomorphic with the direct product $T \times f(N)$.

Next, let $\mathscr{G}^*$, $\overline{\mathscr{G}^*}$, $\mathscr{N}^*$ and $\mathscr{T}$ denote the Lie algebras of $f(G)$, $\overline{f(G)}$, $f(N)$ and T respectively. Then we have that $\overline{\mathscr{G}^*} = \mathscr{T} + \mathscr{N}^*$, $\mathscr{T} \cap \mathscr{N}^* = \{0\}$, and $\mathscr{G}^* \supset \mathscr{N}^*$. Hence denoting $\mathscr{G}^* \cap \mathscr{T} = \mathscr{V}^*$ we have that $\mathscr{G}^* = \mathscr{V}^* + \mathscr{N}^*$, $\mathscr{V}^* \cap \mathscr{N}^* = \{0\}$. Let V denote the analytic subgroup of G such that $\mathscr{V}^*$ is the Lie algebra of $f(V)$. Because $G/N = VN/N$ contains no compact subgroup except $\{e\}$, the same is true for V. Therefore, there is a continuous monomorphism g from some $\boldsymbol{R}^k$ onto V. On the other hand, defining

$$(x, a)(y, b) = (x + y, g(y)^{-1}ag(y)b)$$

for $x, y \in \boldsymbol{R}^k$ and $a, b \in N$, $\boldsymbol{R}^k \times N$ becomes a Lie group such that $\boldsymbol{R}^k \times N \ni (x, a) \mapsto g(x)a \in G$ is a continuous monomorphism. Hence $\boldsymbol{R}^k \times N$ and G are homeomorphic and $g(\boldsymbol{R}^k) = V$ is closed in G.

BIBLIOGRAPHY

[1] A. BOREL, Sous-groupes commutatifs et torsion des groups de Lie compact connexes, Tôhoku Math. J., 13 (1961), 216-240.
[2] C. CHEVALLEY, Theory of Lie groups I, Princeton University Press, 1946.
[3] M. GOTO, Faithful representations of Lie groups I, Math. Japonica, 1 (1948), 107-119.

[4] K. IWASAWA, On some types of topological groups, Ann. of Math., 50 (1949), 507-558.
[5] N. STEENROD, The topology of fibre bundles, Princeton University Press 1951.

UNIVERSITY OF PENNSYLVANIA
AND
AARHUS UNIVERSITET