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ON A STOCHASTIC INTEGRAL EQUATION WITH RESPECT TO A WEAK MARTINGALE

N. KAZAMAKI

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1. **Introduction.** We introduced in [4] the concept of a weak martingale, which is a natural extension of a martingale. The weak martingale property is invariant through all changes of time, but the property is not true with the definition of local martingales. This is the reason for considering weak martingales. We also proved that the theory of stochastic integration of locally bounded previsible processes can be extended to these weak martingales (see [4]).

In this paper we shall give some basic properties of weak martingales, and consider stochastic integral equations relative to them.

2. **Definitions.** The reader is assumed to be familiar with the basic notions of the general theory of processes as expounded in [6] and with the theory of stochastic integrals relative to martingales as given in [2].

Now let (Ω, F, P) be a complete probability space, given an increasing, right continuous family $(F_t)_{t \geqq 0}$ of sub σ-fields of F. We assume as usual that F_0 contains all the negligible sets. A notation such that "let $M = (M_t, F_t)$ be a martingale" means that the martingale property is relative to the F_t family. All martingales below are assumed to be right continuous. By a change of time $T = (F_t, \tau_t)$ is meant a family of stopping times τ_t, $t \geqq 0$, of the F_t family, finite-valued, such that for a.e. $\omega \in \Omega$, the sample function $\tau_{.}(\omega)$ is increasing and right continuous. We say that the change of time is normal if these sample functions are strictly increasing, continuous, $\tau_0(\omega) = 0$ and $\tau_\infty(\omega) \equiv \lim_{t \to \infty} \tau_t(\omega) = +\infty$. We do not distinguish two processes X and Y such that for a.e. $\omega \in \Omega$ $X_{.}(\omega) = Y_{.}(\omega)$.

DEFINITION 1. A process $M = (M_t, F_t)$ is said to be a weak martingale if there exists an increasing sequence (T_n) of finite stopping times of the F_t family such that

(i) $\lim_{n \to \infty} T_n = +\infty$ a.s.

(ii) for each n there exists a martingale $M^n = (M_t^n, F_t)$ such that $M = M^n$ on $[0, T_n[$.

Obviously, any local martingale is a weak martingale. We say that such a sequence (T_n) reduces M, and the process M^n is said to be a martingale reduced by T_n. Note that hte word "reduces" is not used here in the same sense as in [2], where it was demanded that $M = M^n$ on the stochastic interval $[0, T_n]$.

In particular, if each M^n can be chosen to be an L^2-bounded martingale (i.e. $\sup_{0 \leq t < +\infty} E[(M_t^n)^2] < +\infty$), then we say that the M is a weakly square integrable martingale.

As is well known, every martingale X can be split into a continuous part X^c, and a purely discontinuous part X^d, orthogonal to all continuous martingales. We say that the weak martingale M is of discontinuous type if each M^n can be chosen to be purely discontinuous.

3. Basic properties of a weak martingale. The local martingale property is not invariant through changes of time, but the weak martingale has the invariance. Therefore a weak martingale is not always a local martingale.

The following proposition gives a necessary and sufficient condition for a weak martingale M to be a local martingale.

PROPOSITION 1. *A weak martingale M is a local martingale if there exists a sequence (T_n) of stopping times reducing M and the martingales M^n reduced by T_n such that for each n $(M_{T_n}^{n+j})_{j=1,2,\dots}$ is uniformly integrable on the set $\{T_n > 0\}$.*

PROOF. From the definition of a weak martingale,

$$\lim_{j\to\infty} M_{T_n}^{n+j} = M_{T_n} \quad \text{a.s.}$$

for each fixed n, but since $(M_{T_n}^{n+j})_{j=1,2,\dots}$ is uniformly integrable on $\{T_n > 0\}$,

$$M_{T_n}^{n+j} I_{\{T_n>0\}} \to M_{T_n} I_{\{T_n>0\}} \quad \text{in} \quad L^1 \quad \text{as} \quad j \to +\infty\ .$$

Thus for each t, $E[M_{T_n}^{n+j} I_{\{T_n>0\}} | F_t] \to E[M_{T_n} I_{\{T_n>0\}} | F_t]$ in L^1. Similary

$$M_{t\wedge T_n}^{n+j} I_{\{T_n>0\}} \to M_{t\wedge T_n} I_{\{T_n>0\}}$$

in L^1 for each t. Since for each fixed n and j the process $(M_{t\wedge T_n}^{n+j}, F_t)$ is a uniformly integrable martingale, then we get

$$E[M_{T_n} | F_t] = M_{t\wedge T_n} \quad \text{on} \quad \{T_n > 0\}\ .$$

Consequently M is a local martingale. This completes the proof.

Furthermore, if $\sup_j E[(M_{T_n}^{n+j})^2 I_{\{T_n>0\}}] < +\infty$ for each n, then the weak martingale M is a locally square integrable martingale.

The next example shows that a weak martingale is not necessarily a weakly square integrable martingale.

EXAMPLE 1. Let $\Omega = R_+$, F the class of all linear Borel sets in Ω and we designate by S the identity mapping of Ω onto R_+. Let F_t^0 be the Borel field generated by $S \wedge t$. We define the probability measure P on Ω by $P(S > t) = e^{-t}$. Let F_t be the P-completion of F_t^0. Note that the family (F_t) is right continuous and has no times of discontinuity. Clearly S is a stopping time of the F_t family.

Now we are going to verify that any weak martingale M is a martingale on this probability space. Denote by T_n, M^n, stopping times and martingales satisfying the conditions of Definition 1. For simplicity, assume that each T_n is bounded. C. Dellacherie [1] proved that

(3.1) $R_+ \ni {}^\exists s_n \uparrow +\infty$, $T_n = s_n$ a.s. if $S \geqq s_n$ and $T_n \geqq S$ a.s. on $\{S < s_n\}$

Then it is easily checked that $F_t = F_{t\wedge S}$ for every t, and so $M_t^n = M_{t\wedge S}^n$. As the set $\{S > t\}$ is an F_t-atom, we get

$$\begin{cases} M_t = M_S I_{\{S\leqq t\}} + C_t I_{\{S>t\}} \\ M_t^n = M_S^n I_{\{S\leqq t\}} + C_t^n I_{\{S>t\}} \end{cases} \tag{3.2}$$

where C_t and C_t^n are constants.

From (3.1) and the definition of a weak martingale, it follows that for each $t < s_n$

$$M_t = M_t^n \quad \text{on} \quad \{s_n < S\} \quad \text{and} \quad \{s_n < S\} \subset \{t < S\}\,.$$

Thus $C_t^n = C_t$ if $t < s_n$. Consequently, by (3.2)

$$\int_{]s,\infty[} M_t^n dP = \int_{]s,\infty[} M_s^n dP = C_s e^{-s}, \quad s < t < s_n$$

from which

$$\int_{]s,t]} M_S^n dP = C_s e^{-s} - C_t e^{-t}, \quad s < t < s_n\,. \tag{3.3}$$

Therefore for every $k = 1, 2, \cdots$

$$\int_{]s,t]} M_S^n dP = \int_{]s,t]} M_S^{n+k} dP, \quad s < t < s_n$$

and so $M_S^n = M_S^{n+k}$ on $\{S < s_n\}$. By letting $k \to +\infty$, we have

$$M_S^n I_{\{S\leqq t\}} = M_S I_{\{S\leqq t\}}, \quad (t < s_n)\,.$$

Then from (3.2) $M_t = M_t^n$ for $t < s_n$. As $\lim_{n\to\infty} s_n = +\infty$, M is a martingale.

Similarly, one can prove that any weakly square integrable martingale is a square integrable martingale.

Then it suffices to choose a martingale which is not a square integrable.

REMARK. I do not know whether a continuous weak martingale is a weakly square integrable martingale (as in the case of local martingales).

PROPOSITION 2. *Let M be a weak martingale such that $M_0 = 0$. Then there exists a unique decomposition of M into a sum of a continuous local martingale M^c and a weak martingale M^d, which is of discontinuous type.*

PROOF. Denote by T_n, M^n, stopping times and martingales satisfying the conditions of Definition 1. Since for each n the process $M^{T_n} = (M_{t\wedge T_n})$ is a semi-martingale (see [4]), one can show the existence of a unique continuous local martingale M^c such that $M^c = (M^n)^c$ on $[0, T_n]$ for each n. Thus the weak martingale $M^d = M - M^c$ is of discontinuous type. The uniqueness of the decomposition is evident.

It should be noted that a continuous weak martingale can be of discontinuous type. We now give such an example.

EXAMPLE 2. Let $X = (X_t, F_t)$ be a continuous martingale such that $X_0 = 0$ and $\limsup_{t\to+\infty} X_t = +\infty$ a.s. Put now

$$\tau_t = \inf\{u;\, X_u > t\}\,.$$

Then $T = (F_t, \tau_t)$ is a change of time, and $X_{\tau_t} = t$ by the sample continuity of X. Since the weak martingale property is invariant through changes of time, the process $M = (t, F_{\tau_t})$ is a weak martingale. By Proposition 2, M can be written in a unique way as $t = M_t^c + M_t^d$. It is clear that the M is not a local martingale, and so the process M^d is a continuous weak martingale which is of discontinuous type.

We can also define:

$$[M, M]_t = \langle M^c\rangle_t + \sum_{s\leq t} (\Delta M_s)^2\,. \tag{3.4}$$

Here $\langle M^c\rangle$ denotes the unique continuous increasing process such that $(M^c)^2 - \langle M^c\rangle$ is a local martingale.

Then it is easy to see that for any weak martingale M the process $(M_t^2 - [M, M]_t, F_t)$ is also a weak martingale. The theory of stochastic integration can be extended to weak martingales as follows. Let $M = (M_t, F_t)$ be a weak martingale and $C = (C_t, F_t)$ a locally bounded previsible process. Let (T_n) be a sequence of stopping times reducing M. Then there exists a unique weak martingale $C\circ M$ such that $C\circ M = C\circ M^{T_n}$ on $[0, T_n[$ for each n. The process $C\circ M$ is said to be the stochastic integral of C relative to M. Unfortunately we can not characterize the stochastic integrals by an identity involving the bracket [,] as in the case of local martingales, because there exist (bounded) continuous weak martingales M such that $M \neq 0$ and $[M, M] = 0$.

4. A stochastic integral equation with respect to a weak martingale. Let $M = (M_t, F_t)$ be a weak martingale and $A = (A_t, F_t)$ an increasing process. Let $f(x)$ and $g(x)$ be two continuous functions defined on R. We shall consider in this section the following stochastic integral equation:

$$X_t = x + \int_0^t f(X_{u-})dM_u + \int_0^t g(X_{u-})dA_u, \quad x \in R \tag{4.1}$$

where the integral by dM_u (resp. dA_u) is understood in the sense of the stochastic integral (resp. the Lebesgue-Stieltjes integral on R_+ for each ω).

We have supposed in a previous paper [5] that f and g are bounded, but this assumption is unnecessary as we shall see below.

DEFINITION 2. Let T be a stopping time of the F_t family. By a local solution of the equation (4.1) on the stochastic interval $[0, T[$, we mean a process $X = (X_t, F_t)$ with paths right continuous and free of oscillatory discontinuities, which satisfies the equation (4.1) on $[0, T[$.

The value $+\infty$ is admissible for T. In particular, if $P(T = +\infty) = 1$, then we simply call it a solution of (4.1).

Let X and Y be two right continuous processes with finite left limits, and suppose that $X = Y$ on $[0, T[$. Then, if X is a local solution on $[0, T[$, so is Y. We say that the uniqueness on $[0, T[$ holds for (4.1) if for any two local solutions X, Y on $[0, T[$ we have $X = Y$ on this interval.

For example, if M is a continuous local martingale, then the local uniqueness holds for the equation:

$$X_t = \int_0^t |X_{u-}|^\alpha dM_u, \quad \frac{1}{2} \leqq \alpha < +\infty \tag{4.2}$$

for any stopping time T. Of course, this uniqueness does not always hold (even if $1/2 \leqq \alpha$). We shall give some non-trivial example such that the uniqueness does not hold.

EXAMPLE 3. The classical problem

$$u(t) = \int_0^t |u(s)|^\alpha ds \tag{4.3}$$

has a unique solution $u = 0$ for $\alpha \geqq 1$, but for $0 < \alpha < 1$ and $1 - \alpha = \beta$, $u(t) = (\beta t)^{1/\beta}$ is also a solution and so is

$$u(t) = \begin{cases} 0: t \leqq t_1 \\ [\beta(t - t_1)]^{1/\beta}: t > t_1 \end{cases} \tag{4.4}$$

for each choice of $t_1 > 0$.

Now let $(\Omega, F, P; F_t)$ be the same probability space as discussed in Example 1. Denote by M a right continuous modification of $E[S - 1 | F_t]$. As the random variable $S-1$ is square integrable, the martingale M is L^2-bounded. An easy computation shows

$$M_t = t \wedge S - I_{\{S \leqq t\}}$$

and so

$$\int_0^t |X_{u-}|^\alpha dM_u = \int_0^t |X_u|^\alpha du$$

on $[0, S[$.

Consequently, from (4.4), the equation (4.2) has an infinite number of local solution on $[0, S[$ for $0 < \alpha < 1$.

In the remainder of this paper, we assume that the functions f and g are Lipschitz-continuous; namely

$$\mathrm{Max}\,\{|f(x) - f(y)|, |g(x) - g(y)|\} \leqq C|x - y|, \quad x, y \in R \tag{4.5}$$

where C is some positive constant. We suppose in addition that the family (F_t) has no times of discontinuity.

Let us agree to say that a sequence of processes $X^{(n)}$ converges uniformly in probability to a process X if for each $\varepsilon > 0$ and each $t > 0$,

$$\lim_{n\to\infty} P\left\{\sup_{0\leqq s\leqq t} |X_s^{(n)} - X_s| \geqq \varepsilon\right\} = 1 .$$

Furthermore, if $P\{\lim_{n\to\infty} \sup_{0\leqq s\leqq t} |X_s^{(n)} - X_s| = 0\} = 1$ for every $t \geqq 0$, then we say that the processes $X^{(n)}$ converge uniformly almost surely to X.

LEMMA 1. *Let $X^{(n)} = (X_t^{(n)}, F_t)$, $n = 1, 2, \cdots$, be a sequence of locally bounded previsible processes, and assume that M is an L^2-bounded martingale. If the processes $X^{(n)}$ converge uniformly almost surely to X, then $X^{(n)} \circ M$ converges uniformly in probability to $X \circ M$.*

Since this lemma is proved in [3], we omit its proof.

LEMMA 2. *Suppose that the process M is an L^2-bounded martingale with $M_0 = 0$, and the process A is continuous. Then the equation* (4.1) *has a unique local solution on* $[0, T[$ *for any stopping time T.*

PROOF. Define now:

$$\lambda_t = t + \langle M\rangle_t + A_t,\ \theta_t = \inf\{u; \lambda_u > t\} .$$

It is easy to see that $\Theta = (F_t, \theta_t)$ and $\Lambda = (F_{\theta_t}, \lambda_t)$ are normal changes of time. Obviously λ_T is a stopping time of the F_{θ_t} family, and the process $(t - \langle M\rangle_{\theta_t} - A_{\theta_t}, F_{\theta_t})$ is increasing. Therefore, in order to show the existence of a unique local solution of the equation (4.1) on $[0, T[$, by considering the time change transformation Θ there is no loss of generality in assuming that the process $(t - \langle M\rangle_t - A_t, F_t)$ is increasing.

(a) Uniqueness: Let X and Y be two local solutions of (4.1) on $[0, T[$. Put now

$$R_n = \inf\{t; |X_t| \vee |Y_t| \geqq n\}, \quad n = 1, 2, \cdots$$

Each R_n is a stopping time of the F_t family. It is easily checked that $R_n \leqq R_{n+1}$, $R_\infty = \lim_{n\to\infty} R_n \geqq T$ a.s. and

$$D(t) = E[(X_t - Y_t)^2 I_{\{t<T\wedge R_n\}}] \leqq 4n^2 \tag{4.6}$$

for each fixed n.

Then $\left\{\int_0^{t\wedge R_n} f(X_{u-})dM_u, F_t\right\}_{t\geqq 0}$ and $\left\{\int_0^{t\wedge R_n} f(Y_{u-})dM_u, F_t\right\}_{t\geqq 0}$ are L^2-bounded martingales. For simplicity, we assume that $C \leqq 1/2$.

Using the inequality $(a+b)^2 \leqq 2(a^2+b^2)$ and the continuity of $\langle M\rangle_\cdot(\omega)$ and $A_\cdot(\omega)$, we get

$$\begin{aligned} D(t) &= E\left[\left\{\int_0^t [f(X_{u-}) - f(Y_{u-})]dM_u + \int_0^t [g(X_{u-}) - g(Y_{u-})]dA_u\right\}^2 I_{\{t<T\wedge R_n\}}\right] \\ &\leqq 2E\left[\int_0^t [f(X_{u-}) - f(Y_{u-})]^2 I_{\{u<T\wedge R_n\}} d\langle M\rangle^u \right.\\ &\qquad\qquad \left. + A_t \int_0^t [g(X_{u-}) - g(Y_{u-})]^2 I_{\{u<T\wedge R_n\}} dA_u\right] \\ &\leqq (1+t)\int_0^t D(u)du \end{aligned}$$

from which $D \equiv 0$ follows. Hence, by making $n \to \infty$, $X = Y$ on $[0, T[$.

(b) Existence: We first remark that a solution of (4.1) is also a local solution for the equation on $[0, T[$. Therefore it suffices to verify the existence of a solution for (4.1); otherwise stated, we may assume that $P(T = +\infty) = 1$.

Define in succession:

$$\begin{cases} X_t^0 = x \\ \cdots\cdots\cdots \\ X_t^{n+1} = x + \int_0^t f(X_{u-}^n)dM_u + \int_0^t g(X_{u-}^n)dA_u \,. \end{cases} \tag{4.7}$$

For each n, $X^n = (X_t^n)$ is a process adapted to the F_t family, with paths right continuous and free of oscillatory discontinuities.

As the process $(t - \langle M\rangle_t - A_t, F_t)$ is increasing, we get

$$\begin{aligned}E[(X_t^{n+1})^2] &\leqq 2x^2 + 4E\left[\int_0^t f(X_{u-}^n)^2 d\langle M\rangle_u + A_t\int_0^t g(X_{u-}^n)^2 dA_u\right] \\ &\leqq 2x^2 + 4(1+t)E\left[\int_0^t \left\{f(X_u^n)^2 + g(X_u^n)^2\right\} du\right] \qquad \text{(by (4.5))} \\ &\leqq 2x^2 + 4(1+t)E\left[\int_0^t (X_u^n)^2 du + 2\{f(0)^2 + g(0)^2\}t\right] \\ &\leqq 2x^2 + 8t(1+t)\{f(0)^2 + g(0)^2\} + 4(1+t)E\left[\int_0^t (X_u^n)^2 du\right]\end{aligned}$$

for every n.

Thus, by an induction argument, we find for every $t > 0$

$$E\left[\int_0^t (X_u^n)^2 du\right] < +\infty, \quad n = 1, 2, \cdots$$

from which

$$\text{(4.8)}\quad E\left[\int_0^t f(X_{u-}^n)^2 d\langle M\rangle_u\right] < +\infty,\ E\left[\int_0^t g(X_u^n)^2 dA_u\right] < +\infty, \quad \forall t > 0,\ \forall n\ .$$

Consequently, for every n, the process $\left\{\int_0^t f(X_{u-}^n) dM_u, F_t\right\}$ is a square integrable martingale. Now, for simplicity, the proof is spelled out for $0 \leqq t \leqq 1$ only. Then from (4.5)

$$\begin{aligned}D_n(t) &= E[(X_t^{n+1} - X_t^n)^2] \\ &\leqq 2E\left[\left(\int_0^t \{f(X_{u-}^n) - f(X_{u-}^{n-1})\} dM_u\right)^2 + \left(\int_0^t \{g(X_{u-}^n) - g(X_{u-}^{n-1})\} dA_u\right)^2\right] \\ &\leqq 2E\left[\int_0^t \{f(X_{u-}^n) - f(X_{u-}^{n-1})\}^2 d\langle M\rangle_u + A_t\int_0^t \{g(X_u^n) - g(X_u^{n-1})\}^2 dA_u\right] \\ &\leqq \int_0^t D_{n-1}(u) du\end{aligned}$$

where $\sup_{0\leqq t\leqq 1} D_0(t) \leqq 2[f(x)^2 + g(x)^2]$.

Therefore we can derive the estimate:

$$\text{(4.9)}\qquad D_n(t) \leqq 2[f(x)^2 + g(x)^2]\frac{t^n}{n!}\ .$$

An application of the extension of Kolmogorov's inequality to martingales now gives

$$(4.10)\qquad \begin{aligned} &P\left\{\sup_{0\leqq t\leqq 1}\left|\int_0^t [f(X^n_{u-}) - f(X^{n-1}_{u-})]dM_u\right| \geqq \varepsilon\right\} \\ &\leqq \varepsilon^{-2}E\left[\int_0^1 \{f(X^n_u) = f(X^{n-1}_u)\}^2 d\langle M\rangle_u\right] \\ &\leqq \frac{\varepsilon^{-2}}{4}\int_0^1 D_{n-1}(u)du \end{aligned}$$

for every $\varepsilon > 0$.

On the other hand, by using the Schwarz inequality

$$(4.11)\qquad \begin{aligned} &P\left\{\sup_{0\leqq t\leqq 1}\left|\int_0^t [g(X^n_u) - g(X^{n-1}_u)]dA_u\right| \geqq \varepsilon\right\} \\ &\leqq P\left\{\sup_{0\leqq t\leqq 1} A_t \int_0^t [g(X^n_u) - g(X^{n-1}_u)]^2 dA_u \geqq \varepsilon^2\right\} \\ &\leqq P\left\{\int_0^1 [g(X^n_u) - g(X^{n-1}_u)]^2 du \geqq \varepsilon^2\right\} \\ &\leqq \frac{\varepsilon^{-2}}{4}\int_0^1 D_{n-1}(u)du\ . \end{aligned}$$

Thus by (4.10) and (4.11) $P\{\sup_{0\leqq t\leqq 1}|X^{n+1}_t - X^n_t| \geqq 2\varepsilon\} \leqq \text{Const.} \times (\varepsilon^{-2}/n!)$. Pick now $\varepsilon^{-2} = (n-2)!$. Then $\varepsilon^{-2}/n!$ is the general term of a convergent series, and so the first Borel-Cantelli lemma shows

$$(4.12)\qquad P\left\{\sup_{0\leqq t\leqq 1}|X^{n+1}_t - X^n_t| \leqq 2[(n-2)!]^{1/2},\ n\uparrow\infty\right\} = 1\ .$$

Because of this, X^n converges uniformly almost surely to some process X. It is clear that the X is an adapted process with paths right continuous and free of oscillatory discontinuities. Moreover, $\{f(X^n_{t-}), F_t\}$ and $\{g(X^n_t), F_t\}$ converge uniformly almost surely to $\{f(X_{t-}), F_t\}$ and $\{g(X_t), F_t\}$ respectively. Then, it is easy to check that $\left\{\int_0^t g(X^n_u)dA_u, F_t\right\}$ also converges uniformly almost surely to $\left\{\int_0^t g(X_u)dA_u, F_t\right\}$. According to Lemma 1, $\left\{\int_0^t f(X^n_{u-})dM_u, F_t\right\}$ converges uniformly in probability to $\left\{\int_0^t f(X_{u-})dM_u, F_t\right\}$. Thus for some subsequence (n_k) the processes $\left\{\int_0^t f(X^{n_k}_{u-})dM_u\right\}_{t\geqq 0}$ converge uniformly almost surely to $\left\{\int_0^t f(X_{u-})dM_u\right\}_{t\geqq 0}$. Consequently, by (4.7), the process X is a solution of (4.1). This completes the proof.

We are now in a position to state our main result.

THEOREM. *Let T be a stopping time of the F_t family. Then for any weak martingale $M = (M_t, F_t)$, $M_0 = 0$, and any quasi-left continuous increasing process $A = (A_t, F_t)$, the equation* (4.1) *has a unique local solution* *on* $[0, T[$. *In particular, if* $P(T = +\infty) = 1$, *then it is a unique solution of* (4.1).

PROOF. As is well known, there exists a unique continuous increasing process $\widetilde{A}_t$ such that the process $\overset{*}{A}_t = A_t - \widetilde{A}_t$ is a martingale. Therefore we can rewrite the equation (4.1) in the following form:

$$X_t = x + \int_0^t f(X_{u-})dM_u + \int_0^t g(X_{u-})d\overset{*}{A}_u + \int_0^t g(X_{u-})d\widetilde{A}_u . \tag{4.13}$$

This allows us to assume that the process A is continuous.

Next, denote by T_n, M^n, stopping times and martingales satisfying the conditions of Definition 1. As is well known, for each n, there exist stopping times $S_{n,m} \uparrow +\infty$ such that the process $(M^n_{t\wedge S_{n,m}})_{t\geqq 0}$ can be written as

$$M^n_{t\wedge S_{n,m}} = H_t + V_t \tag{4.14}$$

where H is an L^2-bounded martingale stopped at $S_{n,m}$ and V is a process with integrable variation. More precisely, V is written in the form:

$$V_t = B^{(1)}_t - B^{(2)}_t + \Delta M^n_{S_{n,m}} I_{\{S_{n,m}\leqq t\}} , \tag{4.15}$$

where $B^{(i)}$ is a continuous increasing process ($i = 1, 2$) (see Proposition 2, p. 99, in [7]). As $\lim_{m\to\infty} S_{n,m} = +\infty$ a.s. for each n, we have

$$P(S_{n,m_n} < T_n) < 1/2^n$$

for some subsequence (m_n), and then by the first Borel-Cantelli lemma

$$\lim_{n\to\infty} S_{n,m_n} = \lim_{n\to\infty} T_n = +\infty \text{ a.s.}$$

Put now: $S^*_n = (\inf_{i\geqq n} S_{i,m_i}) \wedge T_n$. Clearly, $S^*_n \uparrow +\infty$ and $S^*_n \leqq T_n$ for every n. Thus the sequence (S^*_n) reduces the weak martingale M. Therefore without loss of generality, we may assume that $S^*_n = T_n = S_{n,m_n}$ for every n.

Now we shall treat the equation (4.1) on each stochastic interval $[0, T_n[$. By (4.14) and (4.15), we have

$$X_t = x + \int_0^t f(X_{u-})dH_u + \int_0^t f(X_{u-})d(B^{(1)}_u - B^{(2)}_u) + \int_0^t g(X_u)dA_u \tag{4.16}$$

on this interval.

According to Lemma 2, this equation has a unique solution $X^{(n)}$. $X^{(n)}$ and $X^{(n+1)}$ are local solutions of the equation (4.16) on $[0, T\wedge T_n[$. Then from Lemma 2 $X^{(n)} = X^{(n+1)}$ on $[0, T\wedge T_n[$; that is, $X^{(n)} I_{[0,T[} = X^{(n+1)} I_{[0,T[}$ on $[0, T_n[$. This relation therefore enables us to define a process X such that

$$X = X^{(n)} I_{[0,T[} \quad \text{on} \quad [0, T_n[, \quad n = 1, 2, \cdots \tag{4.17}$$

Obviously X is an adapted process with paths right continuous and free of oscillatory discontinuities. Consequently the process X is a local solution of the equation (4.1) on $[0, T[$.

Finally, if X and Y are two local solutions on $[0, T[$ of (4.1), then these two processes are also solutions on $[0, T \wedge T_n[$ for the equation (4.16). Thus, from Lemma 2, $X = Y$ on $[0, T \wedge T_n[$, and letting $n \to +\infty$ we have: $X = Y$ on $[0, T[$. Hence the theorem is established completely.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN