Tôhoku Math. Journ.
28 (1976), 459-465.

# CRITERIA FOR THE NORMALITY OF A COMPACT OPERATOR

KOH-ICHI KITANO

(Received October 14, 1975)

We present in this note some criteria for the normality of a compact operator on the result of J. R. Ringrose [5] that every compact operator has a maximal nest of invariant subspaces. The main result of this note is a generalization of the known result (see, p. 93 of [3] or [1]).

Throughout this note $\mathscr{H}$ denotes a complex Hilbert space, operator and subspace mean a bounded linear operator and a closed linear manifold, respectively. Now suppose that $T$ is a compact operator acting in $\mathscr{H}$. The operator $T^*T$ is positive and compact, that has a unique non-negative square root which is also compact. The characteristic numbers of $T$ are defined to be the eigenvalues $\mu_1, \mu_2, \cdots, \mu_n, \cdots$ of $(T^*T)^{1/2}$ enumerated with their multiplicities; we arrange them in decreasing order. For $1 \leqq p < \infty$, we define $|T|_p$ to be $\{\sum_j \mu_j^p\}^{1/p}$. The von Neumann-Schatten class $\mathscr{C}_p$ is the set of all compact operators for which $|T|_p$ is finite. The class $\mathscr{C}_p$ is a two-sided ideal in $\mathscr{B}(\mathscr{H})$ (the algebra of all operators on $\mathscr{H}$) which is a Banach algebra under the $\mathscr{C}_p$-norm, $|\cdot|_p$ (see, Chap. III of [3]).

Next more general than the characteristic numbers of $T$, we shall consider the non-negative square roots of the eigenvalues $\nu_1, \nu_2, \cdots, \nu_n, \cdots$, arranged in decreasing order and repeated according to their multiplicities, of $\alpha T^*T + (1-\alpha)TT^*$, where $\alpha$ is any constant with $0 \leqq \alpha \leqq 1$ and $T$ is a compact operator on $\mathscr{H}$. We shall denote by $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ the eigenvalues of $T$. Suppose that are arranged in order of decreasing absolute value, repeated according to their multiplicities.

A generalization of Weyl's inequality; a set of inequalities comparing $|\lambda_j|$ with $\nu_j$, has been found in [K. Fan, 2] (Weyl's inequality is the case for $\alpha = 1$).

Let $T$ be an operator on $\mathscr{H}$. A family $\mathscr{F}$ of subspaces of $\mathscr{H}$ which is totally ordered by the inclusion relation, will be termed a nest of subspaces. If in addition each subspace belonging to $\mathscr{F}$ is invariant under $T$ we shall describe $\mathscr{F}$ as an invariant nest. For nests the following terminologies are contained in [5]. If $\mathscr{F}$ is a complete nest and $F \in \mathscr{F}$, we define $F_{-}$ by

$$F_{-} = \bigvee \{G : G \in \mathscr{F}, G \subsetneqq F\}\ .$$

If there is no $G$ in $\mathscr{F}$ such that $G \subsetneqq F$, we define $F_{-} = \{0\}$. If $T$ is a compact operator, then there exists a maximal invariant nest $\mathscr{F}$, i.e., the quotient space $F/F_{-}$ is at most one-dimensional for every $F$ in $\mathscr{F}$ [J. R. Ringrose, 5].

If $\mathscr{F}$ is a set of subspaces of $\mathscr{H}$. A linear combination of two operators that leaves $\mathscr{F}$ invariant is another such operator, and the same is true of their product, the identity operator $I$ leaves every $F$ in $\mathscr{F}$ invariant. In other words, the set of all operators that leave $\mathscr{F}$ invariant is always an algebra containing $I$, it will be denoted in this note by $\mathrm{Alg}\,\mathscr{F}$.

In [J. A. Erdos, 1] a simple proof of Lidskii's theorem [4] is given based on the fact that every compact operator has a maximal invariant nest.

## 1. Preliminary results.

From the theory of triangular forms, the lemma below summarizes the required results [1, 5].

LEMMA 1. *Let $T$ be a compact operator on $\mathscr{H}$ and let $\mathscr{F}$ be a maximal invariant nest for $T$. Then $T$ is uniquely represented in the form*

$$T = D + V$$

*where $D$, $D^*$ and $V$ are all members of* $\mathrm{Alg}\,\mathscr{F}$, *$D$ is normal and $V$ is quasinilpotent.*

*The eigenvalues of $T$ and $D$ coincide and have the same multiplicities.*

*The compact operator $T$ is quasi-nilpotent if and only if $TF \subset F_{-}$ for all $F$ in $\mathscr{F}$ and consequently the compact quasi-nilpotent operator of* $\mathrm{Alg}\,\mathscr{F}$ *form a two-sided ideal of* $\mathrm{Alg}\,\mathscr{F}$.

*Moreover, if the operator $T$ belongs to the class $\mathscr{C}_p$ for $1 \leqq p < \infty$, then $D$ and $V$ belong to the same class $\mathscr{C}_p$.*

LEMMA 2. *If an operator $A$ belongs to the class $\mathscr{C}_p$ for $1 \leqq p < \infty$, then*

$$\sum_{\iota} |(A\phi_\iota, \psi_\iota)|^p \leqq |A|_p^p$$

*where $\{\phi_\iota\}$ and $\{\psi_\iota\}$ are any orthonormal bases of $\mathscr{H}$.*

PROOF. We note if $A \geqq 0$, $1 \leqq p < \infty$ and $\phi$ is any unit vector in $\mathscr{H}$, then we have

$$(A\phi, \phi)^p \leqq (A^p\phi, \phi)\ .$$

In fact, let $E(\cdot)$ denote the spectral resolution of $A$. Then we have from the Hölder's inequality,

$$\begin{aligned}(A\phi, \phi) &= \int_0^\infty \lambda(E(d\lambda)\phi, \phi) \\ &\leqq \left\{\int_0^\infty \lambda^p(E(d\lambda)\phi, \phi)\right\}^{1/p}\left\{\int_0^\infty 1^{p/(p-1)}(E(d\lambda)\phi, \phi)\right\}^{(p-1)/p} \\ &= (A^p\phi, \phi)^{1/p}|\phi|^{2(p-1)/p} = (A^p\phi, \phi)^{1/p} .\end{aligned}$$

By the above considerations and the polar representation of $A$; $A = U(A^*A)^{1/2}$,

$$\begin{aligned}\sum_\iota |(A\phi_\iota, \psi_\iota)|^p &= \sum_\iota |(U(A^*A)^{1/2}\phi_\iota, \psi_\iota)|^p \leqq \sum_\iota |(A^*A)^{1/2}\phi_\iota|^p|\psi_\iota|^p \\ &= \sum_\iota ((A^*A)\phi_\iota, \phi_\iota)^{p/2} \leqq \sum_\iota ((A^*A)^{p/2}\phi_\iota, \phi_\iota) \\ &= \operatorname{tr}(A^*A)^{p/2} = |A|_p^p .\end{aligned}$$

**2. Necessary and sufficient conditions that a compact operator be normal.** The following result has been found by K. Fan [2].

LEMMA 3. *Let $T$ be a compact operator on $\mathscr{H}$. Let the eigenvalues of $T$ and the non-negative square roots of the eigenvalues of*

$$\alpha T^*T + (1-\alpha)TT^*$$

*for any constant $\alpha$, $0 \leqq \alpha \leqq 1$, be denoted by $\lambda_j$ and $\nu_j$ ($j = 1, 2, 3, \cdots$) with their multiplicities. Suppose that are arranged in order of decreasing absolute value, respectively. Then we have*

$$\nu_1\nu_2\cdots\nu_k \geqq |\lambda_1\lambda_2\cdots\lambda_k| \quad (k = 1, 2, 3, \cdots) .$$

By virture of this lemma and Lemma 3.4 [p. 37, 3] is applicable to the numbers $a_j = \log|\lambda_j|$, $b_j = \log\nu_j$ ($j = 1, 2, 3, \cdots$) and the function $\Phi(t) = (\exp t)^p$ for $1 \leqq p < \infty$. From this one obtains the relations

$$\sum_{j=1}^k |\lambda_j|^p \leqq \sum_{j=1}^k \nu_j^p \quad (k = 1, 2, 3, \cdots) .$$

In particular, if a compact operator $T$ belongs to the class $\mathscr{C}_p$ ($1 \leqq p < \infty$), then $T^*T$ and $TT^*$ belong to the class $\mathscr{C}_{p/2}$. Thus we have

$$\sum_{j=1}^\infty |\lambda_j|^p \leqq \sum_{j=1}^\infty \nu_j^p < \infty .$$

Next we will give proofs of some criteria for the normality of a certain compact operator.

THEOREM 4. *If a compact operator $T$ belongs to the class $\mathscr{C}_p$ (for some $p$, $1 \leqq p < \infty$), then the following are equivalent:*

(i) *T is a normal operator*;

(ii) *there exists a constant* $\alpha$ *with* $0 \leqq \alpha \leqq 1$ *such that*

$$\sum_{j=1}^{\infty} |\lambda_j|^p = \sum_{j=1}^{\infty} \nu_j^p \quad (= |\alpha T^*T + (1-\alpha)TT^*|_{p/2}^{p/2}) .$$

*where* $\lambda_j$ *and* $\nu_j$ *are as the same as in Lemma* 3.

PROOF. The fact that (i) implies (ii) is elementary. Let us go on to the proof of the converse implications. Let $\mathscr{F}$ be a maximal invariant nest for $T$ and write $T = D + V$ as in Lemma 1.

(A) We first prove that (ii) implies (i) for $p = 2$. By virture of Lidskii's theorem [4, 1] and Lemma 1 we show that

$$\mathrm{tr}\,(DV) = \mathrm{tr}\,(D^*V) = 0 .$$

We have from the results in Chap. III of [3], $\alpha T^*T + (1-\alpha)TT^* \in \mathscr{C}_1$ and

$$\mathrm{tr}\,(\alpha T^*T + (1-\alpha)TT^*) = \mathrm{tr}\,(D^*D) + \alpha\,\mathrm{tr}\,(V^*V) + (1-\alpha)\,\mathrm{tr}\,(VV^*) ,$$

that is

$$\begin{aligned}\sum_{j=1}^{\infty} \nu_j^2 &= \sum_{j=1}^{\infty} |\lambda_j|^2 + \alpha|V|_2^2 + (1-\alpha)|V^*|_2^2 \\ &= \sum_{j=1}^{\infty} |\lambda_j|^2 + |V|_2^2 .\end{aligned}$$

Therefore (ii) is equivalent to $|V|_2 = 0$, this implies that $V = 0$ and hence $T$ is normal.

(B) For $1 \leqq p < 2$, if $T \in \mathscr{C}_p$ and

$$\sum_{j=1}^{\infty} \nu_j^p = \sum_{j=1}^{\infty} |\lambda_j|^p ,$$

then $T \in \mathscr{C}_2$ and

$$\sum_{j=1}^{\infty} \nu_j^2 = \sum_{j=1}^{\infty} |\lambda_j|^2 .$$

Therefore this case can be reduced the case for $p = 2$.

In fact, Lemma 3.4 [p. 37; 3] is applicable to the numbers $a_j = \log|\lambda_j|$, $b_j = \log \nu_j$ $(j = 1, 2, 3, \cdots)$ and the strictly convex function $\Phi(t) = (\exp t)^p$. Thus the relation (ii) will hold if and only if $|\lambda_j| = \nu_j$ $(j = 1, 2, 3, \cdots)$.

(C) For $p > 2$, let $K = (D^*D)^{p/4-1/2}$. Let $F$ be an arbitrary subspace in $\mathscr{F}$ and let $P$ be the orthogonal projection onto $F$. Since $F$ is invariant under the operators $D$ and $D^*$ (i.e., $F$ is the reducing subspace of $D$), thus $PD = DP$. According the above operator $K$ belongs to the second commutant of $D$, it follows that $K \in \mathrm{Alg}\,\mathscr{F}$. We note the normality of $D$ implies that $KD$ and $DK$ are normal and from the quasi-nilpotentness of $V$ implies that $KV$ and $VK$ are quasi-nilpotent. By definition of $K$, $K \in \mathscr{C}_{2p/(p-2)}$ and $K(\alpha T^*T + (1-\alpha)TT^*)K \in \mathscr{C}_1$. We note that

$$\begin{aligned} \operatorname{tr}(K(\alpha T^*T + (1-\alpha)TT^*)K) &\leqq |K(\alpha T^*T + (1-\alpha)TT^*)K|_1 \\ &\leqq |\alpha T^*T + (1-\alpha)TT^*|_{p/2}\{|K|_{2p/(p-2)}\}^2 . \end{aligned}$$

Next, by virture of Lemma 1 and Lemma 3,

$$\begin{aligned} |K|_{2p/(p-2)}^{p/(p-2)} &\leqq |\{\alpha T^*T + (1-\alpha)TT^*\}^{p/4-1/2}|_{\{2p/(p-2)\}\{4/(p-2)\}^{-1}}^{p/(p-2)} \\ &\leqq |\alpha T^*T + (1-\alpha)TT^*|_{p/2}^{p/4} , \end{aligned}$$

therefore we have

$$\{|K|_{2p/(p-2)}\}^2 \leqq |\alpha T^*T + (1-\alpha)TT^*|_{p/2}^{(p-2)/2} .$$

Summarizing of the aboves we have

$$\operatorname{tr}(K(\alpha T^*T + (1-\alpha)TT^*)K) \leqq |\alpha T^*T + (1-\alpha)TT^*|_{p/2}^{p/2} .$$

Now it is easy to verify that

$$\operatorname{tr}\{(DK)^*(DK)\} = \operatorname{tr}\{(KD)(KD)^*\} = |KD|_2^2 = |D|_p^p .$$

By the considerations of the aboves and Lemma 1,

$$\begin{aligned} \operatorname{tr}(K(\alpha T^*T + (1-\alpha)TT^*)K) &= \sum_{j=1}^{\infty} |\lambda_j|^p + \alpha|VK|_2^2 + (1-\alpha)|KV|_2^2 \\ &\leqq |\alpha T^*T + (1-\alpha)TT^*|_{p/2}^{p/2} . \end{aligned}$$

Hence, if $0 < \alpha < 1$, the given relation shows that $|KV|_2 = 0$ and $|VK|_2 = 0$, that is

$$KV = VK = 0 . \tag{1}$$

If $\alpha = 0$, we have $KV = 0$; if $\alpha = 1$, we have $VK = 0$, respectively. Another by virture of similar arguments one obtains that $VK = 0$ in the case of $\alpha = 0$ and $KV = 0$ in the case of $\alpha = 1$, respectively. Therefore the given condition (ii) implies in any case that the relation (1) hold.

Now, if $E$ is the orthogonal projection onto the range of $D$, we have from the relation (1)

$$VE = EV = 0 . \tag{2}$$

We complete the proof by showing that

$$V(I - E) = 0 \quad \text{or} \quad V^*(I - E) = 0 . \tag{3}$$

Note that for any two orthonormal sets $\{\phi_\iota\}$, $\{\psi_\iota\}$, from Lemma 2

$$\sum_\iota |(\{\alpha T^*T + (1-\alpha)TT^*\}\phi_\iota, \psi_\iota)|^{p/2} \leqq |\alpha T^*T + (1-\alpha)TT^*|_{p/2}^{p/2} .$$

Let $\phi_0$ be an arbitrary unit vector in the range of $I - E$ and let $\phi_\iota = \psi_\iota = \chi_\iota$ where $\{\chi_\iota\}$ is a set of eigenvectors of $D$ which is complete in the range of $D$. Since from the relation (2) and $T = D + V$, for each $\iota$

$$T\chi_\iota = D\chi_\iota = \lambda_\iota \chi_\iota\ , \quad T^*\chi_\iota = D^*\chi_\iota = \bar{\lambda}_\iota \chi_\iota\ ,$$

it follows that

$$\begin{aligned} &|(\{\alpha T^*T + (1-\alpha)TT^*\}\phi_0, \phi_0)|^{p/2} + \sum_\iota |\lambda_\iota|^p \\ &= |(\{\alpha T^*T + (1-\alpha)TT^*\}\phi_0, \phi_0)|^{p/2} + \sum_{j=1}^\infty |\lambda_j|^p \\ &\leqq \sum_{j=1}^\infty \nu_j^p\ . \end{aligned}$$

Since $T\phi_0 = V\phi_0$, $T^*\phi_0 = V^*\phi_0$, we have from the given relation (ii)

$$\alpha(V\phi_0, V\phi_0) + (1-\alpha)(V^*\phi_0, V^*\phi_0) = 0\ .$$

Thus we have the relation (3) (if $\alpha = 0$, $\alpha = 1$ and $0 < \alpha < 1$, then $V^*(I-E) = 0$, $V(I-E) = 0$ and $V(I-E) = V^*(I-E) = 0$, respectively). Hence, by virture of the relations (2) and (3) implies that $V = 0$. Therefore $T = D$, that is normal. This completes the proof of Theorem 4.

REMARK. For a compact operator $T$ belonging to $\mathscr{C}_p$ $(1 \leqq p < \infty)$, if there exists a constant $\alpha$ $(0 \leqq \alpha \leqq 1)$ such that the equality (ii) in Theorem 4 holds, then for any constant $\alpha$ $(0 \leqq \alpha \leqq 1)$ the equality (ii) holds.

Similar arguments in the proof of Theorem 4 implies that the following theorem holds.

THEOREM 5. (cf. p. 58 Theorem 6.1 in Chap. II of [3], [1]) *Let $T$ be a compact operator belonging to the class $\mathscr{C}_p$ $(1 \leqq p < \infty)$. Then the following are equivalent*:

(i) *$T$ is a normal operator*;

(ii) $\sum_{j=1}^\infty |\mathscr{R}\lambda_j|^p = |\mathscr{R}T|_p^p$;

(iii) $\sum_{j=1}^\infty |\mathscr{I}\lambda_j|^p = |\mathscr{I}T|_p^p$,

*where $\mathscr{R}T = (T + T^*)/2$, $\mathscr{I}T = (T - T^*)/2i$ and $\mathscr{R}\lambda_j$, $\mathscr{I}\lambda_j$ are the real part and the imaginary part of $\lambda_j$, respectively.*

## REFERENCES

[1] J. A. ERDOS, On the trace of a trace class operator, Bull. London Math. Soc., 6 (1974), 47–50.

[2] K. FAN, On a theorem of Weyl concerning eigenvalues of linear transformations. II,

Proc. Nat. Acad. Sci. USA., 36 (1950), 31-35.
[3] I. C. GOHBERG AND M. G. KREIN, Introduction to the theory of linear non-selfadjoint operators, Translations of Mathematical Monographs, vol. 18, American Mathematical Society, Providence, R.I., 1969.
[4] V. B. LIDSKII, Non-selfadjoint operators with a trace, Dokl. Akad. Nauk SSSR., 125 (1959), 485-487; English Transl. Amer. Math. Soc. Transl. Ser II., 47 (1965), 43-46.
[5] J. R. RINGROSE, Super-diagonal forms for compact linear operators, Proc. London Math. Soc., Ser. III., 12 (1962), 367-384.

DEPARTMENT OF MATHEMATICS
TÔHOKU UNIVERSITY
COLLEGE OF GENERAL EDUCATION
KAWAUCHI, SENDAI, JAPAN