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THE DEGREE OF CONVERGENCE OF POSITIVE LINEAR OPERATORS

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1. Introduction. Let F be a locally convex Hausdorff space over the field of real numbers and F' its dual space. Let X be a compact convex subset of F, and let $C(X)$ denote the Banach lattice of all real-valued continuous functions on X with the supremum norm $\|\cdot\|$. Let F'_X denote the set of all restrictions of functions in F' to X and 1 the unit function on X. Throughout this paper, (L_α) will be a net of positive linear operators of $C(X)$ into itself.

The following theorem is a generalization of the well-known theorem of Bohman-Korovkin [6]. See, for instance, [1], [2], [5], [7] and [8]:

THEOREM A. *Suppose that $(L_\alpha(g^i))$ converges to g^i for all g in F'_X and for $i = 0, 1, 2$. Then $(L_\alpha(f))$ converges to f for all f in $C(X)$.*

The purpose of this paper is to recast Theorem A in a quantitative form which estimates the rate of convergence of $(L_\alpha(f))$ to f in $C(X)$ in terms of the quantities associated with the system $\{g^i;\ g \in F'_X,\ i = 0, 1, 2\}$.

In the case that F is the m-dimensional real Euclidean space R^m our results will cover the results of O. Shisha and B. Mond [10], E. Censor [3] and the result of R. De Vore [4] concerning the estimate for approximation of differentiable functions on the closed interval in the real line. Furthermore, they will suggest the characterization of the saturation class of positive linear operators satisfying the hypotheses of Theorem 1 in the author [7] and that of the Bernstein-Schnabl operators constructed by M. W. Grossman [5].

2. Definitions and lemmas. We shall begin with the following, which all the derived estimates for $\|L_\alpha(f) - f\|$ recasting Theorem A will involve.

DEFINITION 1. Let $\{g_1, \cdots, g_k\}$ be a finite subset of F'_X, δ a non-negative real number and f in $C(X)$. Then we define

$$\omega(f; g_1, \cdots, g_k, \delta) = \sup\{|f(x) - f(y)|;\ |g_i(x) - g_i(y)| \leqq \delta,\ x, y \in X,\ i = 1, 2, \cdots, k\}$$

and

$$\omega(f, \delta) = \inf \{\omega(f; g_1, \cdots, g_k, \delta);$$
$$g_1, \cdots, g_k \in F'_X, \sum_{i=1}^{k} ||g_i|| = 1, k = 1, 2, \cdots\},$$

which are called the modulus of continuity of f with respect to $g_1, \cdots, g_k$ and the total modulus of continuity of f, respectively.

LEMMA 1. *Let $g_1, \cdots, g_k$ be in F'_X and f in $C(X)$.*
(i) *$\omega(f; g_1, \cdots, g_k, \delta)$ is a non-decreasing function of δ, $\delta \geqq 0$.*
(ii) *$\omega(f; g_1, \cdots, g_k, \lambda\delta) \leqq (1 + \lambda)\omega(f; g_1, \cdots, g_k, \delta)$ for each λ, $\delta \geqq 0$.*
(iii) $\lim_{\delta \to 0} \omega(f, \delta) = 0$.

PROOF. (i) is clear. Since X is compact and F'_X separates the points of X, the original topology on X is identical with the weak topology on X induced by F'_X. Thus (iii) follows from the continuity of f. Next, let us show (ii). Let n be a natural number. Suppose that x, y in X and

$$|g_i(x) - g_i(y)| \leqq n\delta, \qquad i = 1, 2, \cdots, k.$$

Then we divide the segment joining x and y into n equal parts by the points

$$z_j = x + \frac{j}{n}(y - x), \qquad j = 0, 1, \cdots, n,$$

which belong to X since X is convex. It is obvious that

$$f(y) - f(x) = \sum_{j=0}^{n-1} (f(z_{j+1}) - f(z_j)).$$

Since

$$|g_i(z_{j+1}) - g_i(z_j)| \leqq \delta, \qquad i = 1, 2, \cdots, k,$$

we have

$$|f(z_{j+1}) - f(z_j)| \leqq \omega(f; g_1, \cdots, g_k, \delta),$$

and so

$$|f(y) - f(x)| \leqq n\omega(f; g_1, \cdots, g_k, \delta),$$

which yields

$$\omega(f; g_1, \cdots, g_k, n\delta) \leqq n\omega(f; g_1, \cdots, g_k, \delta). \tag{1}$$

Now let n be the largest integer not exceeding λ, i.e., $n \leqq \lambda < n + 1$. Then (ii) follows from (i) and (1).

For a bounded linear operator L of $C(X)$ into itself and g in F'_X, we define

$$\mu(L, g) = \|L((g(x) - g(y))^2, y)\|^{1/2},$$

where the operator L is applied to the variable x and the norm is taken with respect to the variable y. This convention will be used throughout.

LEMMA 2. *Let L be a positive linear operator of $C(X)$ into itself. Then $\mu(L, g) = 0$ for all g in F'_X if and only if $L(f) = fL(1)$ for all f in $C(X)$.*

PROOF. Suppose that $\mu(L, g) = 0$ for all g in F'_X. Let f be in $C(X)$ and $\varepsilon > 0$ be given. Then there exists a finite subset $\{g_1, \cdots, g_k\}$ of F'_X such that for all x, y in X

$$|f(x) - f(y)| \leqq \varepsilon + \sum_{i=1}^{k} (g_i(x) - g_i(y))^2 .$$

Since L is positive and linear, we have

$$\begin{aligned} &|L(f)(y) - f(y)L(1)(y)| \\ &\quad \leqq \varepsilon L(1)(y) + \sum_{i=1}^{k} L((g_i(x) - g_i(y))^2, y) \end{aligned}$$

for all y in X, and so

$$\|L(f) - fL(1)\| \leqq \varepsilon\|L(1)\| + \sum_{i=1}^{k} (\mu(L, g_i))^2 , \tag{2}$$

which establishes $L(f) = fL(1)$. The converse statement follows from the equality

$$(\mu(L, g))^2 = \|L(g^2) - 2gL(g) + g^2L(1)\| .$$

COROLLARY 1. *Let L be a positive linear operator of $C(X)$ into itself with $L(1) = 1$. Then $\mu(L, g) = 0$ for all g in F'_X if and only if L is the identity operator.*

DEFINITION 2. f in $C(X)$ is said to have the mean value property (briefly, *MVP*) if there exist a finite subset $\{f_1, \cdots, f_r\}$ of $C(X)$ and a finite subset $\{h_1, \cdots, h_r\}$ of F'_X such that for all x, y in X

$$f(x) - f(y) = \sum_{i=1}^{r} f_i(\xi)(h_i(x) - h_i(y)) , \tag{3}$$

where ξ is an internal point of the segment joining x and y. We sometimes say that f has *MVP* associated with the system $\{f_1, \cdots, f_r; h_1, \cdots, h_r\}$ if it satisfies (3).

For a finite number of elements $g_1, \cdots, g_k$ in F'_X, for instance, every polynomial in $g_1, \cdots, g_k$ has *MVP*. Note that for $F = R^m$, every continuously differentiable function f has *MVP*; in this case f_i and h_i in (3) can be taken the i-th partial derivative of f and the i-th coordinate linear functional, respectively.

3. Main theorems. From now on let us set $\mu(L_\alpha, g) = \mu_\alpha(g)$. We first have the following.

THEOREM 1. *Suppose that* $(L_\alpha(1))$ *converges to* 1 *and* $(\mu_\alpha(g))$ *converges to zero for all* g *in* F'_X. *Then* $(L_\alpha(f))$ *converges to* f *for all* f *in* $C(X)$.

PROOF. Obviously, we have for all f in $C(X)$ and all α

$$\|L_\alpha(f) - f\| \leqq \|L_\alpha(f) - fL_\alpha(1)\| + \|f\|\|L_\alpha(1) - 1\| . \tag{4}$$

Putting $L = L_\alpha$ in (2), and passing to the limit, the desired result follows from (4) and the hypotheses.

REMARK 1. Since for all g in F'_X and all α

$$(\mu_\alpha(g))^2 \leqq \|L_\alpha(g^2) - g^2\| + 2\|g\|\|L_\alpha(g) - g\| + \|g\|^2\|L_\alpha(1) - 1\| ,$$

Theorem A can be covered with Theorem 1.

In order to estimate the rate of convergence of $(L_\alpha(f))$ to f in $C(X)$, we define

$$\Omega_\alpha(f) = \inf \{\omega(f; g_1, \cdots, g_k, \sum_{i=1}^{k} \mu_\alpha(g_i)); \quad g_1, \cdots, g_k \in F'_X, \sum_{i=1}^{k} \mu_\alpha(g_i) \neq 0, k = 1, 2, \cdots\} ,$$

which converges to zero if $(\mu_\alpha(g))$ converges to zero for every g in F'_X.

THEOREM 2. *We have*

$$\|L_\alpha(f) - f\| \leqq \|f\|\|L_\alpha(1) - 1\| + \|L_\alpha(1) + (L_\alpha(1))^{1/2}\|\Omega_\alpha(f) \tag{5}$$

for all f *in* $C(X)$ *and all* α. *In particular, if* $L_\alpha(1) = 1$, *then* (5) *reduces to*

$$\|L_\alpha(f) - f\| \leqq 2\Omega_\alpha(f) .$$

PROOF. Let f be in $C(X)$ and α be fixed. In view of Lemma 2, it will suffice to show (5) in the case of $\{g; g \in F'_X, \mu_\alpha(g) = 0\} \neq F'_X$. Let $\{g_1, \cdots, g_k\}$ be a finite subset of F'_X with $\sum_{i=1}^{k} \mu_\alpha(g_i) \neq 0$, and let $\delta > 0$. Then by Lemma 1 (ii), for all x, y in X, we have

$$|f(x) - f(y)| \leqq (1 + \delta^{-1} \sum_{i=1}^{k} |g_i(x) - g_i(y)|) \\ \times \omega(f; g_1, \cdots, g_k, \delta) ,$$

and so

$$|L_\alpha(f)(y) - f(y)L_\alpha(1)(y)| \\ \leqq (L_\alpha(1)(y) + \delta^{-1} \sum_{i=1}^{k} L_\alpha(|g_i(x) - g_i(y)|, y)) \\ \times \omega(f; g_1, \cdots, g_k, \delta) .$$

The Cauchy-Schwarz inequality for positive linear operators gives

$$L_\alpha(|g_i(x) - g_i(y)|, y) \leqq \mu_\alpha(g_i)(L_\alpha(1)(y))^{1/2}$$

for all y in X and for $i = 1, \cdots, k$. Therefore, we obtain

$$|L_\alpha(f)(y) - f(y)L_\alpha(1)(y)| \\ \leqq (L_\alpha(1)(y) + \delta^{-1}(L_\alpha(1)(y))^{1/2} \sum_{i=1}^{k} \mu_\alpha(g_i)) \\ \times \omega(f; g_1, \cdots, g_k, \delta) \tag{6}$$

for all y in X. Putting $\delta = \sum_{i=1}^{k} \mu_\alpha(g_i)$ in (6), and taking the norm, we have

$$\|L_\alpha(f) - fL_\alpha(1)\| \leqq \|L_\alpha(1) + (L_\alpha(1))^{1/2}\| \\ \times \omega(f; g_1, \cdots, g_k, \sum_{i=1}^{k} \mu_\alpha(g_i)) .$$

Therefore, (5) now follows from (4) and the above inequality.

COROLLARY 2. *Let $\{g_1, g_2, \cdots, g_k\}$ be a finite subset of F'_X and (λ_α) a net of positive real numbers converging to zero. If $\|L_\alpha(1) - 1\| = O(\lambda_\alpha^2)$ and $\|L_\alpha(g_j^i) - g_j^i\| = O(\lambda_\alpha^2)$ for $i = 1, 2, j = 1, 2, \cdots, k$ and for all α, then there exists a constant C such that for all f in $C(X)$ and for all α*

$$\|L_\alpha(f) - f\| \leqq C(1 + \|f\|) \\ \times (\lambda_\alpha^2 + \omega(f; g_1, g_2, \cdots, g_k, \lambda_\alpha)) . \tag{7}$$

If $L_\alpha(1) = 1$, $L_\alpha(g) = g$ and $\|L_\alpha(g^2) - g^2\| \leqq (M\|g\|\lambda_\alpha)^2$ for all g in F'_X and for all α, then (7) reduces to

$$\|L_\alpha(f) - f\| \leqq (1 + M)\omega(f, \lambda_\alpha) .$$

Concerning the functions having the mean value property we have the following.

THEOREM 3. *If f in $C(X)$ has MVP associated with the system*

$\{f_1, \cdots, f_r; h_1, \cdots, h_r\}$, *then for all* α, *we have*

$$\|L_\alpha(f) - f\| \leqq \|f\| \|L_\alpha(1) - 1\| + \sum_{i=1}^{r} \|f_i\| \|L_\alpha((h_i(x) - h_i(y)), x)\| + (1 + \|L_\alpha(1)\|^{1/2}) \sum_{i=1}^{r} \mu_\alpha(h_i) \Omega_\alpha(f_i) . \tag{8}$$

In particular, if $L_\alpha(1) = 1$ *and* $L_\alpha(g) = g$ *for all* g *in* F'_X *and for all* α, *then* (8) *reduces to*

$$\|L_\alpha(f) - f\| \leqq 2 \sum_{i=1}^{r} \mu_\alpha(h_i) \Omega_\alpha(f_i) .$$

PROOF. Let α be fixed. In view of Lemma 2, it will suffice to show (8) in the case of $\{g; g \in F'_X, \mu_\alpha(g) = 0\} \neq F'_X$. Let $\{g_1, \cdots, g_k\}$ be a finite subset of F'_X with $\sum_{i=1}^{k} \mu_\alpha(g_i) \neq 0$, and let $\delta > 0$. From (3), we have

$$f(x) - f(y) = \sum_{i=1}^{r} f_i(x)(h_i(x) - h_i(y)) + \sum_{i=1}^{r} (f_i(\xi) - f_i(x))(h_i(x) - h_i(y)) ,$$

and so

$$\begin{aligned} & f(x) L_\alpha(1)(x) - L_\alpha(f)(x) \\ & = \sum_{i=1}^{r} f_i(x) L_\alpha((h_i(x) - h_i(y)), x) \\ & \quad + \sum_{i=1}^{r} L_\alpha((f_i(\xi) - f_i(x))(h_i(x) - h_i(y)), x) . \end{aligned} \tag{9}$$

By virtue of Lemma 1 (ii), for $i = 1, \cdots, r$, we have

$$\begin{aligned} & |f_i(\xi) - f_i(x)| |h_i(x) - h_i(y)| \\ & \leqq (1 + \delta^{-1} \sum_{j=1}^{k} |g_j(\xi) - g_j(x)|) |h_i(x) - h_i(y)| \\ & \quad \times \omega(f_i; g_1, \cdots, g_k, \delta) \\ & \leqq (1 + \delta^{-1} \sum_{j=1}^{k} |g_j(x) - g_j(y)|) |h_i(x) - h_i(y)| \\ & \quad \times \omega(f_i; g_1, \cdots, g_k, \delta) , \end{aligned}$$

and therefore, the Cauchy-Schwarz inequality for positive linear operators yields

$$\begin{aligned} & L_\alpha(|(f_i(\xi) - f_i(x))(h_i(x) - h_i(y))|, x) \\ & \leqq (\|L_\alpha(1)\|^{1/2} + \delta^{-1} \sum_{j=1}^{k} \mu_\alpha(g_j)) \mu_\alpha(h_i) \\ & \quad \times \omega(f_i; g_1, \cdots, g_k, \delta) , \end{aligned}$$

which yields

$$\|L_\alpha(|(f_i(\xi) - f_i(x))(h_i(x) - h_i(y))|, x)\| \leqq (1 + \|L_\alpha(1)\|^{1/2})\mu_\alpha(h_i)\Omega_\alpha(f_i) .$$

Therefore, (8) follows from (4) and (9).

COROLLARY 3. *Suppose that f in $C(X)$ has MVP associated with the system $\{f_1, f_2, \cdots, f_r; h_1, h_2, \cdots, h_r\}$. Let (λ_α) be a net of positive real numbers converging to zero and $\{g_1, g_2, \cdots, g_k\}$ a finite subset of F'_X. If $\|L_\alpha(1) - 1\| = O(\lambda_\alpha^2)$, $\|L_\alpha(g_j^i) - g_j^i\| = O(\lambda_\alpha^2)$ and $\|L_\alpha(h_n^i) - h_n^i\| = O(\lambda_\alpha^2)$ for $i = 1, 2$, $j = 1, 2, \cdots, k$, $n = 1, 2, \cdots, r$ and for all α, then there exists a constant C such that for all α*

$$\|L_\alpha(f) - f\| \leqq C(1 + \|f\| + \sum_{i=1}^{r} \|f_i\|) \times (\lambda_\alpha^2 + \lambda_\alpha \sum_{i=1}^{r} \omega(f_i; g_1, g_2, \cdots, g_k, \lambda_\alpha)) . \tag{10}$$

If $L_\alpha(1) = 1$, $L_\alpha(g) = g$ and $\|L_\alpha(g^2) - g^2\| \leqq (M\|g\|\lambda_\alpha)^2$ for all g in F'_X and for all α, then (10) *reduces to*

$$\|L_\alpha(f) - f\| \leqq M(1 + M)\lambda_\alpha \sum_{i=1}^{r} \|h_i\| \omega(f_i, \lambda_\alpha) .$$

REMARK 2. In the case of $F = R^m$ we see that the modulus of continuity with respect to the coordinate linear functionals is majorized by the usual modulus of continuity defined by

$$\sup\{|f(x) - f(y)|; x, y \in X, d(x, y) \leqq \delta\} ,$$

where $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_m)$, $f \in C(X)$, $\delta \geqq 0$ and $d(x, y) = (\sum_{i=1}^m (x_i - y_i)^2)^{1/2}$, and so Theorems 2 and 3 cover Theorems 1 in [3], [10] and Theorem 2.1 in [4], respectively.

4. Degree of convergence of Bernstein-Schnabl operators. Denote by $A(X)$ the space of all real-valued continuous affine functions on X. For a point x in X, an $A(X)$-representing measure for x is a probability measure ν_x on X such that $f(x) = \int_X f d\nu_x$ for all f in $A(X)$. Let E be a closed subset of X, containing the extreme points of X, and let $\mathscr{U}(E) = \{\nu_x\}_{x \in X}$ be a selection of $A(X)$-representing measures supported by E. Let $P = (p_{nj})_{n,j\geqq 1}$ be a lower triangular stochastic matrix, i.e., an infinite real matrix satisfying: $p_{nj} \geqq 0$ for all $n \geqq 1$ and $j \geqq 1$, $p_{nj} = 0$ whenever $j > n$, and $\sum_{j\geqq 1} p_{nj} = 1$ for each $n \geqq 1$. Then the Bernstein-Schnabl operators on $C(X)$ with respect to the matrix P and the selection $\mathscr{U}(E)$ are defined by

$$B_{n,P}^{\mathscr{U}(E)}(f)(x) = \int_X f d\pi_{n,P}(\nu_x^{\otimes n}) , \quad f \in C(X) , \quad x \in X , \qquad n = 1, 2, \cdots ,$$

where $\pi_{n,P}: E^n \to X$ are defined by

$$\pi_{n,P}(x_1, \cdots, x_n) = \sum_{j \geqq 1} p_{nj} x_j ,$$

$\otimes$ denotes tensor product and $\pi_{n,P}(\nu_X^{\otimes n})$ are the induced measures on X([5]). Briefly, we write $(\sum_{j\geqq 1} p_{nj}^2)^{1/2} = \lambda_n$ and $B_{n,P}^{\mathscr{U}(E)} = B_n$.

THEOREM 4. *Suppose that $(\lambda_n)_{n\geqq 1}$ converges to zero. Then we have*

(i) $$\|B_n(f) - f\| \leqq 2\omega(f, \lambda_n)$$

for all f in $C(X)$ and all $n \geqq 1$. In particular, if f in $C(X)$ has MVP associated with the system $\{f_1, \cdots, f_r; h_1, \cdots, h_r\}$, then (i) *reduces to*

(ii) $$\|B_n(f) - f\| \leqq 2\lambda_n \sum_{i=1}^{r} \|h_i\| \omega(f_i, \lambda_n) .$$

PROOF. It can be verified that for all g in $A(X)$ and all $n \geqq 1$,

$$B_n(g) = g ,$$
$$B_n(g^2) = g^2 + \lambda_n^2(B_1(g^2) - g^2) ,$$

and therefore,

$$\|B_n(g^2) - g^2\| \leqq \lambda_n^2 \|g\|^2$$

since the operator norm of B_1 equals one. Therefore, (i) and (ii) follow from Corollaries 2 and 3, respectively.

REMARK 3. In the special case that the entries of P are chosen as follows:

$$p_{nj} = \frac{1}{n} , \qquad (n \geqq 1, 1 \leqq j \leqq n) ,$$
$$p_{nj} = 0 , \qquad (j > n) ,$$

Theorem 4 (i) should be compared with the result of R. Schnabl [9, Satz 3].

The author [7] proved that if $(\lambda_n^2)_{n\geqq 1}$ converges to zero, then $(B_n)_{n\geqq 1}$ is saturated with order $(\lambda_n^2)_{n\geqq 1}$ and its trivial class coincides with $A(X)$. Theorem 4 (ii) suggests the characterization of the saturation class of $(B_n)_{n\geqq 1}$ and in general, Corollary 3 does that of positive linear operators satisfying the hypotheses of Theorem 1 in [7].

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