Tôhoku Math. Journ.
29 (1977), 61-68.

A NOTE ON MAUS' THEOREM ON RAMIFICATION GROUPS

HIROO MIKI

(Received October 24, 1975)

Introduction. Let k be a complete field under a discrete valuation with a perfect residue field $\bar{k}$ of characteristic $p \neq 0$, and let K/k be a fully ramified finite Galois extension with Galois group G. Let G_i denote the i-th ramification group of G. Then it is well known that the sequence $G = G_0 \supseteqq G_1 \supseteqq \cdots \supseteqq G_i \supseteqq G_{i+1} \supseteqq \cdots$ has the following properties:

G_i is normal in G for $i \geqq 0$, and there exists $i_0 > 0$ such that $G_i = 1$ for $i \geqq i_0$; G_0/G_1 is a cyclic group of order prime to p; for $i \geqq 1$, G_i/G_{i+1} is an elementary abelian p-group contained in the center of G_1/G_{i+1}; as a G_0/G_1-module, G_i/G_{i+1} is the direct sum of irreducible submodules which are isomorphic each other, for $i \geqq 1$.

Maus [3] has proved the 'inverse' of the above when k is a finite algebraic extension of the field of p-adic numbers $\boldsymbol{Q}_p$ and when k is of characteristic p, by using local class field theory and Artin-Schreier theory, respectively.

The purpose of this paper is to show that Maus' theorem is also valid when k is a complete field of characteristic 0 under a discrete valuation with a perfect residue field $\bar{k}$ of characteristic p, using Kummer theory.

For a Galois extension K of k, the *sequence of ramification groups* of K/k means the descending sequence of all ramification groups of K/k, without taking ramification numbers into account.

MAUS' THEOREM. *Let k be a complete field of characteristic* 0 *under a discrete valuation with a perfect residue field $\bar{k}$ of characteristic p and with absolute ramification order e_k, i.e., $e_k = \mathrm{ord}_k(p)$, where ord_k is the normalized additive valuation of k. Let $G = G^{(0)} \supseteqq G^{(1)} \supsetneqq \cdots \supsetneqq G^{(r)} \supsetneqq G^{(r+1)} = 1$ be the sequence of finite groups satisfying the following*:

(i) *$G^{(i)}$ is a normal subgroup of G for $i = 0, 1, \cdots, r$;*

(ii) *$G^{(0)}/G^{(1)}$ is a cyclic group of order prime to p;*

(iii) *$G^{(i)}/G^{(i+1)}$ is an elementary abelian p-group contained in the center of $G^{(1)}/G^{(i+1)}$ for $i \geqq 1$;*

(iv) *As a $G^{(0)}/G^{(1)}$-module, $G^{(i)}/G^{(i+1)}$ is the direct sum of irreducible submodules which are isomorphic each other, for $i = 1, 2, \cdots, r$. Then* *there exist a finite algebraic extension k' of k and a fully ramified finite Galois extension K' of k' with Galois group G whose sequence of ramification groups is $G = G^{(0)} \supseteqq G^{(1)} \supsetneqq \cdots \supsetneqq G^{(r)} \supsetneqq G^{(r+1)} = 1$. Moreover, if $e_k \not\equiv 0(\bmod p-1)$, then we can take k' such that $e_{k'} \not\equiv 0(\bmod p-1)$.*

In the above theorem, Maus assumed that $G = G^{(1)}$ or $r = 1$ when $\zeta \in k$, where ζ is a primitive p-th root of unity, but this assumption is not necessary.

The condition $e_k \not\equiv 0(\bmod p-1)$ is slightly stronger than the condition that $\zeta \notin k$. Precisely, it is equivalent to that the ramification index of $k(\zeta)/k$ is greater than 1 (see [5], Lemma 8).

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NOTATIONS

(1) k: a complete field of characteristic 0 under a discrete valuation with an arbitrary residue field of characteristic $p \neq 0$. ord_k: the normalized additive valuation of k. $\mathcal{O}_k$: the ring of integers of k. U_k: the group of units of k. $U_k^{(i)} = \{u \in U_k \mid \mathrm{ord}_k(u-1) \geqq i\}$ for $i \geqq 1$. $\bar{k}$: the residue field of k. e_k: the absolute ramification order of k, i.e., $e_k = \mathrm{ord}_k(p)$. $\bar{a}$ (for $a \in \mathcal{O}_k$): the image of a by the canonical homomorphism of $\mathcal{O}_k$ to $\bar{k}$. G_i (for a fully ramified finite Galois extension K of k with Galois group G): the i-th ramification group of G for an integer $i \geqq 0$, i.e., $G_i = \{\sigma \in G \mid \mathrm{ord}_K(\Pi^\sigma - \Pi) \geqq i+1\}$, where Π is a prime element of K. A *ramification number* t of K/k: a rational integer such that $G_t \supsetneqq G_{t+1}$. The *first ramification number* of K/k: the minimum of all the ramification numbers of K/k. $\psi_{K/k}$: the *Hasse function of* K/k.

(2) $\boldsymbol{Z}$: the ring of all rational integers. $\boldsymbol{N} = \{z \in \boldsymbol{Z} \mid z \geqq 1\}$. $\boldsymbol{F}_p$: the finite field of p elements. $G(K/k)$: the Galois group of a Galois extension K/k. $K^\times$: the multiplicative group of a field K. $\boldsymbol{Z}_p$: the ring of p-adic integers. ζ: a primitive p-th root of unity.

1. A certain filter of subgroups of a complete field. Let p, ζ, k and $U_k^{(i)}$ be as in Notations. Put $k' = k(\zeta)$, and fix a generator σ of $G(k'/k)$. Regard $U_{k'}^{(1)}$ as a $\boldsymbol{Z}_p[G(k'/k)]$-module in the natural way. Let $\eta \in \boldsymbol{Z}_p^\times$ be a unique primitive N_1-th root of unity such that $\zeta^\sigma = \zeta^\eta$, where $N_1 = [k' : k]$. We define subgroups $A_k^{(i)}$ of $U_{k'}^{(i)}$ and an element Ω of $\boldsymbol{Z}_p[G(k'/k)]$ in the following:

DEFINITION. For any integer $i \geqq 1$,

$$A_k^{(i)} = \{x \in U_{k'}^{(i)} \mid x^{\sigma-\eta} = 1\}$$

and

$$\Omega = (\sigma^{N_1-1} + \sigma^{N_1-2}\eta + \cdots + \sigma\eta^{N_1-2} + \eta^{N_1-1})\eta N_1^{-1} .$$

For the properties of the operator Ω we have the following

LEMMA 1 ([7], Lemma 3). *Let notations and assumptions be as above. Then the operator Ω has the properties*:

(1) $A_k^{(i)} = (U_{k'}^{(i)})^{\Omega}$ *for any integer* $i \geqq 1$;

(2) $x^{\Omega} = x$ *for any* $x \in A_k^{(1)}$.

Any element of $A_k^{(1)}$ can be expressed in the following normal form of an infinite product.

PROPOSITION 1. *Let notations and assumptions be as above. Put $e_1 = \mathrm{ord}_{k'}(\zeta - 1)$ and let π_k and $\pi_{k'}$ be prime elements of k and k', respectively. Let N be the ramification index of k'/k. For any $\lambda \in \mathcal{O}_k$ and any $j \in \boldsymbol{Z}$ with $e_1 p + jN \geqq 1$, put $X_j(\lambda) = (1 + \lambda(\zeta - 1)^p \pi_k^j)^{\Omega}$. Then the following are valid*:

(1) $X_j(\lambda) \in A_k^{(e_1 p + Nj)}$ *and* $X_j(\lambda) \equiv 1 + \lambda(\zeta - 1)^p \pi_k^j \pmod{\pi_{k'}^{e_1 p + Nj + 1}}$;

(2) *For any $x \in A_k^{(1)}$, there exist $\lambda_i \in U_k \cup \{0\}$ such that $x = \prod_{i=i_0}^{\infty} X_i(\lambda_i)$, where $i_0 \in \boldsymbol{Z}$ is such that $e_1 p + N i_0 \geqq 1$.*

PROOF. (1) That $X_j(\lambda) \in A_k^{(e_1 p + Nj)}$ follows from (1) of Lemma 1. The second assertion follows easily from the definition and that $((\zeta-1)^p)^{\sigma} \equiv \eta(\zeta - 1)^p \pmod{\pi_{k'}^{e_1 p + 1}}$.

(2) Let k_{ur}/k be the maximum unramified extension of k in k' and let σ' be a generator of $G(k'/k_{ur})$. Then $N = [k' : k_{ur}]$. Put $a = (x - 1)/(\zeta - 1)^p$, then by the definition of x we have easily $a^{\sigma-1} \in U_{k'}^{(1)}$, so $a^{\sigma'-1} \in U_{k'}^{(1)}$. By Serre [10], Chap. IV, §2, Proposition 7 this implies $\mathrm{ord}_{k'}(a) \equiv 0 \pmod{N}$, so $x = 1 + \lambda(\zeta - 1)^p \pi_k^{i_0}$ with $\lambda \in U_{k'}$ and $i_0 \in \boldsymbol{Z}$ such that $e_1 p + i_0 N \geqq 1$. Then $a^{\sigma-1} \in U_{k'}^{(1)}$ implies $\lambda^{\sigma} \equiv \lambda \pmod{\pi_{k'}}$, so $\bar{\lambda} = \bar{\lambda}_{i_0}$ with some $\lambda_{i_0} \in U_k$. Hence by (1), $x/X_{i_0}(\lambda_{i_0}) \in A_k^{(e_1 p + i_0 N + 1)}$. Using this procedure successively, we see that for any $j \geqq i_0$, there exist $\lambda_i \in U_k \cup \{0\}$ such that $x/\prod_{i=i_0}^{j} X_i(\lambda_i) \in A_k^{(e_1 p + jN + 1)}$. Taking the limit, we obtain the assertion (2), since k is complete.

COROLLARY. *Notations and assumptions being as in Proposition 1, the following are valid*:

(i) $A_k^{(e_1 p + jN)} \supsetneqq A_k^{(e_1 p + jN + 1)} = A_k^{(e_1 p + jN + N)}$ *for $j \in \boldsymbol{Z}$ such that $e_1 p + jN \geqq 1$.*

(ii) $A_k^{(e_1 p + jN)}/A_k^{(e_1 p + jN + 1)} \cong \bar{k}$ *by* $X_j(\lambda) \bmod A_k^{(e_1 p + jN + 1)} \mapsto \bar{\lambda}$ *with* $\lambda \in \mathcal{O}_k$, *for $j \in \boldsymbol{Z}$ such that $e_1 p + jN \geqq 1$.*

(iii) $A_k^{(e_1p-jN)}(k'^\times)^p/A_k^{(e_1p-jN+1)}(k'^\times)^p \cong \bar{k}$ *by* $X_{-j}(\lambda) \bmod A_k^{(e_1p-jN+1)}(k'^\times)^p \mapsto \bar{\lambda}$ *with* $\lambda \in \mathcal{O}_k$, *for* $j \in \boldsymbol{Z}$ *such that* $j \not\equiv 0 (\bmod p)$ *and* $1 \leqq j < e_k p/(p-1)$.

PROOF. The assertions (i) and (ii) follow directly from Proposition 1. The assertion (iii) follows from Proposition 1 and the following Lemma 2.

For the connection of $A_k^{(1)}$ and $(k'^\times)^p$, we have the following

LEMMA 2. *Let notations and assumptions be as above. Let* $x \in A_k^{(1)}$ *be such that* $1 \leqq \operatorname{ord}_{k'}(x-1) < e_1 p$. *Then the following are valid:*

(1) *If* $x \in (k')^p$, *then* $\operatorname{ord}_{k'}(x-1) \equiv 0 (\bmod p)$ *and* $x/X_{j'p}(\lambda^p) \in A_k^{(e_1p+j'p+1)}$ *with some* $j' \in \boldsymbol{Z}$ *and some* $\lambda \in U_k$.

(2) *If* $x \equiv 1 + \lambda^p(\zeta-1)^p \pi_k^{j'p} (\bmod \pi_{k'}^{e_1p+j'pN+1})$ *with some* $\lambda \in U_k$, *then* $xy^p \in A_k^{(e_1p+j'pN+N)}$ *with some* $y \in A_k^{(1)}$.

PROOF. (1) Write $x = z^p$ with $z \in k'$. Since $x \in U_{k'}^{(1)}$, $z \in U_{k'}^{(1)}$. If $\operatorname{ord}_{k'}(z-1) \geqq e_1$, then $\operatorname{ord}_{k'}(z^p-1) \geqq e_1 p$. Since $\operatorname{ord}_{k'}(x-1) < e_1 p$, this implies that $1 \leqq \nu < e_1$, where $\nu = \operatorname{ord}_{k'}(z-1)$. Write $z \equiv 1 + \alpha\pi_{k'}^{\nu}$ $(\bmod \pi_{k'}^{\nu+1})$ with $\alpha \in U_{k'}$. Since $\nu < e_1$, $z^p \equiv 1 + \alpha^p \pi_{k'}^{\nu p} (\bmod \pi_{k'}^{\nu p+1})$, so by (2) of Proposition 1, $x \equiv 1 + \lambda^p(\zeta-1)^p\pi_k^{j'p} (\bmod \pi_{k'}^{e_1p+Nj'p+1})$ with some $\lambda \in U_k$, so by (1) of Proposition 1, $x \equiv X_{j'p}(\lambda^p)(\bmod \pi_{k'}^{e_1p+j'pN+1})$, hence $x/X_{j'p}(\lambda^p) \in A_k^{(e_1p+j'pN+1)}$.

(2) Put $y = (1-\lambda(\zeta-1)\pi_k^{j'})^{\Omega}$. Then by (1) of Lemma 1, $y \in A_k^{(e_1+Nj')}$. Since $(\zeta-1)^\sigma \equiv \eta(\zeta-1)(\bmod \pi_{k'}^{e_1+1})$, $y \equiv 1 - \lambda(\zeta-1)\pi_k^{j'}(\bmod \pi_{k'}^{e_1+Nj'+1})$. Since $e_1 + Nj' < e_1$, $y^p \equiv 1 - \lambda^p(\zeta-1)^p\pi_k^{j'p}(\bmod \pi_{k'}^{e_1p+j'pN+1})$. Hence $xy^p \in A_k^{(e_1p+j'pN+1)}$, so by (2) of Proposition 1, $xy^p \in A_k^{(e_1p+j'pN+N)}$. q.e.d.

2. Proof of Maus' Lemma 2.7 and Satz 2.8 when $\bar{k}$ is perfect. In this section, we prove Maus' Lemma 2.7 and Satz 2.8 when $\bar{k}$ is perfect, using §1.

PROOF OF MAUS' LEMMA 2.7 AND SATZ 2.8 WHEN $\bar{k}$ IS PERFECT. If Lemma 2.7 is proved when $\bar{k}$ is perfect, then Maus' proof of Satz 2.8 is still valid when $\bar{k}$ is perfect; so it is enough to prove Lemma 2.7. Let $\gamma(\sigma') \in \boldsymbol{Z}_p^\times$ be a unique $(p-1)$-th root of unity such that $\zeta^{\sigma'} = \zeta^{\gamma(\sigma')}$ for $\sigma' \in G(E'/k)$, where $E' = E(\zeta)$. Then $\gamma \in \operatorname{Hom}(G(E'/k), \boldsymbol{Z}_p^\times)$. Regard $A_E^{(i)}$ as a $G(E/k)$-module by $x^\tau = x^{\tau'\gamma(\tau')^{-1}}$ for $x \in A_E^{(i)}$ and $\tau \in G(E/k)$, where $\tau' \in G(E'/k)$ is such that $\tau'|E = \tau$. This is well defined. In fact, let $\tau'' \in G(E'/k)$ be such that $\tau''|E = \tau$, then $\tau'\tau''^{-1} = \tilde{\sigma} \in G(E'/E)$, and by the definition of $A_E^{(i)}$, $x^{\tilde{\sigma}\gamma(\tilde{\sigma})^{-1}} = x$, so $x^{\tau'\gamma(\tau')^{-1}} = x^{\tau''\gamma(\tau'')^{-1}}$. Regard $A_E^{(i)}(E'^\times)^p/(E'^\times)^p$ as a $G(E/k)$-module, by $(x \bmod (E'^\times)^p)^\tau = x^\tau \bmod (E'^\times)^p$ with $x \in A_E^{(i)}$ and $\tau \in G(E/k)$. Put $F_t = A_E^{(e_1p-Nt)}(E'^\times)^p/(E'^\times)^p$, where N is the ramification index of E'/E and $e_1 = \operatorname{ord}_{E'}(\zeta-1)$. Since F_t is a completely reducible $G(E/k)$-module containing F_{t-1} as a $G(E/k)$-submodule, there exists a $G(E/k)$-submodule $D/(E'^{\times})^p$ of F_t such that $F_t = D/(E'^{\times})^p \times F_{t-1}$ (direct product). Put $K' = E'(\sqrt[p]{x} \mid x \in D)$. Then by [5], Corollary to Proposition 2, there exists a unique abelian extension K/E whose Galois group is an elementary abelian p-group such that $K(\zeta) = K'$. We see that K/k is a Galois extension. In fact, since $D^{\tau} = D$ for all $\tau \in G(E'/k)$, we see by Kummer theory that K'/k is a Galois extension; for any $\tilde{\sigma} \in G(K'/k)$, $E \subset K^{\tilde{\sigma}} \subset K'$ and $K^{\tilde{\sigma}}(\zeta) = K'$, so by the uniqueness of such K, $K^{\tilde{\sigma}} = K$, hence K/k is a Galois extension. Identify $G(K'/E')$ and $G(K/E)$ by the restriction from K' to K. By Kummer theory, $D/(E'^{\times})^p$ is isomorphic to the character group $X(G(K'/E'))$ of $G(K'/E')$ in the canonical way. As usual, regard $G(K/E)$ as a $G(E/k)$-module by $\tau \circ \xi = \tilde{\tau}\xi\tilde{\tau}^{-1}$ for $\tau \in G(E/k)$ and $\xi \in G(K/E)$, where $\tilde{\tau} \in G(K/k)$ is such that $\tilde{\tau} \mid E = \tau$, and regard $X(G(K/E))$ as a $G(E/k)$-module by $(\tau \circ \chi)(g) = \chi(\tau^{-1} \circ g)$ with $\tau \in G(E/k)$, $\chi \in X(G(K/E))$ and $g \in G(K/E)$. Then it is easily verified that the canonical isomorphism $D/(E'^{\times})^p \cong X(G(K'/E'))$ is a $G(E/k)$-isomorphism. Thus $X(G(E/k)) \cong M_{-t,\theta_0}(G(E/k), \bar{k})$ as a $G(E/k)$-module, where $M_{-t,\theta_0}(G(E/k), \bar{k})$ is as in Maus [3], §1.2. In fact, it is easily verified that the isomorphism from $A_E^{(e_1p-Nt)}(E'^{\times})^p/A_E^{(e_1p-Nt+1)}(E'^{\times})^p (= F_t/F_{t-1})$ onto $\bar{k}$ $(=\bar{E})$ defined in (iii) of Corollary to Proposition 1 is a $G(E/k)$-isomorphism from F_t/F_{t-1} onto $M_{-t,\theta_0}(G(E/k), \bar{k})$, so $D/(E'^{\times})^p \cong M_{-t,\theta_0}(G(E/k), \bar{k})$, hence $X(G(E/k)) \cong M_{-t,\theta_0}(G(E/k), \bar{k})$ as a $G(E/k)$-module. Hence by the duality theorem of Pontrjagin, $G(K/E) \cong M_{t,\theta_0}(G(E/k), \bar{k})$ as a $G(E/k)$-module. In general, it is easily verified that $E'(\sqrt[p]{x})/E'$ has the ramification number $(e_1p - \nu)$ if $x \in U_{E'}^{(\nu)}, \notin U_{E'}^{(\nu+1)}$ with $1 \leqq \nu < e_1p$, $\nu \not\equiv 0 (\mathrm{mod}\, p)$. Since $D \cap A_E^{(e_1p-Nt+1)}(E'^{\times})^p = (E'^{\times})^p$, by this remark and [5], Lemma 10, we see that any sub-extension of K/E of degree p has the ramification number t; so by Serre [10], Chap. IV, §1, Proposition 3, we see easily that K/E has the only one ramification number t.

REMARK. When $\bar{k}$ is algebraically closed, Maus' proof of Lemma 2.7 is still valid if we replace local class field theory by local class field theory of Serre [9]. However, we adopt the elementary method, not using class field theory.

3. Proof of Maus' Korollar 5.10 when $\bar{k}$ is perfect. In this section, we prove Theorem which corresponds to Maus [3], Korollar 5.10, and for its proof we use Wyman [11], Corollary 29, Maus [2], (3.3), (3.7), (3.9) and the following Lemmas 3 and 4.

LEMMA 3. *Let p, k and e_k be as in Notations. Assume moreover that $\bar{k}$ is algebraically closed. Let $t \in N$ be such that* $1 \leqq t < e_kp/(p-1)$

and $t \not\equiv 0 \pmod p$). *Then for any integer* n *there exists a fully ramified cyclic extension* k_n *of* k *of degree* p^n *whose first ramification number is* t.

PROOF. By MacKenzie-Whaples [12], there exists a cyclic extension k_1 of k of degree p whose ramification number is t. It is well known that the Galois group of the maximal p-extension of k is free pro-p-group. Hence there exists a cyclic extension k_n of k of degree p^n containing k_1. By Serre [10], Chap. IV, §1, Proposition 3, the first ramification number of k_n/k is t.

REMARK. It is verified by using [5], Corollary to Proposition 3 and Serre [10], Chap. V that Lemma 3 is also valid when $\bar{k}$ is perfect.

LEMMA 4. *Let* p *be a prime number. Put* $\tilde{M}(e) = \{t' \in N \mid t' \not\equiv 0 \pmod p),\ e/(p-1) \leqq t' < ep/(p-1)\}$ *and* $M(t, e, m) = \{t, t+e, \cdots, t+(m-1)e\}$ *for* $e \in N$, $m \in N$ *and* $t \in N$. *Let* $n \in N$ *be such that* $n < e(p-1)/p$ *and let* $r_1 < r_2 < \cdots < r_n$ *be a sequence of non-negative rational numbers. Fix* e. *Then there exists* $t \in \tilde{M}(e)$ *such that* $r_i \notin M(t, e, m)$ *for* $i = 1, 2, \cdots, n$ *and for all* $m \in N$.

PROOF. Put $\tilde{M} = \tilde{M}(e)$ and $M_t = \bigcup_{m=1}^{\infty} M(t, e, m)$. It is easily verified that $M_t \cap M_{t'} = \varnothing$ with $t \neq t'$, $t \in \tilde{M}$ and $t' \in \tilde{M}$. Since $n < e(p-1)/p < \#(\tilde{M})$, there exists $t \in \tilde{M}$ such that $M_t \cap \{r_1, \cdots, r_n\} = \varnothing$. q.e.d.

For a Galois extension K/k, we call s an *upper ramification number* of K/k when $\psi_{K/k}(s)$ is a ramification number of K/k.

THEOREM. *Let* p, k *and* e_k *be as in Notations and let* K/k *be a finite fully ramified Galois extension. Moreover suppose that* $\bar{k}$ *is perfect. Then there exists a finite algebraic extension* k'/k *satisfying the following properties* (1) *and* (2):

(1) *The sequence of the ramification groups of* K/k *can be identified with that of* K'/k' *by the restriction homomorphism of* $G(K'/k')$ *onto* $G(K/k)$, *where* $K' = k'K$.

(2) *All the upper ramification numbers of* K'/k' *are smaller than* $e_{k'}/(p-1)$.

Moreover, if $e_k \not\equiv 0 \pmod{p-1}$, *then we can take* k' *such that* $e_{k'} \not\equiv 0 \pmod{p-1}$.

PROOF. Serre [10], Chap. V, §4, Lemma 7, we may suppose that $\bar{k}$ is algebraically closed. Let $r_1 < r_2 < \cdots < r_n$ be the sequence of all the upper ramification numbers of K/k. By taking a suitable tamely ramified extension of k of degree prime to $[K:k]$ and $(p-1)$, we may suppose that $n < e_k(p-1)/p$. By Lemma 4, there exists $t \in \tilde{M}(e_k)$ such that

$r_i \notin M(t, e_k, m)$ for $i = 1, 2, \cdots, n$ and for all $m \in N$. Put $s_m = t + (m-1)e_k$ and fix $m \in N$ such that $r_n < s_m$. By Lemma 3, there exists a fully ramified cyclic extension k'/k of degree p^m whose first ramification number is t. By Wyman [11], Corollary 29, the set of all the upper ramification numbers of k'/k is $M(t, e_k, m)$. Since $r_i \notin M(t, e_k, m)$ for $i = 1, \cdots, m$, by Maus [2], (3.3) (3.7) (3.9), the sequence of the ramification groups of K/k is isomorphic to that of K'/k', and $r'_n = \psi_{k'/k}(r_n)$, where r'_n is the maximum of the upper ramification numbers of K'/k'. Since $r_n < s_m$, we have $\psi_{k'/k}(r_m) < \psi_{k'/k}(s_m) = t + (e_k p/(p-1))(p^{m-1} - 1) < e_{k'}/(p-1)$, hence $r'_n < e_{k'}/(p-1)$.

4. Proof of Maus' theorem quoted in the introduction when $\bar{k}$ is perfect. Using §2 and 3, Maus [3], Satz 3.4 and Lemma 4.3, we can prove Maus' theorem when $\bar{k}$ is perfect. Note that *Satz* 3.4[(1)] *is valid when $\bar{k}$ is algebraically closed without the assumption that E is regular and that Maus' Lemma* 4.3 *is valid when $\bar{k}$ is perfect.* In fact, since the Galois group of the maximal p-extension of E is free pro-p-group and since Maus' Lemma 2.7 is valid when $\bar{k}$ is perfect by §2 of this paper, Maus' proof of Satz 3.4 is also valid; since Maus' Lemma 2.7 is valid when $\bar{k}$ is perfect by §2, Maus' proof of Lemma 4.3 is still valid.

PROOF OF MAUS' THEOREM WHEN $\bar{k}$ IS PERFECT. By Serre [10], Chap. V, §4, Lemma 7, we may suppose from the beginning that $\bar{k}$ is algebraically closed. We shall prove the theorem by induction on r. If $r = 1$, then the assertion follows from §2. Suppose $r > 1$. We shall prove this case in the following four steps (I) $\sim$ (IV).

(I) By the induction hypothesis, there exist a finite algebraic extension k_1/k and a finite fully ramified Galois extension K_1/k_1 whose sequence of ramification groups is $G^{(0)}/G^{(r)} \supseteqq G^{(1)}/G^{(r)} \supsetneqq \cdots \supsetneqq G^{(r-1)}/G^{(r)} \supsetneqq 1$.

(II) By Maus' Satz 3.4 (see the above remark), there exists a finite Galois extension K/k_1 containing K_1 such that $G(K/k_1) = G^{(0)}$ and $G(K/K_1) = G^{(r)}$.

(III) By §3, there exists a finite algebraic extension k'/k_1 satisfying the following (i) and (ii):

(i) The sequence of the ramification groups of Kk'/k' is isomorphic to that of K/k_1 in the natural way.

(ii) All the upper ramification numbers of Kk'/k' are smaller than $e_{k'}/(p-1)$.

(IV) Let E'/k' be the maximum tamely ramified extension of k' in K_1k'. Let $t \in N$ be such that $e_{k'}/(p-1) < t$ and $t \in \bar{V}_p(e_{k'}, e_0, \bar{t})$, where $\bar{V}_p(e_{k'}, e_0, \bar{t})$ is as in Maus' Lemma 4.3 for $Kk' \supset K_1k' \supset E' \supset k'$. Then by

(1) This theorem is generalized in [8], Theorems 7 and 8.

Maus' Lemma 4.3, there exists a finite fully ramified Galois extension K'/k' satisfying the following (iii) and (iv):

(iii) $K' \supset K_1 k'$, $G(K'/k') = G^{(0)}$ and $G(K'/K_1 k') = G^{(r)}$.

(iv) $K'/K_1 k'$ has the only one ramification number $\psi_{K'/E'}(t)$.

The conditions (ii) and (iv) imply that the only one ramification number $\psi_{K'/E'}(t)$ of $K'/K_1 k'$ is greater than all ramification numbers of $K_1 k'/k'$. Hence by Maus [3], Lemma 4.2, the sequence of the ramification groups of K'/k' is $G^{(0)} \supseteqq G^{(1)} \supsetneqq \cdots \supsetneqq G^{(r)} \supsetneqq G^{(r+1)} = 1$. The last assertion is verified in each step in the above.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF TOKYO
HONGO, TOKYO
113 JAPAN

PRESENTLY AT
DEPARTMENT OF MATHEMATICS
FACULTY OF ENGINEERING
YOKOHAMA NATIONAL UNIVERSITY
156, TOKIWADAI, HODOGAYA
YOKOHAMA
240 JAPAN