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ON THE ISOMETRIC STRUCTURE OF RIEMANNIAN MANIFOLDS OF NON-NEGATIVE RICCI CURVATURE CONTAINING A COMPACT HYPERSURFACE WITHOUT FOCAL POINT

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1. Introduction. In their paper [2], J. Cheeger and D. Gromoll proved the following:

THEOREM (Cheeger-Gromoll). *Let M be a connected, complete and non-compact Riemannian manifold of non-negative Ricci curvature. If M contains a line, then M is isometric to the Riemannian product $N \times \boldsymbol{R}$, where N is a totally geodesic hypersurface in M.*

Recall that a line is a normal geodesic $l: (-\infty, \infty) \to M$, any segment of which is minimal.

The above theorem says that the existence of suitable geometric objects in M determines the isometric structure of M. In the present paper, we shall consider the case where M contains a compact hypersurface without focal point. Our results are the following:

THEOREM A. *Let M be a connected, complete and non-compact Riemannian manifold of non-negative Ricci curvature. If M contains a compact hypersurface N without focal point, then N is totally geodesic, and M is isometric to a flat line bundle on N or on $N/\boldsymbol{Z}_2$.*

THEOREM B. *Let M be a connected, compact Riemannian manifold of non-negative Ricci curvature. If M contains a compact hypersurface N without focal point, then N is totally geodesic, and M is isometric to a Riemannian manifold $\perp_{[0,r]}N/i$.*

The Riemannian manifold $\perp_{[0,r]}N/i$ is defined as follows: For $r>0$, let $\perp_{[0,r]}N$ be a flat line bundle on N with fibre $[-r, r]$. Let $i: \perp_r N \to \perp_r N$ be a fixed-point free isometric involution on the boundary $\perp_r N$ of $\perp_{[0,r]}N$. Then identifying the boundary points u and $i(u)$, we obtain the Riemannian manifold $\perp_{[0,r]}N/i$.

2. Preliminaries. Let M be an n-dimensional connected and complete Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$ and Levi-Civita connection ∇. For $p \in M$, let M_p be the tangent space to M. Let $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ be the Riemannian curvature tensor. For $u, v \in M_p$, let $K(u, v)$ be the sectional curvature of the plane spanned by u and v. If u and v are mutually orthogonal unit vectors, recall that $K(u, v) = \langle R(u, v)u, v\rangle$. For a unit vector $u \in M_p$, $\mathrm{Ric}(u) = \sum_{k=1}^{n-1} K(u, e_k)$ is the Ricci curvature of M with respect to u, where $e_1, \cdots, e_{n-1}, u$ is an orthonormal basis of M_p. Let N be a connected and complete hypersurface in M. Let $\nu\colon \perp N \to N$ and $\nu\colon \perp_1 N \to N$ be the flat normal bundle and the unit normal bundle on N respectively. For $u \in \perp_1 N$, $p = \nu(u)$, let $S_u\colon N_p \times N_p \to \boldsymbol{R}$ be the second fundamental form of N with respect to u. $S_u(X, Y) = -\langle u, \nabla_X Y\rangle$ for tangent vector fields X and Y on N. The mean curvature of N with respect to u is given by $m(u) = \sum_{k=1}^{n-1} S_u(e_k, e_k)$, where $e_1, \cdots, e_{n-1}$ is an orthonormal basis of N_p. Let $\exp\colon TM \to M$ be the exponential map. Let $\exp_N\colon \perp N \to M$ and $\exp_N\colon \perp_1 N \to M$ be the restrictions of exp on $\perp N$ and on $\perp_1 N$ respectively. A geodesic c is called normal if its tangent vector $\dot{c}$ is of unit length. For $u \in \perp_1 N$, the map $c\colon [0, \infty) \to M$ defined by $c(t) = \exp_N(tu)$ is a normal geodesic starting from N and perpendicular to N at $t = 0$. A cut point $c(\tau)$ of N along c is a point such that the restriction $c|[0, \tau]$ is a minimal geodesic from N to $c(\tau)$, but $c|[0, \tau']$ is not for any $\tau' > \tau$. The cut locus $C(N)$ of N is the set of cut points of N along all geodesics starting from N and perpendicular to N. $C(N)$ is a closed set in M. A Jacobi field $J\colon [0, \infty) \to TM$ along c is said to be transversal to N at $t = 0$ if it satisfies

(i) J is perpendicular to c,

(ii) $\langle \nabla_u J(0), v\rangle = -S_u(J(0), v)$ for any $v \in N_p$,

where $u = \dot{c}(0)$. A deformation $\mathscr{V}\colon (-\varepsilon, \varepsilon) \times [0, \infty) \to M$ of c is said to be transversal to N at $t = 0$ if it satisfies

(i) $\mathscr{V}(0, t) = c(t)$ for $t \in [0, \infty)$,

(ii) the curve $t \mapsto \mathscr{V}(s, t)$ is a normal geodesic that is perpendicular to N at $t = 0$, for each $s \in (-\varepsilon, \varepsilon)$.

It is well-known that the Jacobi field associated to a transversal deformation is transversal. Conversely, any transversal Jacobi field is associated to at least one transversal deformation. Actually, for a transversal Jacobi field J, let $u\colon (-\varepsilon, \varepsilon) \to \perp_1 N$ be a map such that $u(0) = \dot{c}(0)$, and the tangent vector to the curve $s \mapsto \nu \circ u(s)$ at $s = 0$ is $J(0)$. Then the map $\mathscr{V}\colon (-\varepsilon, \varepsilon) \times [0, \infty) \to M$ defined by $\mathscr{V}(s, t) = \exp_N(tu(s))$ is a transversal deformation, and the Jacobi field associated to $\mathscr{V}$ coincides with J. See Hermann [3] or Bishop-Crittenden [1]. A focal point $c(\tau)$ of N along c is a point such that $\exp_N$ is singular at $\tau\dot{c}(0) \in \perp N$. $c(\tau)$ is a focal point of N along c if and only if there exists a Jacobi field J along

c that is transversal to N at $t=0$, $J(0)\neq 0$ and $J(\tau)=0$. The focal locus $F(N)$ of N is the set of focal points of N along all geodesics starting from N and perpendicular to N. For fixed $\tau>0$, a map $\mathscr{W}:(-\varepsilon, \varepsilon)\times[0,\tau]\to M$ will be called a proper deformation of $c|[0,\tau]$ between N and $c(\tau)$ if it satisfies

(i) $\mathscr{W}(0, t)=c(t)$ for $t\in[0,\tau]$,

(ii) $\mathscr{W}(s, 0)\in N$ for $s\in(-\varepsilon,\varepsilon)$,

(iii) $\mathscr{W}(s,\tau)=c(\tau)$ for $s\in(-\varepsilon,\varepsilon)$,

(iv) the tangent vector $X(t)$ to the curve $s\mapsto\mathscr{W}(s,t)$ at $s=0$ is perpendicular to c, for each $t\in[0,\tau]$.

A vector field $X:[0,\tau]\to TM$ along $c|[0,\tau]$ will be called a proper infinitesimal deformation of $c|[0,\tau]$ between N and $c(\tau)$ if it satisfies

(i) $X(\tau)=0$,

(ii) $X(t)$ is perpendicular to c for $t\in[0,\tau]$.

For any such X, there exists a proper deformation $\mathscr{W}$ of $c|[0,\tau]$ between N and $c(\tau)$ such that the associated vector field coincides with X. Let $L(s)$ denote the length of the curve $t\mapsto\mathscr{W}(s,t)$. Then $L:(-\varepsilon,\varepsilon)\to\boldsymbol{R}$ is smooth in a neighbourhood of 0, and

$$\frac{d^2L(0)}{ds^2}=\int_0^\tau(\langle X', X'\rangle-\langle R(X,\dot c)X,\dot c\rangle)dt+S_u(X(0),X(0))\ ,$$

where X' denotes the covariant derivative of X along c, and $u=\dot c(0)$. Let $I(X)$ denote the right hand side of the above formula.

BASIC LEMMA. *If N has no focal point along $c|[0,\tau]$. Then*

$$I(X)\geqq 0\ ,$$

for any proper infinitesimal deformation X of $c|[0,\tau]$ between N and $c(\tau)$, moreover equality occurs if and only if $X\equiv 0$.

For the proof, see Bishop-Crittenden [1].

Let $\rho: M\times M\to\boldsymbol{R}$ denote the distance function on M. The distance function $\rho_N: M\to\boldsymbol{R}$ from N is given by $\rho_N(p)=\inf\{\rho(p,q)\,|\,q\in N\}$. ρ_N is continuous on M, and smooth on $M-N-C(N)$. If $c([0,\tau])\cap C(N)=\emptyset$ for some $\tau>0$, then $c|(0,\tau]$ is an integral curve of the gradient vector field $\operatorname{grad}\rho_N$ of ρ_N. $\rho_N(c(t))=t$ for $t\in[0,\tau]$. Since $\operatorname{grad}\rho_N(c(\tau))\neq 0$, $N'=\rho_N^{-1}(\{\tau\})\cap U$ is a piece of hypersurface in M, where U is a small neighbourhood of $c(\tau)$ in M. c is perpendicular to N' at $t=\tau$. Moreover, for any $u'\in\perp_1 N$ which is sufficiently close to $\dot c(0)$, the geodesic $c':[0,\infty)\to M$ defined by $c'(t)=\exp_N(tu')$ is perpendicular to N' at $t=\tau$.

3. The isometric structure of M. From now on, we shall assume that M is of non-negative Ricci curvature, and N is a connected and compact hypersurface in M, which has no focal point, that is, $F(N)=\emptyset$.

LEMMA 1. *N is a minimal hypersurface.*

PROOF. For any $u\in\perp_1 N$, we shall prove that the mean curvature $m(u)$ of N with respect to u vanishes. Define $c\colon[0,\infty)\to M$ by $c(t)=\exp_N(tu)$. Let $e_1,\cdots,e_{n-1},\dot{c}$ be parallel orthonormal vector fields along c. Fix any $\tau>0$, and define proper infinitesimal deformations X_k, $k=1,\cdots,n-1$, of $c\,|\,[0,\tau]$ between N and $c(\tau)$ by $X_k(t)=((\tau-t)/\tau)e_k(t)$. Since N has no focal point along c, we have, by Basic Lemma in §2,

$$\begin{aligned}
0&\leqq\sum_{k=1}^{n-1}I(X_k)\\
&=\sum_{k=1}^{n-1}\int_0^\tau(\langle X_k',X_k'\rangle-\langle R(X_k,\dot{c})X_k,\dot{c}\rangle)dt+\sum_{k=1}^{n-1}S_u(X_k(0),X_k(0))\\
&=\frac{n-1}{\tau}-\int_0^\tau\left(\frac{\tau-t}{\tau}\right)^2\mathrm{Ric}\,(\dot{c}(t))dt+m(u)\\
&\leqq\frac{n-1}{\tau}+m(u)\,.
\end{aligned}$$

Letting $\tau\to\infty$, we have $m(u)\geqq 0$. Similarly we have $0\leqq m(-u)=-m(u)$, and the lemma follows.

Fix $p\in M-N-C(N)$, and choose a small neighbourhood U of p in $M-N-C(N)$. Then $N'=\rho_N^{-1}(\{\tau\})\cap U$ is a piece of hypersurface through p, where $\tau=\rho_N(p)$.

LEMMA 2. *N' is a piece of minimal hypersurface.*

PROOF. Let $c\colon(-\infty,\infty)\to M$ be a normal geodesic which is perpendicular to N at $t=0$, and $c\,|\,[0,\tau]$ is a minimal geodesic from N to $p=c(\tau)$. Then c is perpendicular to N' at $t=\tau$. It is sufficient to prove that the mean curvature of N' with respect to $\dot{c}(\tau)$ vanishes. Let $c_+=c\,|\,[0,\infty)$, and $c_-\colon[0,\infty)\to M$; $c_-(t)=c(-t)$. For each $v\in N_{c(0)}$, $v\neq 0$, let J_+ and J_- be the Jacobi fields along c_+ and c_- respectively that are transversal to N at $t=0$, and $J_+(0)=J_-(0)=v$. Since N has no focal point along c_+ and c_-, J_+ and J_- do not vanish everywhere. Define $J\colon(-\infty,\infty)\to TM$ by $J(t)=J_+(t)$ for $t\geqq 0$, and $J(t)=J_-(-t)$ for $t<0$. Then J is a smooth Jacobi field along c, which does not vanish everywhere. Recall that the Jacobi equation is of second order. Since N' is a "level surface" of ρ_N, the Jacobi fields $J_1\colon[0,\infty)\to TM$; $J_1(t)=J(t+\tau)$ and $J_2\colon[0,\infty)\to TM$; $J_2(t)=J(-t+\tau)$ are transversal to N' at $t=0$. It follows easily that N' has no focal point along c. Then, by Lemma 1, the mean curvature of N' with respect to $\dot{c}(\tau)$ vanishes.

LEMMA 3. *ρ_N is harmonic in $M-N-C(N)$.*

PROOF. Let p, U and N' be as above. Let $E_1, \cdots, E_{n-1}$, $E_n = \operatorname{grad}(\rho_N | U)$ be orthonormal vector fields in U. Then the restrictions $E_k | N'$, $k=1, \cdots, n-1$, are tangent to N', and $E_n | N'$ is perpendicular to N'. The integral curves of E_n are geodesics, $\nabla_{E_n} E_n = 0$. Hence we have,

$$\begin{aligned}\Delta\rho_N(p) &= -\sum_{k=1}^{n} \langle \nabla_{E_k} E_n, E_k \rangle |_p \\ &= -\sum_{k=1}^{n-1} \langle E_n, \nabla_{E_k} E_k \rangle |_p \\ &= m(E_n(p)) \\ &= 0\,,\end{aligned}$$

by Lemma 2, where $m(E_n(p))$ is the mean curvature of N' with respect to $E_n(p)$.

The following lemma is due to Cheeger-Gromoll [2].

LEMMA 4. *$\operatorname{grad} \rho_N$ is parallel in $M-N-C(N)$.*

PROOF. Let p and U be as above. Let $E_1, \cdots, E_{n-1}$, $E_n = \operatorname{grad}(\rho_N | U)$ be orthonormal vector fields in U which are parallel along the integral curves of E_n. Then in U,

$$\begin{aligned}\operatorname{Ric}(E_n) &= \sum_{k=1}^{n-1} \langle R(E_n, E_k)E_n, E_k \rangle \\ &= \sum_{k=1}^{n-1} \langle \nabla_{[E_n, E_k]} E_n - \nabla_{E_n} \nabla_{E_k} E_n + \nabla_{E_k} \nabla_{E_n} E_n, E_k \rangle \\ &= -\sum_{k=1}^{n-1} (\langle \nabla_{\nabla_{E_k} E_n} E_n, E_k \rangle + \langle \nabla_{E_n} \nabla_{E_k} E_n, E_k \rangle) \\ &= -\sum_{j,k=1}^{n-1} \langle \nabla_{E_j} E_n, E_k \rangle \langle \nabla_{E_k} E_n, E_j \rangle - \sum_{k=1}^{n-1} E_n \langle \nabla_{E_k} E_n, E_k \rangle \\ &= -\langle \nabla E_n, \nabla E_n \rangle + E_n(\Delta\rho_N) \\ &= -\langle \nabla E_n, \nabla E_n \rangle\,,\end{aligned}$$

by Lemma 3, where ∇E_n is the covariant differential of E_n. Since we have assumed that the Ricci curvature of M is non-negative, it follows that $\nabla E_n = 0$.

Let V' be a small neighbourhood of p in N', and $V' \times (-\varepsilon, \varepsilon)$ be the Riemannian product of V' and $(-\varepsilon, \varepsilon)$, for small $\varepsilon > 0$. Then, by Lemma 4, the map $\iota' \colon V' \times (-\varepsilon, \varepsilon) \to M-N-C(N)$; $\iota'(q, t) = \exp(tE_n(q))$ is an isometric imbedding. See Kobayashi-Nomizu [4]. For fixed $q \in V'$, $t \mapsto \iota'(q, t)$ is an integral curve of E_n. For fixed $t \in (-\varepsilon, \varepsilon)$, $\iota'(V' \times \{t\})$ coincides with $\rho_N^{-1}(\{\tau + t\}) \cap \iota'(V' \times (-\varepsilon, \varepsilon))$, where $\tau = \rho_N(p)$. Similarly, for $p \in N$, let

V be a small neighbourhood of p in N, and $V\times(-\varepsilon, \varepsilon)$ be the Riemannian product. Let X_V be a unit normal vector field on V. Then the map $\iota: V\times(-\varepsilon, \varepsilon)\to M-C(N)$; $\iota(q, t)=\exp_N(tX_V(q))$ is an isometric imbedding. For fixed $q\in V$, $t\mapsto\iota(q, t)$ and $t\mapsto\iota(q, -t)$ are integral curves of E_n, for $t>0$. For fixed $t\in(-\varepsilon, \varepsilon)$, $\iota(V\times\{t\})$ coincides with $\rho_N^{-1}(\{t\})\cap\iota(V\times(-\varepsilon, \varepsilon))$.

The following lemma is essentially due to Shiohama [6].

LEMMA 5. *If N has a cut point, then the cut locus $C(N)$ is a compact totally geodesic hypersurface without boundary.*

PROOF. Since N is compact, the distance $r=\rho(N, C(N))$ between N and $C(N)$ is greater than zero. Let $p_r\in C(N)$ be a point such that $\rho_N(p_r)=r$. First we shall prove that, for a small neighbourhood U of p_r in M, $C(N)\cap U$ is a piece of totally geodesic hypersurface, and $\rho_N|C(N)\cap U\equiv r$. Let $c: (-\infty, \infty)\to M$ be a normal geodesic such that $c|[0, r]$ is a minimal geodesic from N to $C(N)$, $c(r)=p_r$. Since N has no focal point, there are precisely two minimal geodesic from N to p_r. $c_1=c|[0, r]$ and $c_2: [0, r]\to M$; $c_2(t)=c(2r-t)$. See Ōmori [5] and also Shiohama [6]. Let V_j, $j=1, 2$, be small neighbourhoods of $c_j(0)$ in N. Let $X_j: V_j\to\perp_1 N$ be unit normal vector fields on V_j such that $X(c_j(0))=\dot{c}_j(0)$. Define $\Phi_j: V_j\times(-\infty, \infty)\to M$ by $\Phi_j(q, t)=\exp_N(tX_j(q))$. Then Φ_j are immersions, and $\Phi_j|V_j\times(-r, r)$ are isometric imbeddings. It follows that $\Phi_j(V_j\times\{r\})$ are totally geodesic hypersurfaces which are perpendicular to $\dot{c}(r)$. Hence $H=\Phi_1(V_1\times\{r\})\cap\Phi_2(V_2\times\{r\})$ is also a totally geodesic hypersurface through p_r. For any $p\in H$, there are two minimal geodesics, of length r, from p to N. Hence $H\subset C(N)$. By taking U suitably, we have $H=C(N)\cap U$. Next, let $p'\in\bar{H}$, where $\bar{H}$ denotes the closure of H in M. Then $p'\in C(N)$ and $\rho_N(p')=r$. Therefore, as above, $C(N)\cap U'$ is a piece of totally geodesic hypersurface, $\rho_N|C(N)\cap U'\equiv r$, where U' is a small neighbourhood of p' in M. Let C_0 denote the connected component of $C(N)$ which contains p_r. Then we have shown that C_0 is a compact totally geodesic hypersurface without boundary, here the compactness of C_0 follows from that of N. It is easy to see that $C(N)$ has at most two connected components C_0, in the direction of $\dot{c}(0)$, and $C(N)-C_0$, in the direction of $-\dot{c}(0)$. It is proved by the same way as above that if $C(N)-C_0$ is non-empty, then it is also a compact totally geodesic hypersurface without boundary.

REMARK. (i) If there does not exist a unit normal vector field $X: N\to\perp_1 N$ defined globally on N, then $C(N)$ is connected.

(ii) If $C(N)$ consists of two connected components, then M is compact.

COROLLARY. *$C(N)$ is locally isometric to N.*

3. Non-compact case, Proof of Theorem A. In this section, we shall consider the case where M is non-compact. If N has no cut point, then M is isometric to the flat normal bundle $\perp N$. The isometry is given by $\exp_N: \perp N \to M$. On the other hand, if N has a cut point, then $C(N)$ is a connected and compact totally geodesic hypersurface without boundary. There exists a unit normal vector field $X: N \to \perp_1 N$ defined globally on N such that there is no cut point in the direction of $-X$. Define $i_N: N \to N$ by $i_N(q) = \exp_N(2rX(q))$, where $r = \rho(N, C(N))$. Then i_N is an isometric involution on N. Since for each $p \in C(N)$, there are precisely two minimal geodesics from p to N, i_N has no fix point. Define $j: N \to C(N)$ by $j(q) = \exp_N(rX(q))$, then j is an isometric double covering. $j(q) = j(i_N(q))$ for $q \in N$. $C(N)$ is isometric to the quotient space $N/_{\{i_N\}} = N/\boldsymbol{Z}_2$. As a hypersurface, $C(N)$ has no cut point. Therefore M is isometric to the flat normal bundle $\perp C(N)$ on $C(N)$. $\perp C(N)$ is a non-trivial line bundle. Thus we obtain Theorem A.

4. Compact case, Proof of Theorem B. In this section, we shall consider the case where M is compact. For $r>0$, let $\perp_{[0,r)}N = \{u \in \perp N \mid \langle u, u\rangle < r^2\}$, $\perp_{[0,r]}N = \{u \in \perp N \mid \langle u, u\rangle \leqq r^2\}$ and $\perp_r N = \{u \in \perp N \mid \langle u, u\rangle = r^2\}$ be Riemannian submanifolds in the flat normal bundle $\perp N$. For a fixed-point free isometry $i: \perp_r N \to \perp_r N$, let $\perp_{[0,r]}N/i$ denote the Riemannian manifold obtained from $\perp_{[0,r]}N$ by identifying the boundary points $u \in \perp_r N$ with $i(u)$. Now, if $C(N)$ is connected, then $C(N) = \rho_N^{-1}(\{r\})$ and $M - C(N)$ is isometric to $\perp_{[0,r)}N$, where $r = \rho(N, C(N))$. Define $i: \perp_r N \to \perp_r N$ by $i(u) = v$, where v is such that $\exp_N(v) = \exp_N(u)$, $v \neq u$, which is determined uniquely. Then i is a fixed-point free isometric involution on $\perp_r N$. It is easy to see that M is isometric to $\perp_{[0,r]}N/i$. Next, if $C(N)$ consists of two connected components C_0 and C_1. Then, for the sake of simplicity, we may assume $r = \rho(N, C_0) = \rho(N, C_1)$. Then $C(N) = \rho_N^{-1}(\{r\})$, and $M - C(N)$ is isometric to $\perp_{[0,r)}N$. Let $i: \perp_r N \to \perp_r N$ be as above. Then i is a fixed-point free isometric involution on each of the connected components of $\perp_r N$. It is easy to see that M is isometric to $\perp_{[0,r]}N/i$. Thus we obtain Theorem B.

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