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ON THE SEMI-SIMPLE GAMMA RINGS

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1. Introduction. N. Nobusawa [1] introduced the notion of a Γ-ring, more general than a ring, and proved analogues of the Wedderburn-Artin theorems for simple Γ-rings and for semi-simple Γ-rings; Barnes [2] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings; Luh [3], [4] gave a generalization of the Jacobson structure theorems for primitive Γ-rings having minimum one-sided ideals, and obtained several other structure theorems for simple Γ-rings; Coppage-Luh [5] introduced the notion of Jacobson radical, Levitzki nil radical, nil radical and strongly nilpotent radical for Γ-rings and Barnes' [2] prime radical was studied further. Also, inclusion relations for these radicals were obtained, and it was shown that the radicals all coincide in the case of a Γ-ring which satisfies the descending chain condition on one-sided ideals. The author [6] gave a characterization of the prime radical of a Γ-ring M by introducing the notion of semi-primeness, and obtained close radical properties between a Γ-ring M and its right operator ring R.

In this paper, first we introduce the notion of a Γ-ring M-module and define Jacobson radical $J(M)$ along with the ideas of irreducible modules, while in [5] and [6] $J(M)$ was defined by the ideas of rqr elements. Properties of $J(M)$ and its relation with $J(R)$ are considered here, and it is also shown that our definition coincides with the one in [5] and [6]. After the semi-simplicity is defined by $J(M) = (0)$, the relation between semi-simple M and semi-simple R is considered. Defining the direct sum of Γ-rings S_i, $i \in \mathfrak{A}$, and the primitivity and getting the analogous results of corresponding part in ring theory, we have that a Γ-ring is semi-simple if and only if it is isomorphic to a subdirect sum of primitive Γ-rings.

For all notions relevant to ring theory we refer to [7].

2. Preliminaries. Let M and Γ be additive abelian groups. If for all $a, b, c \in M$ and $\gamma, \delta \in \Gamma$ the following conditions are satisfied, (1) $a\gamma b \in M$, (2) $(a + b)\gamma c = a\gamma c + b\gamma c$, $a(\gamma + \delta)b = a\gamma b + a\delta b$, $a\gamma(b + c) = a\gamma b + a\gamma c$ (3) $(a\gamma b)\delta c = a\gamma(b\delta c)$, then M is called a Γ-ring. If A and B are subsets of a Γ-ring M and $\Theta \subseteq \Gamma$, we denote $A\Theta B$, the subset of M consisting of all finite sums of the form $\sum_i a_i\gamma_i b_i$, where $a_i \in A$, $b_i \in B$, and $\gamma_i \in \Theta$. For singleton subsets we abbreviate this notation, for example, $\{a\}\Theta B = a\Theta B$. A right (left) ideal of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M \subseteqq I(M\Gamma I \subseteqq I)$. If I is both a right and a left ideal, then we say that I is an ideal, or a two-sided ideal of M. For each a of a Γ-ring M, the smallest right ideal containing a is called the principal right ideal generated by a and is denoted by $|a\rangle$. Similarly we define $\langle a|$ and $\langle a\rangle$, the principal left and two-sided (respectively) ideals generated by a. A subring of a Γ-ring M is an additive subgroup S of M such that $S\Gamma S \subseteqq S$.

Let M be a Γ-ring and F be the free abelian group generated by $\Gamma \times M$. Then

$$A = \left\{\sum_i n_i(\gamma_i, x_i) \in F \mid a \in M \Longrightarrow \sum_i n_i a\gamma_i x_i = 0\right\}$$

is a subgroup of F. Let $R = F/A$, the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. It can be verified easily that $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$ and $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication in R by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i\beta_j y_j] .$$

Then R forms a ring. If we define a composition on $M \times R$ into M by

$$a \sum_i [\alpha_i, x_i] = \sum_i a\alpha_i x_i \qquad \text{for} \quad a \in M, \sum [\alpha_i, x_i] \in R ,$$

then M is a right R-module, and we call R the right operator ring of Γ-ring M. In ordinary ring theory, if $Mr = (0)$ forces $r = 0$, then M is said to be a faithful R-module. For subsets $N \subseteqq M$, $\Phi \subseteqq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i [\gamma_i, x_i]$ in R, where $\gamma_i \in \Phi$, $x_i \in N$. Thus, in particular, $R = [\Gamma, M]$. For $P \subseteqq R$ we define $P^* = \{a \in M \mid [\Gamma, a] = [\Gamma, \{a\}] \subseteqq P\}$. It follows that if P is a right (left) ideal of R, then P^* is a right (left) ideal of M. Also for any collection $\mathscr{C}$ of sets in R, $\bigcap_{P\in\mathscr{C}} P^* = (\bigcap_{P\in\mathscr{C}} P)^*$. For $Q \subseteqq M$ we define

$$Q^{*\prime} = \{\sum_i [\alpha_i, x_i] \in R \mid M(\sum_i [\alpha_i, x_i]) \subseteqq Q\} .$$

Then it follows that if Q is a right (left) ideal of M, then $Q^{*\prime}$ is a right (left) ideal of R. Also for any collection $\mathscr{D}$ of sets in M, $\bigcap_{Q\in\mathscr{D}} Q^{*\prime} = (\bigcap_{Q\in\mathscr{D}} Q)^{*\prime}$.

If M_i is a Γ_i-ring for $i = 1, 2$ then an ordered pair (θ, ϕ) of mappings is called a homomorphism of M_1 onto M_2 if it satisfies the following properties:

(1) θ is a group homomorphism from M_1 onto M_2.
(2) ϕ is a group isomorphism from Γ_1 onto Γ_2.
(3) For every $x, y \in M_1$, every $\gamma \in \Gamma_1$,

$$(x\gamma y)\theta = (x\theta)(\gamma\phi)(y\theta) .$$

The kernel of the homomorphism (θ, ϕ) is defined to be $K = \{x \in M_1 \mid x\theta = 0\}$. Clearly K is an ideal of M_1. If θ is a group isomorphism from M_1 onto M_2, i.e., if $K = (0)$, then (θ, ϕ) is called an isomorphism from the Γ_1-ring M_1 onto the Γ_2-ring M_2.

Let (θ, ϕ) be a homomorphism from the Γ_1-ring M_1 onto the Γ_2-ring M_2 and B a right (resp. left, two-sided) ideal of M_2. Then $B\theta^{-1} = \{x \in M_1 \mid x\theta \in B\}$ is also a right (resp. left, two-sided) ideal of M_1. Similarly, if (θ, ϕ) is a homomorphism of the Γ_1-ring M_1 onto the Γ_2-ring M_2 and A is any right (resp. left, two-sided) ideal of M_1, then $A\theta = \{a\theta \mid a \in A\}$ is a right (resp. left, two-sided) ideal of M_2.

Let A be an ideal of a Γ-ring M. Then $M/A = \{x + A \mid x \in M\}$, the set of cosets of A, is again a Γ-ring with respect to the operations

$$(x + A) + (y + A) = (x + y) + A$$
$$(x + A)\gamma(y + A) = x\gamma y + A ,$$

as may be verified by a straightforward computation. We call M/A the residue class Γ-ring of M with respect to A. The mapping (τ, ι) from a Γ-ring M onto the Γ-ring M/A, where τ is defined by $x\tau = x + A$ and ι is the identity mapping of Γ, is a homomorphism called the natural homomorphism from M onto M/A.

The proof of the following fundamental theorem of homomorphism for Γ-rings is minor modifications of the proof of the corresponding theorem in ordinary ring theory, and will be omitted.

THEOREM 2.1. *If (θ, ϕ) is a homomorphism from the Γ_1-ring M_1 onto the Γ_2-ring M_2 with kernel K, then M_1/K and M_2 are isomorphic.*

3. The Jacobson radical. The additive group N is said to be a Γ-ring M-module if there is a Γ-mapping (Γ-composition) from $N \times \Gamma \times M$ to N (sending (n, γ, m) to $n\gamma m$) such that:
(1) $n\gamma(a + b) = n\gamma a + n\gamma b$
(2) $(n_1 + n_2)\gamma a = n_1\gamma a + n_2\gamma a$
(3) $(n\gamma a)\delta b = n\gamma(a\delta b)$,
for all $n, n_1, n_2 \in N$, all $a, b \in M$ and all $\gamma, \delta \in \Gamma$. For the sake of brevity we shall drop the "Γ-ring" in a Γ-ring M-module and refer it merely as an M-module.

EXAMPLES. Let M be any Γ-ring and N be a right ideal of M. We impose on N a natural M-module structure by defining the action of M on N to coincide with the product of elements of M. Let M be any Γ-ring and M/P be a residue class ring of M, where P is an ideal of M. If we define $(x+P)\gamma m = x\gamma m + P$ for all $x+P \in M/P$, all $m \in M$ and all $\gamma \in \Gamma$, then the additive group M/P forms an M-module.

We say that N is a Γ-faithful M-module if $N\Gamma x = (0)$ forces $x = 0$. For an M-module N, we define $A_M(N) = \{x \in M \mid N\Gamma x = (0)\}$.

LEMMA 3.1. *If N is an M-module then $A_M(N)$ is a two-sided ideal of M. Moreover, N is a Γ-faithful $M/A_M(N)$-module.*

PROOF. That $A_M(N)$ is a right ideal of M is immediate from the axioms for an M-module. To see that it is also a left ideal we proceed as follows: $N\Gamma(M\Gamma A_M(N)) = (N\Gamma M)\Gamma A_M(N) \subseteqq N\Gamma A_M(N) = (0)$, hence $M\Gamma A_M(N) \subseteqq A_M(N)$. Thus, $A_M(N)$ is a two-sided ideal of M. We now make of N an $M/A_M(N)$-module by defining, for $n \in N$, $\gamma \in \Gamma$ and $m + A_M(N) \in M/A_M(N)$, the action $n\gamma(m + A_M(N)) = n\gamma m$. If $m_1 + A_M(N) = m_2 + A_M(N)$ then $m_1 - m_2 \in A_M(N)$ hence $a\gamma(m_1 - m_2) = 0$ for all $a \in N$, all $\gamma \in \Gamma$, that is, $a\gamma m_1 = a\gamma m_2$. Thus, the action of $M/A_M(N)$ on N has been shown to be well defined. The verification that this defines the structure of an $M/A_M(N)$-module on N may be completed by a straightforward computation. Finally, to see that N is a Γ-faithful $M/A_M(N)$-module we note that if $n\gamma(m + A_M(N)) = (0)$ for all $n \in N$ and all $\gamma \in \Gamma$ then by definition $n\gamma m = 0$ hence $m \in A_M(N)$. This says that only the zero element of $M/A_M(N)$ annihilates all of N.

A submodule of an M-module N is an additive subgroup S of N such that $S\Gamma M \subseteqq S$.

N is said to be an irreducible M-module if $N\Gamma M \neq (0)$ and if the only submodules of N are (0) and N.

LEMMA 3.2. *N is an irreducible M-module if and only if N is an irreducible R-module.*

PROOF. Let N be an irreducible M-module. Then $N\Gamma M = N$. We make of N an R-module by defining, for $n \in N$, $\sum [\gamma_i, x_i] \in R$, the composition $n\sum [\gamma_i, x_i] = n\sum(\gamma_i, x_i)$, which is defined by $\sum n\gamma_i, x_i$. If $\sum(\gamma_i, x_i) + A = \sum(\delta_j, y_j) + A$ then $\sum(\gamma_i, x_i) - \sum(\delta_j, y_j) \in A$. Since $NA = (N\Gamma M)A = N\Gamma(0) = (0)$ we have $n(\sum(\gamma_i, x_i) - \sum(\delta_j, y_j)) = 0$, that is, $n\sum(\gamma_i, x_i) = n\sum(\delta_j, y_j)$. Thus the composition from $N \times R$ to N is well-defined. The verification that this defines the structure of an R-module on N may be completed by a straightforward computation. Let N' be an additive subgroup of N such that $N'R \subseteqq N'$. Since $N'R = N'[\Gamma, M] = N'\Gamma M$, we get $N'\Gamma M \subseteqq N'$. Therefore, N' is a submodule of an M-module N. Since N is irreducible, N' must be N or (0). Thus, N is an irreducible R-module. On the other hand, let N be an irreducible R-module. If we define the action $n\gamma x = n[\gamma, x]$, a similar argument as in the proof above will show that N is an irreducible M-module. Thus, the proof is completed.

Let R be the right operator ring of a Γ-ring M. A right ideal ρ of R is said to be regular if there is an $a \in R$ such that $x - ax \in \rho$ for all $x \in R$.

LEMMA 3.3. *Let R be the right operator ring of a Γ-ring M. If N is an irreducible M-module then N is isomorphic as an R-module to R/ρ for some maximal regular right ideal ρ of R. Conversely, for every maximal regular right ideal ρ of R, R/ρ is an irreducible R-module.*

PROOF. Since N is irreducible, by the above definition we must have that $N\Gamma M \neq (0)$. Since $S = \{n \in N \mid n\Gamma M = (0)\}$ is a submodule of N and is not N, it must be (0). Equivalently, if $n \neq 0$ is in N then $n\Gamma M \neq (0)$. However, $n\Gamma M$ is a submodule of N hence $n\Gamma M = N$. By Lemma 3.2 N is an R-module and so we can define $\psi\colon R \to N$ by $\psi(r) = nr$ for every $r \in R$. We see at once that ψ is a homomorphism of R into N as R-modules; since $nR = n\Gamma M = N$ we have that ψ is surjective. Finally, $\operatorname{Ker}\psi = \{r \in R \mid nr = 0\}$ is a right ideal ρ: by standard homomorphism theorem we have that N is isomorphic to R/ρ as an R-module. Any right ideal of R which properly contains ρ maps, under ψ, into a submodule of N. Hence ρ is a maximal right ideal in R. Since $nR = N$ there is an element $a \in R$ such that $na = n$. Therefore for any $x \in R$ $nax = nx$, which is to say $n(x - ax) = 0$. This puts $x - ax$ in ρ. As the converse will be shown easily, we omit the proof.

The Jacobson radical of a Γ-ring M, written as $J(M)$, is the set of all elements of M which annihilate all the irreducible M-modules. If M has no irreducible modules we put $J(M) = M$. We note that $J(M) = \bigcap A_M(N)$, where this intersection runs over all irreducible M-modules N. Since the $A_M(N)$ are two-sided ideals of M by Lemma 3.1, we see that $J(M)$ is a two-sided ideal of M.

In ordinary ring theory, for the right operator ring R of a Γ-ring M we have also that $J(R) = \bigcap A_R(N)$ where this intersection runs over all irreducible R-modules N, and where $A_R(N) = \{r \in R \mid Nr = (0)\}$. We have that $A_M(N)^{*'} = A_R(N)$, for $A_M(N)^{*'} = \{r \in R \mid Mr \subseteqq A_M(N)\} = \{r \in R \mid N\Gamma Mr = (0)\} = \{r \in R \mid Nr = (0)\} = A_R(N)$. Also $A_R(N)^* = A_M(N)$, for $A_R(N)^* =$

$\{x \in M \mid [\Gamma, x] \subseteqq A_R(N)\} = \{x \in M \mid N\Gamma x = (0)\} = A_M(N)$. By these facts and Lemma 3.2 we have $J(M)^{*'} = (\bigcap A_M(N))^{*'} = \bigcap A_M(N)^{*'} = \bigcap A_R(N) = J(R)$. Similarly, $J(R)^* = (\bigcap A_R(N))^* = \bigcap A_R(N)^* = \bigcap A_M(N) = J(M)$. Hence we have

THEOREM 3.1. *If M is a Γ-ring and R is the right operator ring of M then $J(M) = J(R)^*$ and $J(R) = J(M)^{*'}$.*

A Γ-ring M is said to be semi-simple if $J(M) = (0)$.

THEOREM 3.2. *If a Γ-ring M is semi-simple then the right operator ring R of M is also semi-simple.*

PROOF. Let $(0)_M$ be the zero ideal in M, and $(0)_R$ be the zero ideal in R. $J(M)^{*'} = (0)_M^{*'} = \{r \in R \mid Mr = (0)_M\} = (0)_R$, for M is a faithful R-module. Hence by Theorem 3.1 we have $J(R) = (0)_R$.

THEOREM 3.3. *Let a Γ-ring M be a Γ-faithful M-module, that is, $M\Gamma x = (0)$ implies $x = 0$. If R is semi-simple then M is semi-simple.*

PROOF. Since M is Γ-faithful, $J(R)^* = (0)_R^* = \{a \in M \mid [\Gamma, a] = (0)_R\} = \{a \in M \mid M\Gamma a = (0)_M\} = (0)_M$. Thus, by Theorem 3.1, we get $J(M) = (0)_M$.

For the right operator ring R of a Γ-ring M, we define $(\rho\colon R) = \{x \in R \mid Rx \subseteqq \rho\}$, where ρ is a right ideal of R.

LEMMA 3.4. *$A_R(N) = (\rho\colon R)$ is the largest two-sided ideal of R which lies in ρ, where ρ is a maximal regular right ideal of R and N denotes R/ρ.*

PROOF. If $x \in A_R(N)$ then $Nx = (0)$, which is to say, $(r + \rho)x = \rho$ for all $r \in R$. This says that $Rx \subseteqq \rho$, hence $A_R(N) \subseteqq (\rho\colon R)$. Similarly $(\rho\colon R) \subseteqq A_R(N)$ whence $A_R(N) = (\rho\colon R)$. Since ρ is regular there is an $a \in R$ with $x - ax \in \rho$ for all $x \in R$; in particular, if $x \in (\rho\colon R)$ then since $ax \in Rx \subseteqq \rho$ we get $x \in \rho$. Thus the proof is completed.

By Lemma 3.3 and Lemma 3.4 we get $A_M(N) = (\rho\colon R)^*$, and so by the definition of $J(M)$, we have

THEOREM 3.4. *$J(M) = \bigcap (\rho\colon R)^*$, where ρ runs over all the maximal regular right ideals of R, and where $(\rho\colon R)$ is the largest two-sided ideal of R lying in ρ.*

An element a of a Γ-ring M is said to be right-quasi-regular (abbreviated rqr) if for any $\gamma \in \Gamma$ the element $[\gamma, a]$ of the right operator ring R of M is right-quasi-regular in the usual sense. That is to say, a is rqr, if for any $\gamma \in \Gamma$, there exists $\sum_{i=1}^{n} [\gamma_i, x_i]$ in R such that

$$[\gamma, a] + \sum_{i=1}^{n} [\gamma_i, x_i] - [\gamma, a] \sum_{i=1}^{n} [\gamma_i, x_i] = 0 \ ,$$

i.e.,

$$x\gamma a + \sum_{i=1}^{n} x\gamma_i x_i - \sum_{i=1}^{n} (x\gamma a)\gamma_i x_i = 0 \qquad \text{for all} \quad x \in M \ .$$

A subset S of M is rqr if every element in S is rqr.

THEOREM 3.5. *$J(M)$ is a right-quasi-regular ideal and contains all right-quasi-regular right ideals of M.*

PROOF. Let R be the right operator ring of M. The ordinary ring theory shows that $J(R)$ is a rqr ideal of R and contains all the rqr right ideals of R (c.f., [7] p. 12). As has already been shown in this paper, we have

$$J(M) = J(R)^* = \{a \in M \mid [\Gamma, a] \subseteq J(R)\} \ .$$

If $a \in J(M)$, then for any $\gamma \in \Gamma$ $[\gamma, a] \in J(R)$, so $[\gamma, a]$ is rqr, that is, a is rqr. Let N be a rqr right ideal of M. There remains to show $[\gamma, N] \subseteq J(R)$, where γ is any element of Γ. If $n \in N$, then n is rqr, so that $[\gamma, n]$ is rqr. Since N is a right ideal of M, we have $[\gamma, N][\Gamma, M] = [\gamma, N\Gamma M] \subseteq [\gamma, N]$ and hence $[\gamma, N]$ is a right ideal of R. Therefore, $[\gamma, N]$ is a rqr right ideal of R. Thus $[\gamma, N] \subseteq J(R)$ and the proof of the theorem is completed.

As an immediate consequence of Theorem 3.5 the Jacobson radical $J(M)$ of M can be characterized as follows:

$$J(M) = \{a \in M \mid \langle a \rangle \quad \text{is} \quad \text{rqr}\} \ .$$

This is the definition of $J(M)$ given in [5] and [6].

4. **Semi-simple Γ-rings.** Let S_i, $i \in \mathfrak{A}$, be a family of Γ-rings indexed by the set $\mathfrak{A}$. By the direct sum (complete direct sum) of the Γ-rings S_i, $i \in \mathfrak{A}$, we mean the set $S = \prod_{i \in \mathfrak{A}} S_i = \{a\colon \mathfrak{A} \to \bigcup_{i \in \mathfrak{A}} S_i \mid a(i) \in S_i, \text{ all } i \in \mathfrak{A}\}$. We give a Γ-ring structure to S by defining

$$(a + b)(i) = a(i) + b(i) \qquad (4.1)$$
$$(a\gamma b)(i) = a(i)\gamma b(i)$$

for all $a, b \in S, \gamma \in \Gamma$ and $i \in \mathfrak{A}$.

If S is the direct sum of Γ-rings S_i, $i \in \mathfrak{A}$, with each element i of $\mathfrak{A}$ we may associate a mapping (θ_i, ι) of S onto S_i as follows:

$$a\theta_i = a(i) \ , \quad a \in S \qquad (4.2)$$
$$\gamma\iota = \gamma \ , \qquad \gamma \in \Gamma \ .$$

Clearly, $S\theta_i = S_i$. Moreover, it follows immediately from (4.1) that (θ_i, ι) is a homomorphism of S onto S_i. If, now, T is a subring of S, $T\theta_i$ is a subring of S_i for each $i \in \mathfrak{A}$.

Let T be a subring of the direct sum of Γ-rings S_i, $i \in \mathfrak{A}$, and for each $i \in \mathfrak{A}$ let (θ_i, ι) be the homomorphism of S onto S_i defined by (4.2). If $T\theta_i = S_i$ for every $i \in \mathfrak{A}$, T is said to be a subdirect sum of the Γ-rings S_i, $i \in \mathfrak{A}$.

LEMMA 4.1. *A Γ-ring M is isomorphic to a subdirect sum of Γ-rings S_i, $i \in \mathfrak{A}$, if and only if for each $i \in \mathfrak{A}$ there exists a homomorphism (ϕ_i, ι) of M onto S_i such that if a is an arbitrary nonzero element of M, then $a\phi_i \neq 0$ for at least one $i \in \mathfrak{A}$.*

The proof may be established by very easy modifications of the proof of Theorem 3.6 in [8], so this will be omitted.

In view of Theorem 2.1, if (ϕ_i, ι) is a homomorphism of M onto S_i, then $S_i \cong M/K_i$, where K_i is the kernel of (ϕ_i, ι). Therefore, we may formulate Lemma 4.1 as follows:

LEMMA 4.2 *A Γ-ring M is isomorphic to a subdirect sum of Γ-rings S_i, $i \in \mathfrak{A}$, if and only if for each $i \in \mathfrak{A}$ there exists in M a two-sided ideal K_i such that $M/K_i \cong S_i$, moreover $\bigcap_{i \in \mathfrak{A}} K_i = (0)$.*

A Γ-ring M is said to be primitive if it has a Γ-faithful irreducible module.

THEOREM 4.1. *A Γ-ring M is primitive if and only if the right operator ring R is primitive and $M\Gamma x = (0)$ forces $x = 0$.*

PROOF. Let M be primitive and N be a Γ-faithful irreducible M-module. By Lemma 3.2 N is an irreducible R-module. If $Nr = (0)$, then since $N\Gamma M = N$ we have $N\Gamma Mr = (0)$, and so $Mr = (0)$, thus $r = 0$. Therefore, N is faithful. If $M\Gamma x = (0)$, we get that $(N\Gamma M)\Gamma x = (0)$, and $N\Gamma x = (0)$, thus $x = 0$.

Conversely, let N be a faithful irreducible R-module. By Lemma 3.2 N is an irreducible M-module. To show that N is Γ-faithful we assume that $N\Gamma x = (0)$. Then we have that $N[\Gamma, x] = (0)$, and $[\Gamma, x] = (0)$. Hence $M\Gamma x = (0)$, so $x = 0$. Thus, the proof is completed.

THEOREM 4.2. *A Γ-ring M is primitive if and only if there exists a maximal regular right ideal ρ in R such that $(\rho : R)^* = (0)$, where R denotes the right operator ring of M. A primitive Γ-ring is semi-simple.*

PROOF. Let M be primitive, then there exists a Γ-faithful irreducible M-module N. By Lemma 3.3 there exists a maximal regular right ideal

ρ in R such that N is isomorphic to R/ρ as an R-module. Lemma 3.4 shows that $(\rho: R)^* = A_M(N)$. Since N is Γ-faithful we get $A_M(N) = (0)$. Thus, $(\rho: R)^* = (0)$. Let ρ be a maximal regular right ideal of R. Put $N = R/\rho$. Since $A_M(N) = (\rho: R)^* = (0)$ N is Γ-faithful, thus M is primitive.

Finally, $J(M) = \bigcap (\rho: R)^*$, where ρ runs over all maximal regular right ideals of R, hence if $(\rho: R)^* = (0)$ for one such ρ we have $J(M) = (0)$, and the proof is completed.

THEOREM 4.3. *A Γ-ring M is semi-simple if and only if it is isomorphic to a subdirect sum of primitive Γ-rings.*

PROOF. Let M be a semi-simple Γ-ring. As is shown in Theorem 3.4, $J(M) = \bigcap (\rho: R)^*$, where ρ runs over the maximal regular right ideals of R. Since M is semi-simple $\bigcap (\rho: R)^* = (0)$. By Lemma 4.2 M is isomorphic to a subdirect sum of the $M/(\rho: R)^*$. By Lemma 3.1 and Lemma 3.4 $M/(\rho: R)^*$ is primitive. Therefore M is isomorphic to a subdirect sum of primitive Γ-rings. On the other hand, suppose that M is isomorphic to a subdirect sum of the rings $M_\phi = M/K_\phi$. Therefore $\bigcap K_\phi = (0)$. If the rings M_ϕ are all primitive, then they are semi-simple. Since $J(M)$ maps into a quasi-regular right ideal of M_ϕ it must map into (0). Thus $J(M) \subseteqq K_\phi$ for each ϕ, hence $J(M) \subseteqq \bigcap K_\phi = (0)$ proving that M is semi-simple. Thus, the proof is completed.

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