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A REMARK ON THE ZEROS OF SOME L-FUNCTIONS

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1. Introduction. In [2] and [4] we have given a comparative study of the zeros of L-functions of absolute abelian extensions. Here extending this we shall give a remark to Uchida's work [9] on Artin's L-functions under certain assumption. Our assumption which will be described later is much weaker than the Riemann hypothesis and was already proved for the Riemann zeta function and Dirichlet L-functions.

Let F be an algebraic number field of finite degree. Let $L(s, \psi, K_\psi/F)$ be a primitive abelian Hecke's L function, where ψ is an one dimensional character of cyclic extension K_ψ/F and the conductor of ψ is that of K_ψ/F. Let $N(t, \psi, F)$ be the number of the zeros of $L(s, \psi, K_\psi/F)$ in $0 < \operatorname{Re} s < 1$ and $0 \leqq \operatorname{Im} s \leqq t$, possible zeros on the boundary being counted with weight one-half. Then the moments

$$
\begin{aligned}
&M_k(T, h; \psi, \psi', F) \\
&\quad = \int_T^{2T} \{N(t+h, \psi, F) - N(t, \psi, F) - (N(t+h, \psi', F) - N(t, \psi', F))\}^k dt
\end{aligned}
$$

for $k \geqq 1$ may be considered as the measure of the independence of the distribution of the zeros of $L(s, \psi, K_\psi/F)$ and $L(s, \psi', K_{\psi'}/F)$, where $h > 0$, $\psi' \neq \psi$ is a primitive character having the conductor of $K_{\psi'}/F$ as the conductor of ψ'. We shall see later that if

$$
\begin{aligned}
\pi_F(x; \psi, \psi') &\equiv \sum_{p \leqq x} \Big| \sum_{N_F\mathfrak{p} = p} (\psi(\mathfrak{p}) - \psi'(\mathfrak{p})) \Big|^2 \\
&\sim C_F(\psi, \psi')\pi(x) ,
\end{aligned}
$$

then for each $k \geqq 1$, when $h \log T \to \infty$ as $T \to \infty$, we have

$$
\begin{aligned}
&M_{2k}(T, h; \psi, \psi', F) \\
&\quad \sim \frac{2k!}{(2\pi)^{2k} k!} T(2C_F(\psi, \psi') \log (h \log T))^k ,
\end{aligned}
$$

where $N_F\mathfrak{p}$ is the absolute norm of a prime ideal $\mathfrak{p}$ of F and $\pi(x)$ is the number of rational primes p less than x as usual. So the degree of the independence of the distribution of the zeros of $L(s, \psi, K_\psi/F)$ and $L(s, \psi', K_{\psi'}/F)$ may be considered as a function of $C_F(\psi, \psi')$. Moreover if $F = \boldsymbol{Q}$, then

$$\begin{aligned}\pi_F(x; \psi, \psi') &= \sum_{p \leqq x} |\psi(p) - \psi'(p)|^2 \\ &\sim 2\pi(x) - \sum_{p \leqq x} \psi\bar{\psi}'(p) - \sum_{p \leqq x} \bar{\psi}\psi'(p) \\ &\sim 2\pi(x) .\end{aligned}$$

So $C_{\boldsymbol{Q}}(\psi, \psi') = 2$ reflects the fact that rational primes are uniformly distributed in the arithmetic progressions.

Our problem is related with Artin's conjecture on Artin's L-functions. Here we shall quote an example from Uchida's [8]. Let G be a group of order 24 generated by σ, τ and ρ whose relations are as follows: $\sigma^4 = \rho^3 = 1$, $\sigma^2 = \tau^2$, $\sigma\tau\sigma^{-1} = \tau^{-1}$, $\rho\sigma\rho^{-1} = \tau$, $\rho\tau\rho^{-1} = \tau\sigma$. G has seven conjugate classes which are represented by 1, σ^2, σ, ρ, ρ^2, $\rho\sigma^2$ and $\rho^2\sigma^2$. G has three different 2-dimensional characters χ_1, χ_2 and χ_3 satisfying $\chi_i(1) = 2$, $\chi_i(\sigma^2) = -2$ and $\chi_i(\sigma) = 0$ for every i, and $\chi_1(\rho) = \chi_1(\rho^2) = -1$, $\chi_1(\rho\sigma^2) = \chi_1(\rho^2\sigma^2) = 1$, $\chi_2(\rho) = -\omega$, $\chi_2(\rho^2) = -\omega^2$, $\chi_2(\rho\sigma^2) = \omega$, $\chi_2(\rho^2\sigma^2) = \omega^2$ and $\chi_3 = \bar{\chi}_2$, where ω is a primitive cube root of unity. Let K be a Galois extension of $\boldsymbol{Q}$ with the Galois group G. Thus we have three multiplicatively independent Artin's L-functions $L(s,\chi_i,K/\boldsymbol{Q})$ for $i = 1, 2, 3$ as usual. For simplicity we write $L(s, \chi_i) = L(s, \chi_i, K/\boldsymbol{Q})$. Now let H be the subgroup of G generated by $\rho\sigma^2$. Let φ and ψ be one-dimensional characters of H such that $\varphi(\rho\sigma^2) = -\omega$ and $\psi(\rho\sigma^2) = -1$. Then as is noticed in Theorem 2 of [9],

$$L(s, \chi_1)^2 = L(s, \varphi)L(s, \bar{\varphi})/L(s, \psi) , \qquad L(s, \chi_2)^2 = L(s, \bar{\varphi})L(s, \psi)/L(s, \varphi)$$

and $L(s, \chi_3)^2 = L(s, \varphi)L(s, \psi)/L(s, \bar{\varphi})$, where

$$L(s, \varphi) = L(s, \varphi^G, K/\boldsymbol{Q}) = L(s, \varphi, K/F) , \qquad L(s, \bar{\varphi}) = L(s, \bar{\varphi}, K/F)$$

and $L(s, \psi) = L(s, \psi, K_\psi/F)$ with the induced character φ^G on G, the intermediate field F corresponding to H and the quadratic extension K_ψ of F. Our particularly interesting problem is to find asymptotic laws for $N_{\psi\varphi}(T)$, $N_{\varphi\psi}(T)$ and $N_{\varphi\bar{\varphi}}(T)$ and where

$$N_{\psi\varphi}(T) = \sum_{\substack{m_\psi(\rho) > m_\varphi(\rho) \\ 0 \leqq \gamma \leqq T}} m_\psi(\rho)$$

with $m_\psi(\rho) =$ "the multiplicity of ρ as a zero of $L(s, \psi)$." In this paper we shall give lower bounds for $N_{\psi\varphi}(T)$, $N_{\varphi\psi}(T)$ and $N_{\varphi\bar{\varphi}}(T)$. In the following of this paper we shall fix our situation as above and we shall ignore the dependence on the fields and characters if it is not necessary, although we can discuss generally.

To state our results more precisely, let $N(\alpha, T; \varphi)$ be the number of the zeros of $L(s, \varphi)$ in $\alpha < \mathrm{Re}\, s < 1$ and $0 \leqq \mathrm{Im}\, s \leqq T$. We denote the following estimate by $(\varphi, K/F)$; $N(\alpha, T; \varphi) \ll T^{1+b(1-2\alpha)} \log T$ for some positive constant b and uniformly for $\alpha \geqq 1/2$. For simplicity we put $N_\varphi(t) = N(t, \varphi, F)$. We may remark here that $(\varphi_0, \boldsymbol{Q}/\boldsymbol{Q})$ with the principal character φ_0 (namely, for the Riemann zeta-function) was demonstrated by A. Selberg [5]. Now in this paper we shall prove

THEOREM. *Suppose that* $(\varphi, K/F)$, $(\bar{\varphi}, K/F)$ *and* $(\psi, K_\psi/F)$ *hold. Then*

$$N_{\psi\varphi}(T) \geqq C_1 N_\psi(T) , \qquad N_{\varphi\bar{\varphi}}(T) \geqq C_2 N_\varphi(T)$$

and $N_{\varphi\psi}(T) \geqq C_3 N_\varphi(T)$ *with some positive absolute constants* C_1, C_2 *and* C_3 .

2. Proof of Theorem.

2-1. We shall prove only $N_{\psi\varphi}(T) \geqq C_1 N_\psi(T)$, since other cases come similarly. From the functional equation of $L(s, \varphi)$, one gets for $t > 0$,

$$N_\varphi(t) = b_1 t \log t - b_2 t + S(t, \varphi) + b_3(\varphi) + O(1/(1 + t)) ,$$

where b_1 depends only on the degree of F, b_2 may depend on F and the norm of the conductor of φ, $b_3(\varphi)$ may depend on φ and F, and

$$S(t, \varphi) = (1/\pi) \arg L(1/2 + it, \varphi) \qquad \text{as usual.}$$

Similarly, one gets

$$N_\psi(t) = b_1 t \log t - b_2' t + S(t, \psi) + b_3(\psi) + O(1/(1 + t)) .$$

We put $g_h(t) = N_\psi(t + h) - N_\psi(t) - (N_\varphi(t + h) - N_\varphi(t))$ and

$$E_M' = \{t \in (T, 2T);\, g_h(t) > M\}$$

for positive M. Suppose that $|E_M'| \geqq AT$. Then, of the intervals $(T, T + h), (T + h, T + 2h), \cdots$, at least $[AT/h]$ must contain a point of E_M'. If $(T + hn, T + h(n + 1))$ contains t of E_M', then there must exist a $\rho = \beta + i\gamma$ such that $m_\psi(\rho) > m_\varphi(\rho)$ and $t \leqq \gamma \leqq t + h$. Hench there exists a ρ satisfying $m_\psi(\rho) > m_\varphi(\rho)$ in $T + hn < \gamma < T + h(n + 2)$. So, $N_{\psi\varphi}(T) \gg N_\psi(T)$ if we can take $h = C/\log T$ with some positive constant C. In the following we shall prove that $|E_M'| \gg T$ for $h = C/\log T$. By asymptotic formulas for $N_\psi(t)$ and $N_\varphi(t)$ this is reduced to get $|E_M| \gg T$ for $h = C/\log T$, where

$$E_M = \{t \in (T, 2T);\, f_h(t) > M\}$$

and

$$f_h(t) = S(t + h, \psi) - S(t, \psi) - (S(t + h, \varphi) - S(t, \varphi)) .$$

We shall carry out this step by step. From 2-3 to 2-6 below we shall assume Riemann hypothesis to $L(s, \psi)$ and $L(s, \varphi)$ instead of $(\psi, K_\psi/F)$ and $(\varphi, K/F)$. It is clear from Selberg's argument in [5] how to modify our argument. Our argument below follows pp. 308-314 of [7].

2-2. We put $\operatorname{Re} s = \mu$ and $\alpha = \operatorname{Max}(2, 1+\mu)$. We start from the following integral as in pp. 308-314 of [7]

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{z-s} - x^{2(z-s)}}{(z-s)^2} \frac{L'}{L}(z, \psi)dz .$$

We get on the one hand

$$\log x \sum_{N\mathfrak{A} \leqq x^2} \frac{\Lambda_x(\mathfrak{A})\psi(\mathfrak{A})}{(N\mathfrak{A})^s} ,$$

where $N\mathfrak{A}$ is the absolute norm of integral ideal $\mathfrak{A}$ of F and

$$\Lambda_x(\mathfrak{A}) = \begin{cases} \Lambda(\mathfrak{A}) & \text{for} \quad N\mathfrak{A} \leqq x \\ \dfrac{\Lambda(\mathfrak{A}) \log (x^2/(N\mathfrak{A}))}{\log x} & \text{for} \quad x \leqq N\mathfrak{A} \leqq x^2 \end{cases}$$

with

$$\Lambda(\mathfrak{A}) = \begin{cases} \log N\mathfrak{p} & \text{if} \quad \mathfrak{A} = \mathfrak{p}^m \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by the theorem of residues, we get

$$-\log x \frac{L'}{L}(s, \psi) + (\nu + r_2) \sum_{k=0}^{\infty} \frac{x^{-2k-1-s} - x^{-2(2k+1+s)}}{(2k+1+s)^2}$$
$$+ (r_1 + r_2 - \nu) \sum_{k=1}^{\infty} \frac{x^{-2k-s} - x^{-2(2k+s)}}{(2k+s)^2} + \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2} ,$$

where ρ runs over non-trivial zeros of $L(s, \psi)$, ν is the number of real places ramified at K_ψ/F, r_1 and r_2 are usual notations. Hence we get under the same notations

LEMMA 1.

$$\frac{L'}{L}(s, \psi) = -\sum_{N\mathfrak{A} \leqq x^2} \frac{\Lambda_x(\mathfrak{A})\psi(\mathfrak{A})}{(N\mathfrak{A})^s} + \frac{(\nu + r_2)}{\log x} \sum_{k=0} \frac{x^{-2k-1-s} - x^{-2(2k+1+s)}}{(2k+1+s)^2}$$
$$+ \frac{r_1 + r_2 - \nu}{\log x} \sum_{k=1}^{\infty} \frac{x^{-2k-s} - x^{-2(2k+s)}}{(2k+s)^2} + \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2} .$$

2-3. By the logarithmic derivative of Weierstrass' product formula for $L(s, \psi)$, one gets for $t > t_0$

$$\frac{L'}{L}(s, \psi) = \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + O(\log t)\,.$$

We call this Lemma 1′. Using Lemmas 1 and 1′, one gets as in the proof of Theorem 14.21 of [7],

$$\begin{aligned}\frac{L'}{L}(\mu + it, \psi) = &- \sum_{N\mathfrak{A} \leqq x^2} \frac{\Lambda_x(\mathfrak{A})\psi(\mathfrak{A})}{(N\mathfrak{A})^{\mu+it}} \\ &+ O\left(x^{-\mu+1/2}\left|\sum_{N\mathfrak{A} \leqq x^2} \frac{\Lambda_x(\mathfrak{A})\psi(\mathfrak{A})}{(N\mathfrak{A})^{a+it}}\right|\right) + O(x^{-\mu+1/2}\log t)\end{aligned}$$

for $\mu \geqq a$, where $a = 1/2 + (\log x)^{-1}$, $t > 2$ and $4 \leqq x \leqq t^2$. Now

$$\begin{aligned}\arg L(1/2 + it, \psi) &= -\int_{1/2}^{\infty} \operatorname{Im} \frac{L'}{L}(\mu + it, \psi)d\mu \\ &= -\int_{a}^{\infty} \operatorname{Im} \frac{L'}{L}(\mu + it, \psi)d\mu - (a - 1/2)\operatorname{Im}\frac{L'}{L}(a + it, \psi) \\ &\quad + \int_{1/2}^{a} \operatorname{Im}\left(\frac{L'}{L}(a + it, \psi) - \frac{L'}{L}(\mu + it, \psi)\right)d\mu \\ &= J_1 + J_2 + J_3\,,\end{aligned}$$

say. J_1 and J_2 can be treated using the above formula for $\mu \geqq a$. J_3 can be treated using Lemma 1′, and we get

LEMMA 2. *For $t > 2$, $4 \leqq x \leqq t^2$, $a = 1/2 + (1/\log x)$, we have*

$$\begin{aligned}S(t, \psi) = &\frac{1}{\pi}\operatorname{Im}\sum_{N\mathfrak{A} \leqq x^2} \frac{\Lambda_x(\mathfrak{A})\psi(\mathfrak{A})}{(N\mathfrak{A})^{a+it}\log(N\mathfrak{A})} \\ &+ O\left(\frac{1}{\log x}\left|\sum_{N\mathfrak{A} \leqq x^2} \frac{\Lambda_x(\mathfrak{A})\psi(\mathfrak{A})}{(N\mathfrak{A})^{a+it}}\right|\right) + O(\log t/\log x)\,.\end{aligned}$$

From this we get further

$$\begin{aligned}S(t, \psi) &- \frac{1}{\pi}\operatorname{Im}\sum_{N\mathfrak{p} \leqq x^2} \frac{\psi(\mathfrak{p})}{(N\mathfrak{p})^{1/2+it}} \\ &= \frac{1}{\pi}\operatorname{Im}\sum_{N\mathfrak{p} \leqq x^2} \frac{(\Lambda_x(\mathfrak{p})(N\mathfrak{p})^{-a+1/2} - \Lambda(\mathfrak{p}))\psi(\mathfrak{p})}{(N\mathfrak{p})^{1/2+it}\log(N\mathfrak{p})} \\ &\quad + O\left(\frac{1}{\log x}\left|\sum_{N\mathfrak{p} \leqq x^2} \frac{\Lambda_x(\mathfrak{p})\psi(\mathfrak{p})}{(N\mathfrak{p})^{a+it}}\right|\right) \\ &\quad + O\left(\frac{1}{\log x}\left|\sum_{N\mathfrak{p}^2 \leqq x^2} \frac{\Lambda_x(\mathfrak{p}^2)\psi(\mathfrak{p}^2)}{(N\mathfrak{p})^{2(a+it)}}\right|\right) \\ &\quad + O\left(\left|\sum_{N\mathfrak{p}^2 \leqq x^2} \frac{\Lambda_x(\mathfrak{p}^2)\psi(\mathfrak{p}^2)}{(N\mathfrak{p})^{2(a+it)}\log(N\mathfrak{p})}\right|\right) \\ &\quad + O\Big(\sum_{r>2}\sum_{\mathfrak{p}}(N\mathfrak{p})^{-r/2}\Big) + O(\log t/\log x)\,.\end{aligned}$$

The last two terms are bounded for $T \leqq t \leqq 2T$, $x \geqq T^c$ with a fixed positive c. The other terms of the right hand side are one of the following types:

$$\sum_{N\mathfrak{p}\leqq y} \frac{a(\mathfrak{p})}{(N\mathfrak{p})^{1/2+it}} \quad \text{with} \quad |a(\mathfrak{p})| \ll \frac{\log(N\mathfrak{p})}{\log y} \quad \text{for} \quad N\mathfrak{p} \leqq y$$

and

$$\sum_{N\mathfrak{p}\leqq y} \frac{a'(\mathfrak{p})}{(N\mathfrak{p})^{1+2it}} \quad \text{with} \quad |a'(\mathfrak{p})| \ll 1 \quad \text{for} \quad N\mathfrak{p} \leqq y\,.$$

2–4. Here we shall prove

LEMMA 3. *Suppose that* $T > T_0$, $1 < y \ll T^{1/k}$ *and* $|a(\mathfrak{p})| < \log(N\mathfrak{p})/\log y$, $|a'(\mathfrak{p})| \ll 1$ *for* $N\mathfrak{p} \leqq y$. *Then for each* $k \geqq 1$, *we have*

$$\int_T^{2T} \left| \sum_{N\mathfrak{p}\leqq y} \frac{a(\mathfrak{p})}{(N\mathfrak{p})^{1/2+it}} \right|^{2k} dt \ll T$$

and

$$\int_T^{2T} \left| \sum_{N\mathfrak{p}\leqq y} \frac{a'(\mathfrak{p})}{(N\mathfrak{p})^{1+2it}} \right|^{2k} dt \ll T\,.$$

PROOF.

$$\left(\sum_{N\mathfrak{p}\leqq y} \frac{a(\mathfrak{p})}{(N\mathfrak{p})^{1/2+it}}\right)^k = \sum_{N(\mathfrak{p}_i)\leqq y, 1\leqq i\leqq k} \frac{a(\mathfrak{p}_1)a(\mathfrak{p}_2)\cdots a(\mathfrak{p}_k)}{(N(\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_k))^{1/2+it}}\,.$$

Hence the first integral is

$$\begin{aligned}
&= \sum_{N\mathfrak{p}_i\leqq y, N\mathfrak{q}_i\leqq y} \frac{a(\mathfrak{p}_1)\cdots a(\mathfrak{p}_k)\overline{a(\mathfrak{q}_1)}\cdots\overline{a(\mathfrak{q}_k)}}{(N(\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_k\mathfrak{q}_1\mathfrak{q}_2\cdots\mathfrak{q}_k))^{1/2}} \int_T^{2T} \left(\frac{N(\mathfrak{q}_1\cdots\mathfrak{q}_k)}{N(\mathfrak{p}_1\cdots\mathfrak{p}_k)}\right)^{it} dt \\
&\ll T\cdot \sum_{\substack{N\mathfrak{p}_i\leqq y, N\mathfrak{q}_i\leqq y \\ N(\mathfrak{p}_1\cdots\mathfrak{p}_k)=N(\mathfrak{q}_1\cdots\mathfrak{q}_k)}} \frac{|a(\mathfrak{p}_1)\cdots a(\mathfrak{p}_k)\overline{a(\mathfrak{q}_1)}\cdots\overline{a(\mathfrak{q}_k)}|}{N(\mathfrak{p}_1\cdots\mathfrak{p}_k)} \\
&\quad + \sum_{\substack{N(\mathfrak{p}_1\cdots\mathfrak{p}_k)\neq N(\mathfrak{q}_1\cdots\mathfrak{q}_k) \\ N\mathfrak{p}_i\leqq y, N\mathfrak{q}_i\leqq y}} \frac{|a(\mathfrak{p}_1)\cdots a(\mathfrak{p}_k)\overline{a(\mathfrak{q}_1)}\cdots\overline{a(\mathfrak{q}_k)}|}{(N(\mathfrak{p}_1\cdots\mathfrak{p}_k\mathfrak{q}_1\cdots\mathfrak{q}_k))^{1/2}\left|\log\dfrac{N(\mathfrak{p}_1\cdots\mathfrak{p}_k)}{N(\mathfrak{q}_1\cdots\mathfrak{q}_k)}\right|} \\
&= TS_1 + S_2\,,
\end{aligned}$$

say. $S_1 \ll (\log y)^{-k}(\sum_{N\mathfrak{p}\leqq y}(\log N\mathfrak{p})/N\mathfrak{p})^k \ll 1$ by the prime ideal theorem. Now we put $b_i(m) = |\{\mathfrak{p}_1\cdots\mathfrak{p}_i;\ N\mathfrak{p}_j \leqq y \text{ for } j = 1, \cdots, i,\ N(\mathfrak{p}_1\cdots\mathfrak{p}_i) = m\}|$ for $i = 1, 2, \cdots, k$. Then we have $b_k(m) \ll 1$ if $m = p_1^{f_1}\cdots p_k^{f_k}$, where p's may not be different, $f_i \geqq 1$ and $p_i^{f_i} < y$. We have also $b_k(m) = 0$ for other cases. Now

$$S_2 \ll \sum_{m,n\leqq y^k, n\leqq m/2} b_k(m)b_k(n)(mn)^{-1/2}|\log(n/m)|^{-1}$$
$$+ \sum_{m\leqq y^k, m/2<n<m} b_k(m)b_k(n)(mn)^{-1/2}|\log(n/m)|^{-1} = S_3 + S_4 ,$$

say. $S_3 \ll y^k \ll T$.

$$\begin{aligned} S_4 &\ll \sum_{m\leqq y^k} \sum_{r<m/2} b_k(m)b_k(m-r)m^{-1/2}(m-r)^{-1/2}mr^{-1} \\ &\ll \sum_{m\leqq y^k} b_k(m) \sum_{r<m/2} 1/r \ll \log y \sum_{m\leqq y^k} b_k(m) \\ &\ll \log y(\sum_{p^f\leqq y} b_1(p^f))^k \ll \log y(\pi(y))^k \\ &\ll \log y(y/\log y)^k \ll y^k \ll T. \end{aligned}$$

Hence $S_2 \ll T$ and

$$\int_T^{2T} \left|\sum_{N\mathfrak{p}\leqq y} \frac{a(\mathfrak{p})}{(N\mathfrak{p})^{1/2+it}}\right|^{2k} dt \ll T .$$

The second part of our Lemma 3 comes in a similar manner. q.e.d.

2-5. Using Lemma 3 we get

LEMMA 4.

$$\int_T^{2T} \left(f_h(t) - \frac{1}{\pi} \operatorname{Im} \sum_{N\mathfrak{p}\leqq x^2} \frac{a(\mathfrak{p})}{(N\mathfrak{p})^{1/2+it}}\right)^{2k} dt \ll T$$

for each $k \geqq 1$, $x \ll T^{1/2k}$ *and* $h > 0$, *where* $f_h(t)$ *is the same as in* 2-1 *and we put* $a(\mathfrak{p}) = (\psi(\mathfrak{p}) - \varphi(\mathfrak{p}))(e^{-ih\log(N\mathfrak{p})} - 1)$.

2-6. Hence we get

$$\begin{aligned} \int_T^{2T} f_h(t)^{2k} dt &= \frac{1}{\pi^{2k}} \int_T^{2T} \left(\operatorname{Im} \sum_{N\mathfrak{p}\leqq x^2} \frac{a(\mathfrak{p})}{(N\mathfrak{p})^{1/2+it}}\right)^{2k} dt \\ &+ O\left(\left(\int_T^{2T} \left(\operatorname{Im} \sum_{N\mathfrak{p}\leqq x^2} \frac{a(\mathfrak{p})}{(N\mathfrak{p})^{1/2+it}}\right)^{2k} dt\right)^{1-1/2k} T^{1/2k}\right) \\ &+ O(T) . \end{aligned}$$

Here we shall use the following

LEMMA 5. *Let* $F_\alpha(x) = \sum_{p\leqq x} |b(p)|^{2\alpha}/p^\alpha$ *for positive* α. *Suppose that* $F_\alpha(x) \ll 1$ *for* $\alpha \geqq 2$ *and* $F_{1/2}(x) \ll x^c$ *with some positive* c. *Then for* $x = T^{c'/k}$ *and for each* $k \geqq 1$,

$$\int_T^{2T} \left|\operatorname{Im} \sum_{p\leqq x} \frac{b(p)}{p^{1/2+it}}\right|^{2k} dt = C(k)T(\pi^2 F_1(x))^k + O(TF_1(x)^{0\vee(k-2)})$$

provided $F_1(x) \to \infty$ *as* $x \to \infty$, *where* $C(k) = 2k!/(2\pi)^{2k}k!$ *and* $0 \vee (k-2) =$ Max $(0, k-2)$.

(Cf. Lemma 3 of [3].)

Thus to estimate our integral we have only to know about

$$F_1(x) = \sum_{p \leqq x} |b'(p)|^2 p^{-1} |e^{-ih\log p} - 1|^2 ,$$

where we put

$$b'(p) = \sum_{N\mathfrak{p}=p} (\psi(\mathfrak{p}) - \varphi(\mathfrak{p})) .$$

We may remark here that as above we may ignore prime ideals of degree greater than 2. We may also ignore rational primes p which ramifies at $K/\boldsymbol{Q}$ since the number of such primes is finite. Now we have for our p,

$$\sum_{N\mathfrak{p}=p} \psi(\mathfrak{p}) = \psi^G(p) \quad \text{and} \quad \sum_{N\mathfrak{p}=p} \varphi(\mathfrak{p}) = \varphi^G(p) .$$

On the other hand since we can take 1, σ, τ, $\sigma\tau$ as representatives of G/H, we have $\varphi^G(1) = 4$, $\varphi^G(\sigma^2) = -4$, $\varphi^G(\sigma) = 0$, and $\varphi^G(\xi) = \varphi(\xi)$ if $\xi = \rho\sigma^2$ or $\xi = \rho = (\rho\sigma^2)^4$ or $\xi = \rho^2 = (\rho\sigma^2)^2$ or $\xi = \rho^2\sigma^2 = (\rho\sigma^2)^5$. The same is true for $\psi^G(\xi)$. Hence

$$b'(p) = \begin{cases} 0 & \text{if} \quad \sigma_p \in \{1, \sigma, \sigma^2\} \\ \psi(\sigma_p) - \varphi(\sigma_p) & \text{otherwise} , \end{cases}$$

where σ_p is a representative of the conjugate class determined by the Frobenius element $\sigma_{\mathscr{B}}$ of $\mathscr{B}$ in K which divides p and $\sigma_p \in \{1, \sigma, \sigma^2\}$ means that the conjugate class of σ_p can be represented by one among 1, σ and σ^2. Hence by Tschebotareff's density theorem, we get

$$\begin{aligned} \sum_{p \leqq x} |b'(p)|^2 &= 3 \sum_{\substack{p \leqq x \\ \sigma_p \in \{\rho\sigma 2, \rho, \rho^2, \rho^2\sigma^2\}}} 1 \\ &= (3A/24)x/(\log x) + O(x/(\log x)^2) , \end{aligned}$$

where A is the cardinal of the union of the conjugate classes represented by $\rho\sigma^2$, ρ, ρ^2, $\rho^2\sigma^2$. Hence we get

$$F_1(x) = (2\cdot 3A/24)\log(h\log x) + O(1) ,$$

provided $h\log x \longrightarrow \infty$ as $x \longrightarrow \infty$.

2-7. Hence we get

THEOREM. *For each $k \geqq 1$ and $T > T_0$,*

$$\begin{aligned} \int_T^{2T} f_h(t)^{2k} dt &= C(k)T(2(3A/24)\log(3 + h\log T))^k \\ &\quad + O(T(\log(3 + h\log T))^{k-1/2}) . \end{aligned}$$

(Similarly we get

THEOREM. *For each $k \geqq 1$ and $T > T_0$,*

$$\int_T^{2T} S(t, \varphi)^{2k} dt = C(k)T(((4^2A' + A)/24) \log(\log T))^k + O(T(\log(\log T))^{k-1/2}) \, ,$$

where A' is the cardinal of the union of the conjugate classes represented by 1 and σ^2, and A is the same as before.

The same is true for $S(t, \psi)$).

2-8. We need one more mean value theorem.

LEMMA 6.

$$\int_T^{2T} S(t, \varphi)dt \ll \log T \, .$$

PROOF. From Lemma 1′ in 2-3 and

$$\int_0^T S(t, \varphi)dt = \frac{1}{\pi}\int_{1/2}^2 \log |L(\mu + iT, \psi)| d\mu + O(1) \, ,$$

we get our conclusion as in pp. 187-189 of [7].

2-9. From the first theorem in 2-7, we get

$$\int_T^{2T} f_h(t)^{2k} dt \cong T(\log(h \log T))^k$$

for sufficiently large $h \log T$ and for each $k \geqq 1$. From Lemma 6 we get

$$\int_T^{2T} f_h(t)dt \ll \log T \quad \text{for} \quad h > 0 \, .$$

2-10. Now we can complete our proof of the main theorem. We suppose that $h = C/\log T$ with a sufficiently large constant C. We write $f(t)$ instead of $f_h(t)$. Let $E_M = \{t \in (T, 2T); f(t) > M\}$ for non-negative M as in 2-1. Let $\varphi_M(t)$ be the characteristic function of E_M. Then we have

$$\begin{aligned}\int_T^{2T} f(t)\varphi_0(t)dt &= \int_T^{2T} f(t)\varphi_M(t)\varphi_0(t)dt + \int_T^{2T} f(t)(1 - \varphi_M(t))\varphi_0(t)dt \\ &\leqq (E_M)^{1/2}\left(\int_T^{2T} f(t)^2 dt\right)^{1/2} + MT \, .\end{aligned}$$

On the other hand

$$\begin{aligned}\int_T^{2T} f(t)\varphi_0(t)dt &= 1/2\int_T^{2T} |f(t)|\, dt + 1/2\int_T^{2T} f(t)dt \\ &\geqq 1/2\left(\int_T^{2T} |f(t)|^2 dt\right)^{3/2}\Big/\left(\int_T^{2T} |f(t)|^4 dt\right)^{1/2} + O(\log T) \, .\end{aligned}$$

Using 2-9 we get

$|E_M| \gg T$ for sufficiently large C depending on M.

q.e.d. of the main Theorem.

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