Tôhoku Math. Journ.
30 (1978), 211-221.

SEQUENCES SUMMABLE BY SOME RIESZ MEAN

B. KUTTNER

(Received April 1, 1976)

1. We will use throughout s to denote a sequence $\{s_n\}$ (of complex numbers, except where otherwise stated). We also suppose throughout that s is the sequence of partial sums of the series

$$\sum_{n=0}^{\infty} a_n$$

so that

$$s_n = a_0 + a_1 + \cdots + a_n .$$

Let $\lambda = \{\lambda_n\}$ denote a sequence of non-negative numbers increasing to ∞. Let $k > 0$. With the usual terminology, the sequence s is said to be summable (R, λ, k) to σ if

$$\frac{1}{u^k} \sum_{\lambda_\nu < u} (u - \lambda_\nu)^k a_\nu \to \sigma \tag{1}$$

as $u \to \infty$. Writing

$$\phi_k(u) = \begin{cases} u^k & (u > 0) ; \\ 0 & (u \leqq 0) , \end{cases}$$

we note that (1) can also be written

$$\frac{1}{u^k} \sum_{\nu=0}^{\infty} (\phi_k(u - \lambda_\nu) - \phi_k(u - \lambda_{\nu+1}))s_\nu \to \sigma \tag{2}$$

as $u \to \infty$.

Instead of considering the properties of (R, λ, k) for fixed λ, k, we consider the following problem. Given $k > 0$, what sequences s have the property of being summable (R, λ, k) for some λ? Let R_k denote the set of all such sequences. Thus R_k is defined as the set of all s for which there is some λ and some complex number σ for which (1) (or, what is the same thing, (2)) holds. Note that, when s is real, the values $\sigma = \pm\infty$ are not allowed. Thus the problem considered is that of investigating the sequence set R_k. While the present paper leaves a number of questions unsettled, it constitutes a contribution to the study of this problem.

It should be remarked that some results connected with this problem have been obtained by L. V. Grepachevskaya[†] [2]. But Grepachevskaya considers only the case $k = 1$, and restricts himself to real sequences.

It is convenient to state for reference a result which follows at once from the well known fact that (R, λ, k) is totally regular.

PROPOSITION A. *If s is real, and if $s \to +\infty$ or $s_n \to -\infty$ as $n \to \infty$, then, for any $k > 0$, $s \notin R_k$.*

We shall prove the following result.

THEOREM 1. *Let $k > 0$. If s has any subsequence belonging to R_k, then $s \in R_k$.*

Evidently, any convergent sequence belongs to R_k. The assertion that s has a convergent subsequence is equivalent to the assertion that

$$|s_n| \not\to \infty \tag{3}$$

as $n \to \infty$, and thus Theorem 1 includes the following result.

PROPOSITION B. *If (3) holds, then $s \in R_k$ for any $k > 0$.*

The following result is an immediate corollary of Propositions A and B.

PROPOSITION C. *For any $k > 0$, R_k is not a linear space.*

In fact, if $\{s_n\} \in R_k$, $\{t_n\} \in R_k$, it does not necessarily follow that $\{s_n + t_n\} \in R_k$. Suppose for example, that

$$s_n = \begin{cases} 0 & (n \text{ even})\ ; \\ n & (n \text{ odd})\ , \end{cases} \qquad t_n = \begin{cases} n & (n \text{ even})\ ; \\ 0 & (n \text{ odd})\ . \end{cases}$$

It follows from Proposition B that, for all $k > 0$, $\{s_n\} \in R_k$, $\{t_n\} \in R_k$. But, by Proposition A, $\{s_n + t_n\} \notin R_k$ for any k. We have also the following result.

THEOREM 2. *Let $k > 0$. In order that $s \in R_k$, it is necessary that*

$$\sum_{n=0}^{\infty} \frac{1}{|s_n|^{1/k}} = \infty\ . \tag{4}$$

Here we adopt the convention that, if $s_n = 0$ for a finite set of values of n, these values are to be ignored in considering (4). But we regard (4) as holding if $s_n = 0$ for an infinity of n.

It will be noted that (4) involves only the moduli of the terms s_n. I conjecture that, roughly speaking, (4) is also sufficient as far as $|s_n|$ is concerned. More precisely, the conjecture is that, if we are given $|s_n|$

[†] I am indebted to Professor M. R. Parameswaran for drawing my attention to this paper.

but are free to choose arg s_n, then, if (4) holds, we can choose arg s_n so that $s \in R_k$. However I have so far been able to prove this conjecture only in the cases $0 < k \leqq 1$ and $k = 2$. As I am hoping, at some future time, to obtain a proof at any rate for the case in which k is any positive integer, I do not give the proof for the case $k = 2$ here, but confine myself to the case $0 < k \leqq 1$. Thus we shall prove the following theorem.

THEOREM 3. *Let* $0 < k \leqq 1$. *Suppose that the values of* $|s_n|$ *are given, and that they satisfy* (4). *Then it is possible to choose* arg s_n *so that* $s \in R_k$.

I now consider the special case in which $k = 1$. Even in this case, I have been able to obtain conditions which are both necessary and sufficient for $s \in R_1$ only in the case in which s is restricted to be a real sequence. Supposing that s_n is real, the case in which s_n is ultimately of constant sign may be dealt with by the results already given. For either (3) holds, in which case we may apply Proposition B, or else $s_n \to +\infty$ or $s_n \to -\infty$ as $n \to \infty$, in which case we may apply Proposition A. So this case will be excluded. We may also suppose that, for all n, $s_n \neq 0$. For if $s_n = 0$ for an infinity of n then, again, (3) holds; and the alteration of a finite number of terms of s will not affect the property of belonging, or not belonging, to R_1. Thus we may suppose that the sequence is divided into blocks of consecutive terms, the terms in any one block all of the same sign, this sign being the opposite of that for the next block. Of course, a block may consist of just one term. Let σ_m denote the minimum modulus of the terms in the $m + 1$st block. On this understanding, and, with this notation, we have the following result.

THEOREM 4. *Let s be a real sequence. In order that* $s \in R_1$, *it is necessary and sufficient that*

$$\sum_{m=0}^{\infty} \frac{1}{\sigma_m} = \infty . \tag{5}$$

The sufficiency part of this theorem is included in Theorem 3 of [2]. But I give a proof, partly for the sake of completeness and partly because, by making use of Theorem 1 of this paper, it is possible to prove the result slightly more simply than in [2].

We note that Theorem 4 includes the case $k = 1$ of Theorem 3. For if we are free to choose arg s_n, we may take the terms s_n as all real and alternately positive and negative. Then each block consists of just one term, and the sum (4) (with $k = 1$) reduces to (5).

It is not clear how a condition such as that of Theorem 4 can be extended to complex sequences. In view of the non-linear nature of R_1, we cannot deal with the problem by considering separately real and imaginary parts. Some remarks may, however, be made.

For any real θ, write

$$s_n(\theta) = \mathscr{R}\{e^{-i\theta}s_n\} ;$$

that is to say, $s_n(\theta)$ is the projection of s_n on a line through the origin inclined at an angle θ to the real axis. Let $s(\theta)$ denote the sequence $\{s_n(\theta)\}$. In order that $s \in R_1$, it is clearly necessary that $s(\theta) \in R_1$ for all θ. It would appear to be at least *prima facie* plausible that this condition is also sufficient; and if this were proved to be the case the problem of obtaining necessary and sufficient conditions for $s \in R_1$ with s complex would be solved. It is therefore worth while proving the following result.

THEOREM 5. *There is a complex sequence s such that $s(\theta) \in R_1$ for all real θ, but $s \notin R_1$.*

2. We require a lemma.

LEMMA 1. *Let $k > 0$. Suppose that, for a given s, there is a non-decreasing sequence $\lambda = \{\lambda_n\}$ of non-negative numbers with $\lambda_n \to \infty$ as $n \to \infty$ such that* (2) *holds. Then there is an increasing λ with the same properties.*

Thus the definition of R_k would be unaffected if instead of (as is usual) requiring λ to be increasing, we required only that it should be non-decreasing. The result is proved in [5] (see the first part of §3), though it is not there stated as a formal lemma.

Theorem 1 is an immediate consequence of this lemma. Suppose that the subsequence[†] $\{s(n_r)\}$ is summable (R, λ, k) to σ. Thus

$$\frac{1}{u^k}\sum_{r=0}^{\infty} s(n_r)(\phi_k(u - \lambda_r) - \phi_k(u - \lambda_{r+1})) \to \sigma \tag{6}$$

as $u \to \infty$. Now define

$$\mu_n = \begin{cases} \lambda_0 (n \leqq n_0) ; \\ \lambda_{r+1} (n_r < n \leqq n_{r+1},\ r = 0, 1, 2, \cdots) . \end{cases}$$

Then

[†] To avoid repeated suffixes, we write $s(n)$ in place of s_n whenever n is replaced by an expression itself involving suffixes.

$$\frac{1}{u^k}\sum_{n=0}^{\infty} s_n(\phi_k(u-\mu_n)-\phi_k(u-\mu_{n+1}))$$

reduces to the sum on the left of (6). Thus s is summable (R, μ, k); and μ_n is non-decreasing so that the conclusion follows.

I now give a result which is an immediate corollary of a well known theorem on infinite products, but which we will require to use more than once, and which it is therefore convenient to state as a lemma.

LEMMA 2. *Let $\lambda = \{\lambda_n\}$ be a non-decreasing sequence of non-negative numbers. Then, in order that $\lambda_n \to \infty$ as $n \to \infty$, it is necessary and sufficient that*

$$\sum_{n=1}^{\infty}\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}} \tag{7}$$

should diverge.

For the divergence of (7) is equivalent to the divergence (to zero) of

$$\prod_{n=1}^{\infty}\left(1-\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\right)=\prod_{n=1}^{\infty}\frac{\lambda_n}{\lambda_{n+1}}\,.$$

In order to prove Theorem 2, suppose that $s\in R_k$. Thus, by definition, s is summable (to σ, say) by some (R, λ, k). By the well known limitation theorem ([1], Theorem 1.61 or [3], Theorem 21),

$$s_n-\sigma=o\left\{\left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\right)^k\right\}\,;$$

however, we need only the weaker result that

$$s_n=O\left\{\left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\right)^k\right\}\,.$$

If (4) were false, it would follow that

$$\sum_{n=1}^{\infty}\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}<\infty\,,$$

and, by Lemma 2, this would contradict the requirement that $\lambda_n \to \infty$ as $n \to \infty$.

3. We now prove Theorem 3. By a theorem of Jurkat [4], it is enough to consider the discontinuous Riesz means (R^*, λ, k); so we write

$$t_n=\frac{1}{\lambda_{n+1}^k}\sum_{\nu=0}^{n}\{(\lambda_{n+1}-\lambda_\nu)^k-(\lambda_{n+1}-\lambda_{\nu+1})^k\}s_\nu\,.$$

Suppose that (4) holds. Then we can find a decreasing sequence $\{\eta_n\}$ of positive numbers such that $\eta_n \to 0$ as $n \to \infty$, and such that

$$\sum_{n=0}^{\infty} \left(\frac{\eta_n}{|s_n|}\right)^{1/k} = \infty . \tag{8}$$

We may suppose that $|s_n| \to \infty$ as $n \to \infty$, since otherwise the conclusion follows from Proposition B. Since the alternation of a finite number of terms is irrelevant, we may suppose that, for all n, $s_n \neq 0$. We can then choose η_n so that, further

$$\eta_n < |s_n| . \tag{9}$$

We will take all the s_n's as real. Their signs, and the values of λ_n, will be defined inductively in such a way that

$$t_n = \pm\eta_n .$$

Thus s will be summable (R, λ, k) to 0.

The inductive definition is as follows. Take $\lambda_0 = 0$. Now suppose that $\lambda_0, \lambda_1, \cdots, \lambda_n$ and the signs of $s_0, s_1, \cdots, s_{n-1}$ have been determined. This fixes

$$f_n(\lambda) = \frac{1}{\lambda^k}\sum_{\nu=0}^{n-1} \{(\lambda - \lambda_\nu)^k - (\lambda - \lambda_{\nu+1})^k\} s_\nu .$$

[In the case $n = 0$, $f_n(\lambda)$ is identically zero]. Now consider the set A_n of all solutions greater than λ_n of either of the equations

$$f_n(\lambda) + \left(\frac{\lambda - \lambda_n}{\lambda}\right)^k |s_n| = \eta_n ; \tag{10}$$

$$f_n(\lambda) - \left(\frac{\lambda - \lambda_n}{\lambda}\right)^k |s_n| = -\eta_n . \tag{11}$$

This set is not empty. For[†] $f_n(\lambda_n) = t_{n-1} = \pm\eta_{n-1}$. If $f_n(\lambda_n) = -\eta_{n-1}$ then, since the expression on the left of (10) tends to $|s_n| > \eta_n$ as $n \to \infty$ it follows that (10) has a solution greater than λ_n. Similarly if $f_n(\lambda_n) = \eta_{n+1}$ then (11) has a solution greater than λ_n. Now (10) cannot have arbitrarily large solutions, since the expression on the left tends to $|s_n| > \eta_n$ as $\lambda \to \infty$. Similarly, (11) cannot have arbitrarily large solutions. Hence the set A_n is bounded, and thus, by continuity, it has a largest member. Take λ_{n+1} as this greatest member. If it is a solution of (10), take s_n as positive; otherwise take s_n as negative. Then

$$t_n = \pm\eta_n$$

[†] In the case $n = 0$, we take $t_{-1} = 0$.

as required. Thus it remains only to verify that the requirement that $\lambda_n \to \infty$ as $n \to \infty$ is satisfied.

If λ_{n+1} is a solution of (10), then

$$f_n(\lambda_{n+1}) - \left(\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right)^k |s_n| \leqq -\eta_n \,. \tag{12}$$

For otherwise, (11) would have a solution greater than λ_{n+1}, which would contradict the definition of λ_{n+1} as the greatest member of A_n. It follows that

$$\left(\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right)^k |s_n| \geqq \eta_n \,. \tag{13}$$

A similar argument with signs changed shows that (13) continues to hold when λ_{n+1} is a solution of (11). We now deduce from (8) and (13) that

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}$$

diverges. Hence the conclusion follows from Lemma 2.

4. In order to prove Theorem 4, we require another lemma. This lemma amounts essentially merely to a restatement of the definition of R_1, but it will be convenient for the application to have the definition restated in this form. Although in the application to Theorem 4 we will be considering only real sequences, the lemma applies equally well to complex sequences, so will be stated in the more general form.

LEMMA 3. *In order that the (complex) sequence s should belong to R_1, it is necessary and sufficient that there should exist a sequence $t = \{t_n\}$ such that*

(i) *t converges;*

(ii) $t_n \in [t_{n-1}, s_n)$;

(14) (iii) $\sum^\infty (t_{n-1} - t_n)/(t_n - s_n)$

diverges.

Here $[t_{n-1}, s_n)$ denotes the line segment joining the points t_{n-1}, s_n, taken as closed at t_{n-1} and open at s_n. If $t_{n-1} = s_n$, then (ii) is to be taken as meaning that $t_n = s_n$. Note that, by (ii), (14) is a series of real non-negative numbers. In the case in which $t_{n-1} = s_n$, so that the numerator and denominator of a term in the sum (14) both vanish, the value of this term may be assigned arbitrarily (so long as it is real and non-negative). In the case in which s is real we note that if $|s_n| \to \infty$

as $n \to \infty$ (the only case we need consider, by Proposition B) then, by (i)

$$t_n - s_n \sim -s_n$$

so that (iii) is equivalent to the assertion that

$$\sum^{\infty} \frac{t_{n-1} - t_n}{s_n} \tag{15}$$

diverges.

PROOF. It is enough to consider the discontinuous means, so we write

$$t_n = \frac{1}{\lambda_{n+1}} \sum_{\nu=0}^{n} (\lambda_{\nu+1} - \lambda_\nu) s_\nu \ .$$

Then $s \in R_1$ if and only if, for a suitably chosen λ, (i) holds. We express the requirements that λ_n should be real, non-negative and non-decreasing and that $\lambda_n \to \infty$ as $n \to \infty$ in terms of t. We have

$$(\lambda_{n+1} - \lambda_n)s_n = \lambda_{n+1}t_n - \lambda_n t_{n-1} \ ,$$

so that

$$\lambda_{n+1}(t_n - s_n) = \lambda_n(t_{n-1} - s_n) \ . \tag{16}$$

If $t_{n-1} = s_n$ it follows that $t_n = s_n$ (and λ_{n+1} may be chosen arbitrarily, so long as $\lambda_{n+1} > \lambda_n$). If not, we write (16) in the form

$$\lambda_{n+1} = \lambda_n \left\{1 + \frac{t_{n-1} - t_n}{t_n - s_n}\right\} \ .$$

Since $\lambda_{n+1} > \lambda_n$, the expression

$$\frac{t_{n-1} - t_n}{t_n - s_n}$$

must be real and non-negative, which gives us (ii). Finally the requirement that $\lambda_n \to \infty$ can be put in the form that

$$\prod^{\infty} \frac{\lambda_{n+1}}{\lambda_n}$$

diverges; and (by (16)) this is equivalent to the divergence of (14).

5. We now come to the proof of Theorem 4.

SUFFICIENCY. By Theorem 1, it is enough to prove that s has a subsequence belonging to R_1. We consider the subsequence formed by taking the term of minimum modulus from each block. Changing the notation, and using $s = \{s_n\}$ to denote this subsequence, we are given that

$$\sum_{n=0}^{\infty} \frac{1}{|s_n|}$$

diverges, and that the terms s_n alternate in sign. It follows that we can determine a sequence $\{\eta_n\}$ of positive numbers such that $\eta_n \to 0$ and such that

$$\sum_{n=0}^{\infty} \frac{\eta_n}{|s_n|} \tag{17}$$

diverges. We may further suppose that, for all n,

$$\eta_n < |s_n| \,. \tag{18}$$

Now define

$$t_n = \eta_n \operatorname{sgn} s_n \,.$$

Since $\eta_n \to 0$ as $n \to \infty$, condition (i) of Lemma 3 holds. Further, since s_n alternates in sign, it follows from (18) that (ii) holds. Finally, $|t_n - t_{n-1}| > |t_n|$, and it therefore follows from the divergence of (17) that (iii) holds. Thus all the conditions of Lemma 3 are satisfied.

NECESSITY. Suppose that $s \in R_1$. We may suppose that $|s_n| \to \infty$ as $n \to \infty$, since otherwise the result that (5) holds is trivial. By Lemma 3, there is a sequence t satisfying the conditions of that lemma. Thus, for all sufficiently large n,

$$|t_{n-1}| < |s_n| \; ; \qquad |t_n| < |s_n| \,. \tag{19}$$

Let T_m denote the set of suffixes occurring in the $m + 1$st block of terms. By (19) and condition (ii) of Lemma 3 it follows that, for $n \in T_m$ (where m is sufficiently large), $t_n - t_{n-1}$ is of constant sign, and of the same sign as s_n. Thus

$$\sum_{n \in T_m} |t_n - t_{n-1}| = |\sum_{n \in T_m} (t_n - t_{n-1})| = \varepsilon_m$$

(say), where ε_m is bounded. (In fact, $\varepsilon_m \to 0$ as $m \to \infty$; but only its boundedness is needed.) Hence

$$\sum_{n \in T_m} \frac{t_n - t_{n-1}}{s_n} < \frac{\varepsilon_m}{\sigma_m} \,,$$

so that the divergence of (15) implies that (5) holds.

6. In order to prove Theorem 5, let $\{r_n\}$ denote the sequence

$$0, 0, 1/2, 0, 1/3, 2/3, 0, 1/4, 1/2, 3/4, \cdots .$$

That is to say, the sequence $\{r_n\}$ is formed by taking all numbers of the form p/q with p an integer, $0 \leqq p < q$ (including those for which p, q may have a common factor) first for $q = 1$, then for $q = 2$, then $q = 3$ and so on; for a given q, the numbers p/q are in increasing order of magnitude. Let λ be a constant with $0 < \lambda < 1$. We define

$$s_n = (-1)^n n(\log n)^{1+\lambda} \exp(\pi i r_n) . \tag{20}$$

We take this as meaning 0 when $n = 0$. It is clear that (4), with $k = 1$, is not satisfied. Hence, by Theorem 2, $s \notin R_1$. We will show, however, that $s(\theta) \in R_1$, for every θ.

If θ is a rational multiple of π, then $s_n(\theta) = 0$ for some arbitrarily large n, and the result therefore follows at once from Proposition B. So we may suppose that θ is an irrational multiple of π. It is only the residue of $\theta \bmod \pi$ which is relevant (since altering θ by π merely changes the sign of $s_n(\theta)$); hence we may suppose that $-\pi/2 < \theta < \pi/2$. We note that we can write $s_n(\theta)$ in the form

$$s_n(\theta) = (-1)^{n+1} n(\log n)^{1+\lambda} \sin(\pi r_n - \theta - \pi/2) . \tag{21}$$

Now in the sequence $\{r_n\}$ the set of terms p/q $(0 \leqq p < q)$ will occur for values of n satisfying

$$1 + 2 + \cdots + (q-1) < n \leqq 1 + 2 + \cdots + q ;$$

that is to say, for

$$(q-1)q/2 < n \leqq q(q+1)/2 . \tag{22}$$

Consider the terms $s_n(\theta)$ satisfying (22) and for which, further

$$\pi r_n > \theta + \pi/2 .$$

These terms alternate in sign. Further, since (22) implies that $n < q^2$, they satisfy

$$|s_n(\theta)| \leqq 2^{1+\lambda} q^2 (\log q)^{1+\lambda} (\pi r_n - \theta - \pi/2) . \tag{23}$$

Let the terms of the set now being considered be given by $n = n_0 + h$ $(h = 0, 1, 2, \cdots, H)$, where $r_{n_0} = p/q$; thus

$$\theta + \pi/2 < \pi(p/q) < \theta + \pi/2 + \pi/q$$

whence it follows that

$$\pi r_n - \theta - \pi/2 < \pi(1+h)/q . \tag{24}$$

We also note that, as $q \to \infty$

$$H \sim (1/2 - \theta/\pi)q \tag{25}$$

(and, for fixed θ, the factor of q on the right is a positive constant). Now let $\{\sigma_m(\theta)\}$ be formed from $\{s_n(\theta)\}$ in the same way as $\{\sigma_m\}$ was formed from $\{s_m\}$ in Theorem 4. Since the terms now being considered alternate in sign, each one (except possibly the first and the last) must constitute a block by itself; so their contribution to $\sum 1/\sigma_m(\theta)$ is at least

$$\sum_{n=n_0+1}^{n_0+H-1} \frac{1}{|s_n(\theta)|} \geqq \frac{1}{2^{1+\lambda}\pi q(\log q)^{1+\lambda}} \sum_{h=1}^{H-1} \frac{1}{1+h}, \tag{26}$$

by (23) and (24). It follows from (25) that for fixed θ and sufficiently large q, the expression (26) is at least equal to

$$\frac{c}{q(\log q)^{\lambda}}$$

where c is a positive constant. Hence $\sum 1/\sigma_m(\theta)$ diverges, so that, by Theorem 4, $s(\theta) \in R_1$.

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DEPARTMENT OF PURE MATHEMATICS
THE UNIVERSITY OF BIRMINGHAM
ENGLAND