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DIMENSION OF COHOMOLOGY SPACES OF INFINITESIMALLY DEFORMED KLEINIAN GROUPS

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1. **Introduction.** Let G be the group of all Möbius transformations of $\hat{C} = C \cup \{\infty\}$ of the form $t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$, where $\alpha, \beta, \gamma, \delta \in C$ and $\alpha\delta - \beta\gamma = 1$. Here C is the complex plane. An element $g: t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$, not being the identity, of G is called parabolic if $\mathrm{tr}^2 g = (\alpha + \delta)^2 = 4$.

Let Γ be a subgroup of G and let E be a finite dimensional complex vector space. Let χ be an anti-homomorphism of Γ into $GL(E)$, the group of all non-singular linear mappings of E onto itself. A mapping $z: \Gamma \to E$ is called a cocycle if

$$z(g_1 \circ g_2) = \chi(g_2)(z(g_1)) + z(g_2)$$

for all g_1 and g_2 in Γ. A cocycle z is a coboundary if

$$z(g) = \chi(g)(X) - X$$

for some $X \in E$. We denote by $Z^1_\chi(\Gamma, E)$ the space of all cocycles and by $B^1_\chi(\Gamma, E)$ the space of all coboundaries. A cocycle z is called a parabolic cocycle if, for any parabolic cyclic subgroup Γ_0 of Γ, $z|_{\Gamma_0}$ is an element of $B^1_\chi(\Gamma_0, E)$. We denote by $PZ^1_\chi(\Gamma, E)$ the space of all parabolic cocycles.

The group G is a complex 3-dimensional Lie group isomorphic to $SL(2, C)$ modulo its center. The Lie algebra $\mathfrak{g}$ of G is therefore the algebra of 2×2 complex matrices of trace zero. We identify $\mathfrak{g}$ with the tangent space of G at the identity element e of G.

The adjoint representation Ad of G in $\mathfrak{g}$ is defined by $\mathrm{Ad}(g)(X) = (dA_g)_e(X)$, where $X \in \mathfrak{g}$ and $(dA_g)_e$ is the differential at e of the mapping $A_g: G \ni h \mapsto g^{-1} \circ h \circ g \in G$. The adjoint representation is an anti-homomorphism of G into $GL(\mathfrak{g})$. Hence, for a subgroup Γ of G, we can construct the space of parabolic cocycles $PZ^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g})$.

Let Γ be a subgroup of G and let $\theta: \Gamma \mapsto G$ be a homomorphism of Γ into G. We say that θ is a parabolic homomorphism if $\mathrm{tr}^2\theta(g) = 4$ for any parabolic element g in Γ.

In this paper we prove the following:

THEOREM. *Let Γ be a finitely generated subgroup of G and let $\theta\colon \Gamma \to G$ be a parabolic homomorphism which is sufficiently close to the identity homomorphism. Then*

$$\dim PZ^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g}) \geqq \dim PZ^1_{\mathrm{Ad}}(\Gamma^\theta, \mathfrak{g}) ,$$

where $\Gamma^\theta = \theta(\Gamma)$.

In Section 4 we give an application of this theorem concerning the quasi-conformal deformation of a certain class of finitely generated Kleinian groups.

I would like to express my gratitude to the referee for his informative advice.

2. Linear maps $T^{(\sigma,\omega)}$ and $S^{(\sigma,\omega)}$. Let Γ be a finitely generated subgroup of G with a system of generators $\sigma = \{\sigma_1, \cdots, \sigma_N\}$. Let Λ be the free group with free generators $\{\lambda_1, \cdots, \lambda_N\}$ and let $\pi\colon \Lambda \to \Gamma$ be the homomorphism defined by $\pi(\lambda_k) = \sigma_k$. Denote by $\omega = \omega(\lambda_1, \cdots, \lambda_N)$ an element of Λ, i.e., a word in N letters $\lambda_1, \cdots, \lambda_N$. The kernel of π will be denoted by $\ker \pi$.

We define an anti-homomorphism $\rho\colon \Lambda \to GL(\mathfrak{g})$ by $\rho = \mathrm{Ad}\circ\pi$. Then we can construct, as in the case of $Z^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g})$, the space of cocycles $Z^1_\rho(\Lambda, \mathfrak{g})$, that is, $\tilde{z} \in Z^1_\rho(\Lambda, \mathfrak{g})$ if and only if $\tilde{z}(\lambda\circ\lambda') = \rho(\lambda')(\tilde{z}(\lambda)) + \tilde{z}(\lambda')$ for all λ and λ' in Λ.

Let V_ρ be the subspace of $Z^1_\rho(\Lambda, \mathfrak{g})$ defined by $V_\rho = \{\tilde{z} \in Z^1_\rho(\Lambda, \mathfrak{g})\colon \tilde{z}(\omega) = 0$ for all $\omega \in \ker\pi\}$. By a result in [6], $Z^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g})$ is isomorphic to V_ρ by the map $Z^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g}) \ni z \mapsto z\circ\pi \in V_\rho$. Moreover, $PZ^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g})$ is isomorphic to the subspace PV_ρ of V_ρ defined by $PV_\rho = \{\tilde{z} \in V_\rho$: for any ω with $\pi(\omega)$ parabolic, there exists an $X \in \mathfrak{g}$ with $\tilde{z}(\omega) = \rho(\omega)(X) - X\}$.

Let $\tilde{z} \in Z^1_\rho(\Lambda, \mathfrak{g})$ and let $\tilde{z}(\lambda_k) = X_k$. For a word $\omega = \eta_1\circ\cdots\circ\eta_{n(\omega)}$ in Λ with $\eta_s = \lambda_{k(s)}$ or $\eta_s = \lambda^{-1}_{k(s)}$ for some $k(s)$, $1 \leqq k(s) \leqq N$, we have

$$\begin{aligned}\tilde{z}(\omega) &= \tilde{z}(\eta_1\circ\cdots\circ\eta_{n(\omega)}) \\ &= \sum_{s=1}^{n(\omega)-1} \rho(\eta_{n(\omega)})\circ\cdots\circ\rho(\eta_{s+1})(\tilde{z}(\eta_s)) + \tilde{z}(\eta_{n(\omega)}) \\ &= \sum_{s=1}^{n(\omega)-1} \mathrm{Ad}(\nu_{n(\omega)})\circ\cdots\circ\mathrm{Ad}(\nu_{s+1})(\tilde{z}(\eta_s)) + \tilde{z}(\eta_{n(\omega)})\end{aligned}$$

for $\nu_s = \pi(\eta_s)$. Since $\tilde{z}(\eta_s) = -\rho(\eta_s)(\tilde{z}(\eta_s^{-1})) = -\mathrm{Ad}(\nu_s)(\tilde{z}(\eta_s^{-1}))$, we have

$$\tilde{z}(\omega) = \sum_{s=1}^{n(\omega)} Y_s^{(\sigma,\omega)} ,$$

where

$$Y_s^{(\sigma,\omega)} = \begin{cases} \mathrm{Ad}(\nu_{n(\omega)})\circ\cdots\circ\mathrm{Ad}(\nu_{s+1})(X_{k(s)}) & \text{if} \quad \nu_s = \sigma_{k(s)} \\ -\mathrm{Ad}(\nu_{n(\omega)})\circ\cdots\circ\mathrm{Ad}(\nu_s)(X_{k(s)}) & \text{if} \quad \nu_s = \sigma_{k(s)}^{-1} \end{cases}$$

for s with $1 \leqq s \leqq n(\omega) - 1$ and

$$Y_{n(\omega)}^{(\sigma,\omega)} = \begin{cases} X_{k(n(\omega))} & \text{if} \quad \nu_{n(\omega)} = \sigma_{k(n(\omega))} \\ -\mathrm{Ad}(\nu_{n(\omega)})(X_{k(n(\omega))}) & \text{if} \quad \nu_{n(\omega)} = \sigma_{k(n(\omega))}^{-1} \,. \end{cases}$$

Hence $\tilde{z} \in V_\rho$ if and only if $\sum_{s=1}^{n(\omega)} Y_s^{(\sigma,\omega)} = 0$ for all $\omega \in \ker \pi$ (see also [6]). Moreover, $\tilde{z} \in V_\rho$ is an element of PV_ρ if and only if, for any ω with $\pi(\omega)$ parabolic, there exists an $X \in \mathfrak{g}$ such that $\sum_{s=1}^{n(\omega)} Y_s^{(\sigma,\omega)} = \mathrm{Ad}(\pi(\omega))(X) - X$.

Let L_g, $g \in G$, be the left translation of G and let f be the holomorphic function on G defined by $f(g) = \mathrm{tr}^2 g - 4$. Then we have the following.

LEMMA 1 (Gardiner and Kra [4]). *Let $\omega \in \Lambda$ with $\pi(\omega)$ parabolic and let Y be an element of $\mathfrak{g}$. Then $Y = \mathrm{Ad}(\pi(\omega))(X) - X$ for some $X \in \mathfrak{g}$ if and only if $d(f\circ L_{\pi(\omega)})_e(Y) = 0$ for the tangent linear mapping $d(f \circ L_{\pi(\omega)})_e$ at $e \in G$.*

By this lemma we have immediately the following.

LEMMA 2. *Let $\tilde{z} \in Z_\rho^1(\Lambda, \mathfrak{g})$ and let $\tilde{z}(\lambda_k) = X_k$. Then $\tilde{z}$ is an element of PV_ρ if and only if $\sum_{s=1}^{n(\omega)} Y_s^{(\sigma,\omega)} = 0$ for all $\omega \in \ker \pi$ and $d(f\circ L_{\pi(\omega)})_e(\sum_{s=1}^{n(\omega)} Y_s^{(\sigma,\omega)}) = 0$ for all ω with $\pi(\omega)$ parabolic.*

Let $T_s^{(\sigma,\omega)}$, $1 \leqq s \leqq n(\omega)$ $\omega \in \Lambda$, be the linear mapping of $\mathfrak{g}$ onto itself defined by

$$T_s^{(\sigma,\omega)} = \begin{cases} \mathrm{Ad}(\nu_{n(\omega)}) \circ\cdots\circ \mathrm{Ad}(\nu_{s+1}) & \text{if} \quad \nu_s = \sigma_{k(s)} \\ -\mathrm{Ad}(\nu_{n(\omega)})\circ\cdots\circ\mathrm{Ad}(\nu_s) & \text{if} \quad \nu_s = \sigma_{k(s)}^{-1} \end{cases}$$

for s with $1 \leqq s \leqq n(\omega) - 1$ and

$$T_{n(\omega)}^{(\sigma,\omega)} = \begin{cases} \mathrm{id} & \text{if} \quad \nu_{n(\omega)} = \sigma_{k(n(\omega))} \\ -\mathrm{Ad}(\nu_{n(\omega)}) & \text{if} \quad \nu_{n(\omega)} = \sigma_{k(n(\omega))}^{-1} \,, \end{cases}$$

where id is the identity mapping. We set $T^{(\sigma,\omega)}(k) = \sum_{s,k(s)=k} T_s^{(\sigma,\omega)}$. Here $T^{(\sigma,\omega)}(k_0) = 0$ if $k(s) \neq k_0$ for all s. Let $T^{(\sigma,\omega)}$ be the linear mapping of $\mathfrak{g}^N$ into $\mathfrak{g}$ defined by $T^{(\sigma,\omega)} = (T^{(\sigma,\omega)}(1), \cdots, T^{(\sigma,\omega)}(N))$. For $\omega \in \Lambda$ we denote by $S^{(\sigma,\omega)}$ the linear mapping $d(f\circ L_{\pi(\omega)})_e$ of $\mathfrak{g}$ into $\boldsymbol{C}$.

PROPOSITION. *Let Γ be a finitely generated subgroup of G with a system of generators $\sigma = \{\sigma_1, \cdots, \sigma_N\}$ and let Λ be the free group with free generators $\{\lambda_1, \cdots, \lambda_N\}$ with the homomorphism $\pi: \Lambda \to \Gamma$ defined by*

$\pi(\lambda_k)=\sigma_k$. *Then* $PZ^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g})$ *is isomorphic to the subspace* $W=\{X\in\mathfrak{g}^N\colon T^{(\sigma,\omega)}(X)=0$ *for all* $\omega\in\ker\pi$ *and* $S^{(\sigma,\omega)}\circ T^{(\sigma,\omega)}(X)=0$ *for all* ω *with* $\pi(\omega)$ *parabolic}* *of* $\mathfrak{g}^N$.

PROOF. Set $\tilde{z}(\lambda_k)=X_k$ for $\tilde{z}\in Z^1_\rho(\Lambda, \mathfrak{g})$. Let X be a vector obtained by arranging $X_1, \cdots, X_N$ in a column. Then $Z^1_\rho(\Lambda, \mathfrak{g})$ is isomorphic to $\mathfrak{g}^N$ by the mapping $Z^1_\rho(\Lambda, \mathfrak{g})\ni\tilde{z}\mapsto X\in\mathfrak{g}^N$. So we see by Lemma 2 that PV_ρ is isomorphic to W. Since $PZ^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g})$ is isomorphic to PV_ρ, we are done.

Next we represent linear maps $T^{(\sigma,\omega)}$ and $S^{(\sigma,\omega)}$, $\omega\in\Lambda$, by matrices with respect to the basis

$$(*)\qquad \left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}$$

for $\mathfrak{g}$. Let $\sigma_k(t)=(\alpha_k t+\beta_k)/(\gamma_k t+\delta_k)$. Then

$$\mathrm{Ad}(\sigma_k)=\begin{pmatrix}\alpha_k\delta_k+\beta_k\gamma_k & \gamma_k\delta_k & -\alpha_k\beta_k\\ 2\beta_k\delta_k & \delta_k^2 & -\beta_k^2\\ -2\alpha_k\gamma_k & -\gamma_k^2 & \alpha_k^2\end{pmatrix}$$

with respect to this basis. Hence, by the definition of $T^{(\sigma,\omega)}$, we see that $T^{(\sigma,\omega)}$ is a $3\times 3N$ complex matrix and that each entry of this matrix is a polynomial of $\alpha_k\delta_k+\beta_k\gamma_k$, $\gamma_k\delta_k$, $-\alpha_k\beta_k$, $2\beta_k\delta_k$, δ_k^2, $-\beta_k^2$, $-2\alpha_k\gamma_k$, $-\gamma_k^2$, and α_k^2 with $k=1, \cdots, N$. On the other hand, for

$$Y=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\in\mathfrak{g}$$

and $\omega\in\Lambda$ with $\pi(\omega)(t)=(\alpha t+\beta)/(\gamma t+\delta)$, we have

$$\begin{aligned}d(f\circ L_{\pi(\omega)})_e(Y) &= (d/dx)f\circ L_{\pi(\omega)}(p(x))|_{x=0}\\ &= (d/dx)[\mathrm{tr}^2\{\pi(\omega)\circ p(x)\}-4]|_{x=0}\\ &= (d/dx)[\{\alpha\alpha(X)+\beta\gamma(x)+\gamma\beta(x)+\delta\delta(x)\}^2-4]|_{x=0}\\ &= 2(\alpha+\delta)\{(\alpha-\delta)a+\gamma b+\beta c\}\,,\end{aligned}$$

where $p(x)(t)=(\alpha(x)t+\beta(x))/(\gamma(x)t+\delta(x))$ is a path in G satisfying $p(0)=e$ and $(d/dx)p(x)|_{x=0}=Y$. Hence the matrix $S^{(\sigma,\omega)}$ is of the form

$$S^{(\sigma,\omega)}=(2(\alpha^2-\delta^2)\quad 2(\alpha+\delta)\gamma\quad 2(\alpha+\delta)\beta)\,.$$

Since α, β, γ, and δ are some polynomials of $\alpha_1, \beta_1, \gamma_1, \delta_1, \cdots, \alpha_N, \beta_N, \gamma_N, \delta_N$, we see that $S^{(\sigma,\omega)}$ is a 1×3 complex matrix and each entry is a polynomial of $\alpha_1, \beta_1, \gamma_1, \delta_1, \cdots, \alpha_N, \beta_N, \gamma_N, \delta_N$. Note that the matrices $T^{(\sigma,\omega)}$ and $S^{(\sigma,\omega)}$ are independent of the choice of the representative of σ_k.

3. Proof of main theorem. Let $\theta: \Gamma \to G$ be a parabolic homomorphism. We set $\theta(\Gamma) = \Gamma^\theta$ and $\theta(\sigma_k) = \sigma_k(\theta)$. The group Γ^θ is a subgroup of G with a system of generators $\sigma(\theta) = \{\sigma_1(\theta), \cdots, \sigma_N(\theta)\}$. Let $\pi_\theta: \Lambda \to \Gamma^\theta$ be the homomorphism defined by $\pi_\theta(\lambda_k) = \sigma_k(\theta)$. We set $\sigma_k(\theta)(t) = (\alpha_k(\theta)t + \beta_k(\theta))/(\gamma_k(\theta)t + \delta_k(\theta))$.

By Proposition in Section 2, we see that $PZ^1_{\mathrm{Ad}}(\Gamma^\theta, \mathfrak{g})$ is isomorphic to $W(\theta) = \{X \in \mathfrak{g}^N: T^{(\sigma(\theta),\omega)}(X) = 0$ for all $\omega \in \ker \pi_\theta$ and $S^{(\sigma(\theta),\omega)} \circ T^{(\sigma(\theta),\omega)}(X) = 0$ for all ω with $\pi_\theta(\omega)$ parabolic$\}$.

If θ is a parabolic homomorphism sufficiently close to the identity homomorphism and if $\pi(\omega)$ is parabolic, then $\theta(\pi(\omega)) \neq e$ and $\theta(\pi(\omega))$ is parabolic. Thus we have

$$(**) \qquad \ker \pi \subset \ker \pi_\theta \quad \text{and}$$

$$(***) \qquad \{\omega \in \Lambda: \pi(\omega) \text{ parabolic}\} \subset \{\omega \in \Lambda: \pi_\theta(\omega) \text{ parabolic}\}.$$

Since $\mathfrak{g}^N$ is a finite dimensional vector space, there exist finitely many words $\omega_1, \cdots, \omega_K \in \ker \pi$ and $\omega'_1, \cdots, \omega'_M \in \Lambda$ with $\pi(\omega'_j)$ parabolic such that W is the set of common zeros of those linear mappings $T^{(\sigma,\omega)}$ with ω running through ω_i's and $S^{(\sigma,\omega')} \circ T^{(\sigma,\omega')}$ with ω' running through ω'_j's. Also there exist finitely many words $\omega \in \ker \pi_\theta$ and ω' with $\pi_\theta(\omega')$ parabolic such that $W(\theta)$ is the set of common zeros of those finitely many linear mappings $T^{(\sigma(\theta),\omega)}$ and $S^{(\sigma(\theta),\omega')} \circ T^{(\sigma(\theta),\omega')}$. Since the inclusion relations $(**)$ and $(***)$ hold, we may assume, for θ sufficiently close to the identity, that $W(\theta)$ is the set of common zeros of $T^{(\sigma(\theta),\omega)}$ with ω running through $\omega_1, \cdots, \omega_{K+K(\theta)} \in \ker \pi_\theta$ and $S^{(\sigma(\theta),\omega')} \circ T^{(\sigma(\theta),\omega')}$ with ω' running through $\omega'_1, \cdots, \omega'_{M+M(\theta)}$ with $\pi_\theta(\omega'_j)$ parabolic for $1 \leqq j \leqq M + M(\theta)$.

Let T be the linear mapping of $\mathfrak{g}^N$ into $\mathfrak{g}^K \times C^M$ with T obtained by arranging $T^{(\sigma,\omega_1)}, \cdots, T^{(\sigma,\omega_K)}, S^{(\sigma,\omega'_1)} \circ T^{(\sigma,\omega'_1)}, \cdots, S^{(\sigma,\omega'_M)} \circ T^{(\sigma,\omega'_M)}$ in a column. Also let $T(\theta)$ be the linear mapping of $\mathfrak{g}^N$ into $\mathfrak{g}^{K+K(\theta)} \times C^{M+M(\theta)}$ with $T(\theta)$ obtained by arranging $T^{(\sigma(\theta),\omega_1)}, \cdots, T^{(\sigma(\theta),\omega_{K+K(\theta)})}, S^{(\sigma(\theta),\omega'_1)} \circ T^{(\sigma(\theta),\omega'_1)}, \cdots, S^{(\sigma(\theta),\omega'_{M+M(\theta)})} \circ T^{(\sigma(\theta),\omega'_{M+M(\theta)})}$ in a column. Then we have

$$W = \{X \in \mathfrak{g}^N: T(X) = 0\}$$

and

$$W(\theta) = \{X \in \mathfrak{g}^N: T(\theta)(X) = 0\}\,.$$

LEMMA 3. *Let $\theta: \Gamma \to G$ be a parabolic homomorphism which is sufficiently close to the identity homomorphism and let T and $T(\theta)$ be the linear mappings defined as above. Then*

$$\operatorname{rank} T \leqq \operatorname{rank} T(\theta)\,.$$

PROOF. Let $T^{(\sigma,\omega_i)} = (t^i_{mn})_{1 \leqq m \leqq 3, 1 \leqq n \leqq 3N}$ and let $T^{(\sigma(\theta),\omega_i)} = (t^i_{mn}(\theta))_{1 \leqq m \leqq 3, 1 \leqq n \leqq 3N}$

for $i = 1, \cdots, K$ with respect to the basis $(*)$ for $\mathfrak{g}$. Then, by the construction of the matrices $T^{(\sigma,\omega_i)}$ and $T^{(\sigma(\theta),\omega_i)}$, we have

$$t^i_{mn} = P^i_{mn}(\alpha_1, \beta_1, \gamma_1, \delta_1, \cdots, \alpha_N, \beta_N, \gamma_N, \delta_N)$$

and

$$t^i_{mn}(\theta) = P^i_{mn}(\alpha_1(\theta), \beta_1(\theta), \gamma_1(\theta), \delta_1(\theta), \cdots, \alpha_N(\theta), \beta_N(\theta), \gamma_N(\theta), \delta_N(\theta))$$

for polynomials P^i_{mn} in $4N$ variables. Moreover, if

$$S^{(\sigma,\omega'_j)} \circ T^{(\sigma,\omega'_j)} = (r^j_{1n})_{1 \leqq n \leqq 3N}$$

and if

$$S^{(\sigma(\theta),\omega'_j)} \circ T^{(\sigma(\theta),\omega'_j)} = (r^j_{1n}(\theta))_{1 \leqq n \leqq 3N} \quad \text{for} \quad j = 1, \cdots, M$$

with respect to the basis $(*)$ for $\mathfrak{g}$, then

$$r^j_{1n} = \tilde{P}^j_{1n}(\alpha_1, \beta_1, \gamma_1, \delta_1, \cdots, \alpha_N, \beta_N, \gamma_N, \delta_N)$$

and

$$r^j_{1n}(\theta) = \tilde{P}^j_{1n}(\alpha_1(\theta), \beta_1(\theta), \gamma_1(\theta), \delta_1(\theta), \cdots, \alpha_N(\theta), \beta_N(\theta), \gamma_N(\theta), \delta_N(\theta))$$

for polynomials $\tilde{P}^j_{1n}$ in $4N$ variables. If θ is sufficiently close to the identity homomorphism, then $\alpha_k(\theta), \beta_k(\theta), \gamma_k(\theta)$ and $\delta_k(\theta)$ are sufficiently close to $\alpha_k, \beta_k, \gamma_k$ and δ_k, respectively, for $k = 1, \cdots, N$. Hence the complex numbers $t^i_{mn}(\theta)$ and $r^j_{1n}(\theta)$ are sufficiently close to t^i_{mn} and r^j_{1n}, respectively, and we have the required inequality.

By Proposition in Section 2 and Lemma 3 we see that

$$\dim PZ^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g}) = \dim W = \dim \ker T\,,$$
$$\dim PZ^1_{\mathrm{Ad}}(\Gamma^\theta, \mathfrak{g}) = \dim W(\theta) = \dim \ker T(\theta)$$

and

$$\operatorname{rank} T \leqq \operatorname{rank} T(\theta)$$

for a parabolic homomorphism θ which is sufficiently close to the identity homomorphism.

Now we have the following main theorem announced in the introduction.

THEOREM 1. *Let Γ be a finitely generated subgroup of G and let $\theta: \Gamma \to G$ be a parabolic homomorphism. Assume that θ is sufficiently close to the identity homomorphism. Then*

$$\dim PZ^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g}) \geqq \dim PZ^1_{\mathrm{Ad}}(\Gamma^\theta, \mathfrak{g})\,.$$

PROOF. Since $\dim \ker T = 3N - \operatorname{rank} T$ and $\dim \ker T(\theta) = 3N - \operatorname{rank} T(\theta)$, we have $\dim PZ^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g}) = \dim W = \dim \ker T = 3N - \operatorname{rank} T \geqq 3N - \operatorname{rank} T(\theta) = \dim \ker T(\theta) = \dim W(\theta) = \dim PZ^1_{\mathrm{Ad}}(\Gamma^\theta, \mathfrak{g})$.

4. **An application to Kleinian groups.** In the following, we always assume that Γ is a finitely generated Kleinian group with a system of generators $\sigma=\{\sigma_1, \cdots, \sigma_N\}$. We denote by Π the vector space of complex polynomials of degree at most 2. Let $\chi: G \to GL(\Pi)$ be the anti-homomorphism defined by

$$(\chi(g)(v))(t) = v(g(t))(\gamma t + \delta)^2$$

for $v \in \Pi$ and $g \in G$ of the form $g: t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$. Then the space $PZ^1_\chi(\Gamma, \Pi)$ is isomorphic to the space $PZ^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g})$ (see [4]). Let Γ be non-elementary and assume that Γ^θ is a non-elementary Kleinian group. Then $\dim B^1_\chi(\Gamma, \Pi) = \dim B^1_\chi(\Gamma^\theta, \Pi) = 3$ (see [2]). So, if we consider the parabolic cohomology spaces $PH^1_\chi(\Gamma, \Pi) = PZ^1_\chi(\Gamma, \Pi)/B^1_\chi(\Gamma, \Pi)$ and $PH^1_\chi(\Gamma^\theta, \Pi) = PZ^1_\chi(\Gamma^\theta, \Pi)/B^1_\chi(\Gamma^\theta, \Pi)$, we obtain the following by Theorem 1.

THEOREM 2. *Let Γ be a non-elementary finitely generated Kleinian group and let θ be a parabolic homomorphism which is sufficiently close to the identity homomorphism. Assume that Γ^θ is a non-elementary Kleinian group. Then*

$$\dim PH^1_\chi(\Gamma, \Pi) \geqq \dim PH^1_\chi(\Gamma^\theta, \Pi) .$$

Let $(L_\infty(\boldsymbol{C}))_1$ be the open unit ball in $L_\infty(\boldsymbol{C})$, the space of all measurable functions on $\boldsymbol{C}$ such that the essential supremum, $\|\cdot\|_\infty$, is finite. For an element $\mu \in (L_\infty(\boldsymbol{C}))_1$, we denote by w^μ a unique quasi-conformal self-mapping of $\hat{\boldsymbol{C}}$ which fixes $0, 1, \infty$ and satisfies the Beltrami equation

$$\partial w^\mu/\partial \bar{z} = \mu(\partial w^\mu)/\partial z .$$

Such a quasi-conformal mapping w^μ is said to be compatible with Γ if $w^\mu \circ \Gamma \circ (w^\mu)^{-1} \subset G$. Let $B(\Gamma)$ be the space of all $\mu \in (L_\infty(\boldsymbol{C}))_1$ with $w^\mu \circ \Gamma \circ (w^\mu)^{-1} \subset G$. For $\mu \in B(\Gamma)$, we set $w^\mu \circ g \circ (w^\mu)^{-1} = g(\mu) \in G$ for $g \in \Gamma$. Then the mapping $\mu \mapsto g(\mu)$ is a continuous mapping of $B(\Gamma)$ into G with $g(0) = g$. In fact, this mapping is holomorphic (see [1] and [3]). Hence the isomorphism $\theta(\mu): \Gamma \to G$ defined by $\theta(\mu)(g) = g(\mu)$ is close to the identity homomorphism if $\|\mu\|_\infty$ is close to zero. Moreover, $\theta(\mu)$ is a parabolic homomorphism. We denote the group $\theta(\mu)(\Gamma)$ by Γ^μ. If Γ is a non-elementary Kleinian group, then Γ^μ is also a non-elementary Kleinian group. So we have:

COROLLARY. *Let Γ be a non-elementary finitely generated Kleinian group and let w^μ be a quasi-conformal self-mapping of $\hat{\boldsymbol{C}}$ compatible with Γ, where $\|\mu\|_\infty$ is close to zero. Then*

$$\dim PH^1_\chi(\Gamma, \Pi) \geqq \dim PH^1_\chi(\Gamma^\mu, \Pi) .$$

Let $\Omega(\Gamma)$ be the region of discontinuity of a non-elementary finitely generated Kleinian group Γ and let $A(\Omega(\Gamma), \Gamma)$ be the space of bounded holomorphic quadratic forms on $\Omega(\Gamma)$. Let $\beta^*: A(\Omega(\Gamma), \Gamma) \to PH^1_\chi(\Gamma, \Pi)$ be the so-called Bers map with respect to Γ. For a quasi-conformal mapping w^μ compatible with Γ, we have $\dim A(\Omega(\Gamma), \Gamma) = \dim A(\Omega(\Gamma^\mu), \Gamma^\mu)$. So we can prove the following:

THEOREM 3. *Let Γ be a non-elementary finitely generated Kleinian group with $PH^1_\chi(\Gamma, \Pi) = \beta^*(A(\Omega(\Gamma), \Gamma))$ and let w^μ be a quasi-conformal self-mapping of $\hat{C}$ compatible with Γ, where $\|\mu\|_\infty$ is sufficiently close to zero. Then*

$$PH^1_\chi(\Gamma^\mu, \Pi) = \beta(\mu)^*(A(\Omega(\Gamma^\mu), \Gamma^\mu))$$

for the Bers map $\beta(\mu)^*$ with respect to Γ^μ.

PROOF. By Corollary we see that $\dim PH^1_\chi(\Gamma, \Pi) \geqq \dim PH^1_\chi(\Gamma^\mu, \Pi)$. Since $\beta^*(A(\Omega(\Gamma), \Gamma)) = PH^1_\chi(\Gamma, \Pi)$ and since β^* is injective, we have $\dim A(\Omega(\Gamma), \Gamma) = \dim PH^1_\chi(\Gamma, \Pi)$. Moreover, $\beta(\mu)^*: A(\Omega(\Gamma^\mu), \Gamma^\mu) \to PH^1_\chi(\Gamma^\mu, \Pi)$ is also injective. Hence $\dim A(\Omega(\Gamma), \Gamma) = \dim PH^1_\chi(\Gamma, \Pi) \geqq \dim PH^1_\chi(\Gamma^\mu, \Pi) \geqq \dim A(\Omega(\Gamma^\mu), \Gamma^\mu)$. Since $\dim A(\Omega(\Gamma), \Gamma) = \dim A(\Omega(\Gamma^\mu), \Gamma^\mu)$, we have $\dim PH^1_\chi(\Gamma^\mu, \Pi) = \dim A(\Omega(\Gamma^\mu), \Gamma^\mu)$. By the injectivity of $\beta(\mu)^*$ we are done.

By Theorem 1 in [5], we have the following as an immediate consequence of Theorem 3.

COROLLARY. *Under the same hypothesis as in Theorem 3, Γ^μ is quasi-conformally stable.*

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