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ANALYTIC AUTOMORPHISMS OF THE COMPLEMENT OF AN ALGEBRAIC CURVE IN THE COMPLEX PROJECTIVE PLANE

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Introduction. Every analytic automorphism of a compact Riemann surface punctured at a finite number of points is analytically continued to an automorphism of the compact Riemann surface. On the contrary, there are many examples of compact complex analytic surface S and analytic curve A in S such that the complement $S\setminus A$ has an analytic automorphism which cannot be continued to a bimeromorphic transformation of S. Such an analytic automorphism will be called a *transcendental automorphism* of $S\setminus A$ in this paper. By Sakai [5], the logarithmic Kodaira dimension of $S\setminus A$ having a transcendental automorphism is smaller than two. On the other hand, Wakabayashi [8] has given some necessary conditions on algebraic curves A in the complex projective plane $\boldsymbol{P}^2$ under which the logarithmic Kodaira dimension of $\boldsymbol{P}^2\setminus A$ is smaller than two. In this paper, we show that $\boldsymbol{P}^2\setminus A$ having a transcendental automorphism is very special, in the following sense:

A rational function f on a non-singular complex algebraic surface S is called a rational function *of special type* on S if the irreducible components of almost all level curves $f=$ const. of f in $S\setminus\{$the indetermination points of $f\}$ are biholomorphically equivalent to the Gaussian plane $\boldsymbol{C}$ or to the punctured Gaussian plane $\boldsymbol{C}^*=\boldsymbol{C}\setminus\{0\}$. We can present our principal result as follows (see also Theorem 2 in § 4): If $\boldsymbol{P}^2\setminus A$ has a transcendental analytic automorphism and if A is not a non-singular cubic curve, then there exists a rational function f on $\boldsymbol{P}^2$ such that the restriction $f|_{\boldsymbol{P}^2\setminus A}$ to $\boldsymbol{P}^2\setminus A$ is a rational function of special type on $\boldsymbol{P}^2\setminus A$. If, furthermore, A is irreducible, then A is a level curve of the rational function of special type on $\boldsymbol{P}^2$.

Our principle is as follows: If $\boldsymbol{P}^2\setminus A$ has a transcendental automorphism, then there exists a holomorphic mapping φ of the punctured disc into $\boldsymbol{P}^2\setminus A$ with an essential singularity at the origin whose cluster set $\varphi(0;\boldsymbol{P}^2)$ at the origin in $\boldsymbol{P}^2$ is contained in A. After the minimal resolution of the singularities of A and its normally-crossing minimalization (σ_1 in § 2, 2°), we apply a theorem on the cluster sets due to Nishino and Suzuki [4] to our problem (cf. § 2, 1°).

In § 1, we will give some examples of the domains $\boldsymbol{P}^2\setminus A$ having transcendental automorphisms with rational functions of special type on $\boldsymbol{P}^2$. We shall describe the processes of the minimal resolution of the singularities of A and its normally-crossing minimalization in detail in § 2.

§ 3 is the central part of this paper where we present an application of the result due to Nishino and Suzuki (Theorem 1 in § 3).

Our study on the rational functions of special type on $\boldsymbol{P}^2$ will be published elsewhere. Finally, the author would like to express his hearty thanks to Dr. M. Suzuki for his invaluable suggestions.

1. Transcendental automorphisms of the complement of an algebraic curve in $\boldsymbol{P}^2$ and rational functions of special type on $\boldsymbol{P}^2$. Consider a rational function f on a non-singular algebraic surface S which may be non-compact. Let σ be the set of the indetermination points of f. For a complex value a (which may be ∞), an irreducible component of the level curve $\{p \in S\setminus\sigma;\ f(p) = a\}$ is called a *prime curve* of f in this paper (Nishino). We call f a rational function *of type* $\boldsymbol{C}$ on S if almost all prime curves of f are biholomorphically equivalent to $\boldsymbol{C}$, the Gaussian plane. If almost all prime curves of f are biholomorphically equivalent to the punctured Gaussian plane $\boldsymbol{C}^* = \{z \in \boldsymbol{C};\ z \neq 0\}$, then f is called a rational function *of type* $\boldsymbol{C}^*$ on S. The rational function f on S is said to be *of special type* on S if f is either of type $\boldsymbol{C}$ or of type $\boldsymbol{C}^*$ on S.

Now suppose that $S = \boldsymbol{P}^2$, the complex projective plane. Since the complement $\boldsymbol{P}^2\setminus A$ of an algebraic curve A in $\boldsymbol{P}^2$ is a Stein manifold, we obtain, owing to Suzuki [6, Chap. IV, § 8], the following proposition.

PROPOSITION. *Every level curve of a rational function of type $\boldsymbol{C}$ on $\boldsymbol{P}^2$ is irreducible and non-singular in $\boldsymbol{P}^2\setminus\sigma$ and is biholomorphically equivalent to $\boldsymbol{C}$. Each prime curve of a rational function of type $\boldsymbol{C}^*$ on $\boldsymbol{P}^2$ is a non-singular algebraic curve in $\boldsymbol{P}^2\setminus\sigma$ and is biholomorphically equivalent to $\boldsymbol{C}$ or to $\boldsymbol{C}^*$.*

An analytic automorphism T of a non-singular algebraic surface S is called a *transcendental automorphism* of S if T is not algebraic. Consider the complex Euclidean plane $\boldsymbol{C}^2$ with the coordinate (x, y). We proved in our previous paper [1] the following: If there exists a transcendental automorphism of $\boldsymbol{C}^2$ which transforms some algebraic curve in $\boldsymbol{C}^2$ into another, then there exists a polynomial $P(x, y)$ which defines a regular function of special type on $\boldsymbol{C}^2\setminus A$, and conversely.

In [1], we gave a list of typical examples of algebraic curves in $\boldsymbol{C}^2$ which are invariant under transcendental automorphisms of $\boldsymbol{C}^2$, that is, (I), (II), (III) in § 1 and (IV) in § 2 in [1]. These give also the examples of algebraic curves A whose complement $C^2 \setminus A$ have transcendental automorphisms. In order to complete this list of examples, we add the following examples of algebraic curves whose complements have transcendental automorphisms together with examples of rational functions of special type.

EXAMPLE (V). Suppose that m and n are positive integers. The rational function $x^{-m}y^n$ is of type C^* on C^2. Consider an algebraic curve $A = \bigcup_{j=1}^{k}\{x^{-m}y^n = a_j\}$ where each a_j is a complex number. The following transformation is a transcendental automorphism of $C^2 \setminus A$:

$$x' = x \cdot \exp\left(n \sum_{j=1}^{k} (x^{-m}y^n - a_j)^{-1}\right),$$

$$y' = y \cdot \exp\left(m \sum_{j=1}^{k} (x^{-m}y^n - a_j)^{-1}\right).$$

EXAMPLE (VI). Suppose that l, m and n are positive integers and $P_{l-1}(x)$ is a polynomial of degree at most $l-1$ with $P_{l-1}(0) \neq 0$. The rational function $x^{-m}(x^l y + P_{l-1}(x))^n$ is of type C^* on C^2. Consider an algebraic curve $A = \bigcup_{j=1}^{k}\{x^{-m}(x^l y + P_{l-1}(x))^n = a_j\}$ where each a_j is a complex number. The following transformation is a transcendental automorphism of $C^2 \setminus A$:

$$x' = x \cdot H^n,$$

$$y' = y \cdot H^{m-nl} + x^{-l} \cdot H^{-nl}\{P_{l-1}(x) \cdot H^m - P_{l-1}(x \cdot H^n)\},$$

where $H = H(x, y) = \exp(\sum_{j=1}^{k}\{x^{-m}(x^l y + P_{l-1}(x))^n - a_j\}^{-l})$.

Now we consider an algebraic curve A in P^2 whose complement $P^2 \setminus A$ has a transcendental automorphism. When A contains a complex line as its irreducible component, we have listed all the typical cases above ((I) - (VI)). Here we give other examples. Let $(X: Y: Z)$ be a homogeneous coordinate of P^2. An inhomogeneous coordinate of P^2 is given by $x = X/Z$ and $y = Y/Z$.

EXAMPLE (VII). The rational function f on P^2 defined by $f(x, y) = [(y - x^2)(y - x^2 + 2xy^2) + y^5]^2/(y - x^2)^5$ is of type C on P^2. Consider the algebraic curve A defined by $\{[(y - x^2)(y - x^2 + 2xy^2) + y^5] \cdot (y - x^2) = 0\}$. Then $P^2 \setminus A$ is biregularly isomorphic to the product space $C \times C^*$. Hence there exists a transcendental automorphism of $P^2 \setminus A$. The reader will find out the proof of these facts by the performance of the σ-processes which resolves the indetermination points of f (cf. Lemma 2 in § 3). An intrinsic proof of these facts will be given in another paper where we will also treat other rational functions of special type on P^2.

EXAMPLE (VIII). We can also verify that the rational function f on P^2 defined by

$$f(x, y) = [(y - x^2)(y - x^2 + 2xy^2) + y^5](xy - x^3 + y^3)/(y - x^2)^4$$

is of type C^* on P^2 and that $P^2 \setminus A$ is biregularly isomorphic to the product space $C^* \times C^*$, where A is the algebraic curve defined by $\{f(x, y) = 0\} \cup \{f(x, y) = \infty\}$. Hence $P^2 \setminus A$ has transcendental automorphisms.

The algebraic curve defined by $\{(y - x^2)(y - x^2 + 2xy^2) + y^5 = 0\}$ was first noted by Yoshihara [9] and Wakabayashi in their study on the logarithmic Kodaira dimension.

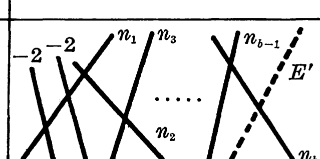
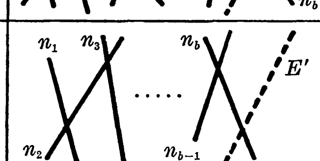
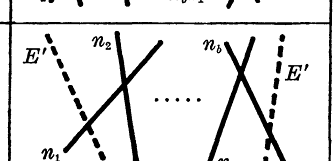
2. **Preliminaries.** 1° *A theorem due to Nishino and Suzuki* [4]. Let S be a non-singular complex analytic surface and let E be a connected analytic curve on S satisfying the following two conditions:

(i) Any singular point of E is an ordinary double point.

(ii) E contains no (compact) exceptional curve of the first kind having at most two intersection with the other irreducible components of E.

Suppose that there is a holomorphic mapping $\varphi: D' \to S \setminus E$ of the punctured disc $D' = \{z \in C;\ 0 < |z| < 1\}$ into $S \setminus E$. Set $G_r = \varphi(D'_r)$ and let $\bar{G}_r$ be the closure of G_r in S, where $D'_r = \{z \in C;\ 0 < |z| < r\}$. The set $\varphi(0; S) = \bigcap_{r>0} \bar{G}_r$ is called the cluster set of φ at $z = 0$ in S. Re-

TABLE I

<table>
<tr><th>Name of type</th><th>Number of points of $C \cap E'$</th><th colspan="2">Explanation of C and E'</th></tr>
<tr><td>$\gamma'(n_1, n_2, \cdots, n_b)$</td><td>1</td><td>
</td><td>$\max\{n_1 + 1, n_2, \cdots, n_b\} \geqq 0$</td></tr>
<tr><td rowspan="2">$\varepsilon(n_1, n_2, \cdots, n_b)$</td><td>1</td><td>
</td><td rowspan="2">$\max\{n_1, n_2, \cdots, n_b\} \geqq 0$</td></tr>
<tr><td>2</td><td>
</td></tr>
</table>

cently, Nishino and Suzuki have given an interesting result as in the following.

THEOREM N-S. *Assume that the cluster set* $C = \varphi(0; S)$ *is contained in* E *and that* C *is a compact set in* S *containing two points at least. Then* C *must consist of irreducible components of* E. *If* $C \neq E$, *then* C *must belong to one of the classes of curves listed in Table* I. *If* $C = E$, *then* C *must belong to one of the classes of curves listed in Table* II, *which we quote from Suzuki* [7] *with notations adjusted.*

In Table I, *each irreducible component of* C *is a non-singular rational curve and is represented by a line. The integer attached to each line represents the self-intersection number of the corresponding irreducible component of* C. *We denote by* E' *the analytic curve consisting of the components of* E *which do not belong to* C, *that is,* E' *is the closure of* $E \setminus C$.

In Table II, *for the types* β_b $(b \geqq 2)$, γ, γ', δ *and* ε, *each irreduible component of* C *is a non-singular rational curve and is represented by a vertex* $\circ$. *Each line represents a point of intersection of irreducible components of* C *corresponding to the vertices. The integer attached to*

TABLE II

	Name of Type	Explication of C	
α	$\alpha(n)$	an irreducible non-singular elliptic curve with the self-intersection number $(C^2) = n \geqq 0$.	
β_1	$\beta(n)$	an irreducible rational curve with only one ordinary double point and $(C^2) = n \geqq 0$.	
β_b	$\beta(n_1, n_2, \cdots, n_b)$ $(b \geqq 2)$	Figure 1,	all $n_i = -2$ or $\max\{n_1, n_2, \cdots, n_b\} \geqq 0$.
γ	$\gamma(n_1, n_2, \cdots, n_b)$ $(b \geqq 1)$	Figure 2,	all $n_i = -2$ or $\max\{n_1+1, n_2, \cdots, n_{b-1}, n_b+1\} \geqq 0$.
γ'	$\gamma'(n_1, n_2, \cdots, n_b)$ $(b \geqq 2)$	Figure 3,	$\max\{n_1+1, n_2, \cdots, n_b\} \geqq 0$.
δ	$\delta(n_0; q_1/l_1, q_2/l_2, q_3/l_3)$	Figure 4,	(i) $n_0 \geqq 2$, (ii) $(l_1, l_2, l_3) = (3, 3, 3), (2, 4, 4)$ or $(2, 3, 6-m)$ with $m = 0, 1, 2, 3$, (iii) for each $i = 1, 2, 3$, (l_i, q_i) is a pair of coprime integers such that $0 < q_i < l_i$ and that $l_i/q_i = n_{i,1} - \underline{1}\rceil n_{i,2} - \cdots - \underline{1}\rceil n_{i,r_i}$ (continued fraction expansion), where $n_{i,j} \geqq 2$ are the integers appearing in Figure 4.
ε	$\varepsilon(n_1, n_2, \cdots, n_b)$ $(b \geqq 1)$	Figure 5,	$\max\{n_1, n_2, \cdots, n_b\} \geqq 0$.

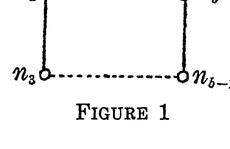


FIGURE 1

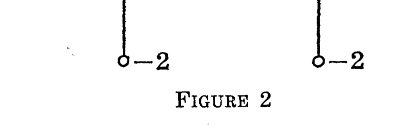


FIGURE 2

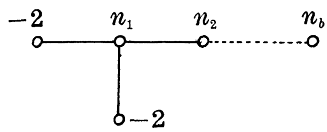


FIGURE 3

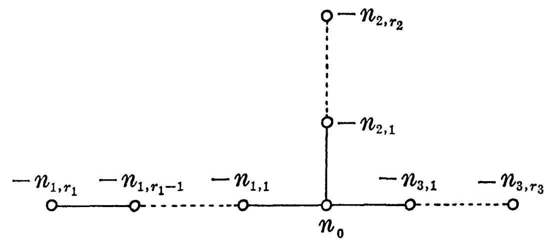


FIGURE 4

FIGURE 5

each vertex is the self-intersection number of the corresponding irreducible component of C.

2° *Non-ordinary singularities of an algebraic curve in* $\boldsymbol{P}^2$. Let A be an algebraic curve in the complex projective plane $\boldsymbol{P}^2$. If a singular point p of A is not an ordinary double point, we call p a non-ordinary singular point of A. Now we describe the processes of the minimal resolution of the non-ordinary singularities of A and its normally-crossing minimalization (σ_1 in the following) in detail to obtain a lemma which plays an important role in the proof of Theorem 1 in § 3.

After the resolution of the non-ordinary singularities of A by a minimal sequence of blowing-up's, we obtain a compact rational surface $\tilde{S}$ and a birational regular mapping $\sigma_0: \tilde{S} \to \boldsymbol{P}^2$ of $\tilde{S}$ onto $\boldsymbol{P}^2$ with the following three properties:

(i) The total preimage $\tilde{E} = \sigma_0^{-1}(A)$ of A is an algebraic curve without a non-ordinary singular point.

(ii) Denote by $p_1, p_2, \cdots, p_m$ the non-ordinary singular points of A. Each irreducible component of the curve $\tilde{B} = \bigcup_{j=1}^{m} \sigma_0^{-1}(p_j)$ is nonsingular and rational. The restriction of the mapping σ_0 to $S \setminus \tilde{B}$ is a biregular mapping of $\tilde{S} \setminus \tilde{B}$ onto $\boldsymbol{P}^2 \setminus \{p_j\}_{j=1}^{m}$.

The self-intersection number of each irreducible component of $\tilde{B}$ is negative. Let $\tilde{B}_j$ be an irreducible component of $\sigma_0^{-1}(p_j)$ with the self-intersection number $(\tilde{B}_j^2) = -1$. $\tilde{B}_j$ intersects at most two other irreducible components of $\tilde{B}$. Since σ_0, the resolution of the non-ordinary singularities of A, is minimal, $\tilde{B}_j$ must intersect the proper transform of A by the mapping σ_0^{-1}. And $\tilde{B}_j$ must intersect other components of E at three points at least. Hence $\tilde{B}$ contains no exceptional curve of the first kind having at most two intersection with the other irreducible components of $\tilde{E}$.

Now, by a finite sequence of the contractions of the exceptional curves of the first kind contained in $\tilde{E}$, we obtain a compact rational surface S and a birational regular mapping $\sigma_1: \tilde{S} \to S$ of $\tilde{S}$ onto S with the following three properties:

(i) The image $E = \sigma_1(\tilde{E})$ of $\tilde{E}$ is an algebraic curve with no non-ordinary singular point and has no exceptional curve of the first kind having at most two intersections with the other irreducible components of E.

(ii) The restriction of the birational mapping $\sigma = \sigma_1 \circ \sigma_0^{-1}: \boldsymbol{P}^2 \to S$ to $\boldsymbol{P}^2 \setminus A$ is a biregular mapping of $\boldsymbol{P}^2 \setminus A$ onto $S \setminus E$.

(iii) Each $B_j = \sigma_1(\tilde{B}_j)$ is a non-singular rational irreducible component of E with the self-intersection number $(B_j^2) \geqq -1$.

We obtain the following lemma easily.

LEMMA 1. (i) *If A has m non-ordinary singular points, then E has, at least, m non-singular rational irreducible components B_j ($j = 1, 2, \cdots, m$) with the self-intersection number $(B_j^2) \geqq -1$.*

(ii) *If there exists an irreducible component E_i of E with the self-intersection number $(E_i^2) \leqq 0$, then A has a non-ordinary singular point.*

(iii) *It is imposible that every irreducible component E_i of E has the self-intersection number $(E_i^2) \leqq -2$.*

(iv) *If an irreducible component $\tilde{E}_i$ of $\tilde{E}$ is an exceptional curve of the first kind having at most two intersections with the other components of $\tilde{E}$, then $\tilde{E}_i$ is an irreducible component of the proper transform of A by the mapping σ_0^{-1}.*

(v) *If the first Betti number $b_1(E)$ of E is equal to zero, then so is $b_1(A)$.*

(vi) *If a level curve of a rational function h on S with the value a is contained in E, the level curve of the rational function $\sigma^*(h)$ on $\boldsymbol{P}^2$ with the value a is contained in A, and conversely.*

The statements (ii) and (iii) are proved because there is no algebraic curve A_1 in $\boldsymbol{P}^2$ with the self-intersection number $(A_1^2) \leqq 0$.

3. An application of Theorem N-S. We prove in this section the following theorem.

THEOREM 1. *Let A be an algebraic curve in the complex projective plane $\boldsymbol{P}^2$. Suppose that there exists a holomorphic mapping ψ of the punctured disc $\Gamma' = \{z \in \boldsymbol{C}; 0 < |z| < 1\}$ into $\boldsymbol{P}^2 \setminus A$ whose cluster set $C' = \psi(0; \boldsymbol{P}^2)$ at $z = 0$ in $\boldsymbol{P}^2$ is contained in A and that C' contains two* *points at least. Then, A must belong to one of the following three classes of curves:*

(i) *A non-singular cubic curve.*

(ii) *The sum of several irreducible components of the level curves of a rational function of special type on $\boldsymbol{P}^2$.*

(iii) *The sum of several irreducible components of the level curves of a rational function f of type $\boldsymbol{C}$ on $\boldsymbol{P}^2$ and an irreducible algebraic curve A_0 such that the restriction $f|_{P^2\setminus A}$ of f on $\boldsymbol{P}^2\setminus A_0$ is a rational function of type $\boldsymbol{C}^*$.*

Unfortunately, we do not know whether the case (i) actually takes place.

Before beginning on the proof, we introduce a lemma due to Kodaira and Spencer [2] which played an important role in our previous paper [1].

LEMMA 2. *Let S be a non-singular compact complex algebraic surface. If an irreducible non-singular rational curve C_0 in S has the self-intersection number $(C_0^2) = 0$, then there exists a holomorphic mapping π of S onto a compact Riemann surface R such that the triple $\mathfrak{F} = (S, \pi, R)$ is an analytic family of rational curves over R having C_0 as its regular fibre.*

REMARKS. Under the same assumption as in Lemma 2, we know the following:

(i) If S is rational, then $R = \boldsymbol{P}^1$, the Riemann sphere.

(ii) An algebraic curve C_1 in S with $C_0 \cap C_1 = \emptyset$ is the sum of several irreducible components of fibres of $\mathfrak{F}$.

Now we start on the proof of Theorem 1. We consider the compact rational surfaces $\tilde{S}$ and S as in §2. The holomorphic mapping $\varphi = \sigma\circ\psi$ of Γ' into $S\setminus E$ satisfies the condition in Theorem N-S in §2. In the following, we denote by C_i the irreducible component of $C = \varphi(0; S)$ with the self-intersection number n_i in Tables I and II. First we suppose that $C \neq E$. Then C must belong to one of the classes listed in Table I.

(1) *The case γ'.* In this case, $\max\{n_1 + 1, n_2, \cdots, n_b\} \geqq 0$.

(i) Suppose that $n_j \geqq 0$ for some $j \neq 1$. Assume, furthermore, that $n_j = (C_j^2) = 0$. By Lemma 2 and by Remark (i) to Lemma 2, there exists a rational function h on S with no indetermination point each prime curve of which is a non-singular rational curve and whose level curve with the value 0 is C_j. By virtue of the graph, we know that C_j must intersect two other irreducible components, which we denote by E_1 and E_2, of E. By Remark (ii) to Lemma 2, the other irreducible components of E are prime curves of h. Since C_j intersects E_1, as well as E_2, transversally at a point, $h|_{S\setminus E}$ is a rational function of type C^* on $S\setminus E$. Since every rational function on P^2 has one indetermination point at least, the proper transform A_1 of E_1 by the mapping σ^{-1} or the proper transform A_2 of E_2 by the mapping σ^{-1} is a point. If each of them is a point, then the rational function $f=\sigma^*(h)$ is a rational function of type C^* on P^2 and A consists of several prime curves of f. If A_2 is not a point, then f is a rational function of type C on P^2 whose restriction $f|_{P^2\setminus A_2}$ to $P^2\setminus A_2$ is a rational function of type C^* on $P^2\setminus A_2$. By Lemma 1 (vi), the level curve of f with the value 0 is contained in A. Because A contains A_2 and $\{f=0\}$, A is reducible.

When $n_j>0$, we obtain, after n_j blowing-up's at the point of intersection $C_j\cap E_1$, the proper transform $\tilde{C}_j$ of C_j with the self-intersection number $(\tilde{C}_j^2)=0$. Hence the same result as in the above follows.

(ii) Suppose that $n_1\geqq -1$. Assume, furthermore, $n_1=-1$. We denote by C_0 and C_0' two irreducible components of C with the self-intersection number -2 on the left hand side of the graph. Contracting C_1 and C_0', we obtain a birational regular mapping $\sigma_2\colon S\to\sigma_2(S)$ of S onto $\sigma_2(S)$ so that the image curve $\sigma_2(C_0)$ of C_0 has the self-intersection number $(\sigma_2(C_0)^2)=0$. By Lemma 2, there exists a rational function h on $\sigma_2(S)$ with no indetermination point each prime curve of which is a non-singular rational curve and whose level curve with the value 0 is $\sigma_2(C_0)$. Since every rational function on P^2 has one indetermination point at least, $\sigma_2(E)$ has an irreducible component I intersecting $\sigma_2(C_0)$ tangentially at a point with the intersection number $(\sigma_2(C_0),I)=2$. By Remark (ii) to Lemma 2, we know that the components of E other than I are prime curves of h. Hence the rational function $h|_{S\setminus E}$ is a rational function of type C^* on $S\setminus E$. The rational function $f=\sigma^*(h)$ is a rational function of type C^* on P^2 with only one indetermination point. The algebraic curve A consists of several prime curves of f.

When $n_1>-1$, we obtain, after the n_1+1 blowing-up's at the intersection of C_1 with the closure of $E\setminus(C_0\cup C_0'\cup C_1)$, the proper transform $\tilde{C}_1$ of C_1 with the self-intersection number $(\tilde{C}_1^2)=-1$. Hence the same result as in the above follows.

(2) *The case* ε. In this case, $(C_j^2)\geqq 0$ for some j. Suppose that $j=1$ and that the cardinality of $C\cap E'$ is one. Assume, furthermore, $(C_1^2)=0$. Then, by Lemma 2, there exists a rational function h on S with no indetermination point each prime curve of which is a non-singular rational curve and whose level curve with the value 0 is C_1. The curve C_1 intersects another irreducible component E_1 of E transversally. By Remark (ii) to Lemma 2, the irreducible components of E other than

E_1 are prime curves of h. Hence the restriction $h|_{S\setminus E}$ to $S\setminus E$ is a rational function of type C on $S\setminus E$. We know that $f=\sigma^*(h)$ is a rational function of type C on P^2. The curve A consists of several prime curves of f.

If $n_1=(C_1^2)>0$, we obtain, after n_1 blowing-up's at the point of intersection $C_1\cap E_1$, the proper transform $\tilde{C}_1$ of C_1 with the self-intersection number $(\tilde{C}_1^2)=0$. Hence the same reasoning as above is applicable, and we obtain the same result as in the above.

In the other case where $j\neq 1$, we can apply the same reasoning as in (1) (i) to obtain the same result as in (1) (i).

Now we suppose that $C=E$. Then C must belong to one of the classes of curves listed in Table II of Theorem N-S.

(3) *The case* α. By Lemma 1 (i), A has no non-ordinary singular point. Since $\sigma_0=\sigma_1=\text{id.}$, A is a non-singular cubic curve in P^2.

(4) *The case* β_1. If A has no non-ordinary singular point, then A is an irreducible cubic with only one ordinary double point. Hence A is transformed by a projective transformation of P^2 into the curve $\{XYZ-X^3+Y^3=0\}$, a prime curve of the rational function of type C^* on P^2 mentioned in §1 (Example (VIII)). If A has a non-ordinary singular point, by Lemma 1 (i), A has only one non-ordinary singular point p_1 and $E=B_1$. As was seen in §1, $\tilde{B}_1$ intersects the other irreducible components of $\tilde{E}$ at more than two points and the self-intersection number $(\tilde{B}_1^2)$ of $\tilde{B}_1$ is -1. Hence we know that the self-intersection number $n=(B_1^2)$ is not smaller than 4. After $n-3$ blowing-up's at the ordinary double point of B_1, we obtain the proper transform B_1' of B_1 with the self-intersection number $(B_1'^2)=0$. Therefore, we can apply the same reasoning as in (1) (i) to obtain the same result as in (1) (i).

(5) *The case* β_b. By Lemma 1 (iii), the case with $n_i=-2$ for all i does not take place. Therefore, for some j, the self-intersection number (C_j^2) of C_j is not negative. Hence the same reasoning as in (1) (i) is applicable, and we obtain the same result as in (1) (i).

(6) *The case* γ. By Lemma 1 (iii), the case with $n_i=-2$ for all i does not take place. Suppose that $b>1$. Then the same result as in (1) follows by the same reasoning as in (1). Suppose that $b=1$. By Lemma 1 (ii) and (i), A has only one non-ordinary singular point p_1. Assume that A is irreducible. As was seen in §2, $\tilde{B}_1$ intersects the proper transform $\tilde{A}$ of A by the mapping σ_0^{-1} and intersects at most two other components of $\tilde{B}$. By Lemma 1 (v), the curve $\tilde{A}$ intersects $\tilde{B}_1$ at only one point. On the other hand, B_1 intersects the other compo-

nents of E at four points. It is a contradiction. Hence A is reducible. Let $A_1, A_2, \cdots, A_k$ ($k \geqq 2$) be the irreducible components of A. By Lemma 1 (v), all A_i must intersect each other at only one point p_0. It is easy to verify this fact when $k = 2$. When $k \geqq 3$, it is proved by the fact that any two algebraic curve in $\boldsymbol{P}^2$ must intersect each other. There exists a rational function g with one indetermination point p_0 whose level curve with the value 0 is A_1 and whose level curve with the value ∞ is A_2. Consider the rational function $h = (\sigma^{-1})^*(g)$ on S. By Lemma 1 (iv), the level curve Σ_1 of h with the value 0 and the level curve Σ_2 of h with the value ∞ are contained in E. The curves Σ_1 and Σ_2 must be connected, respectively. Hence, looking at the graph of C, we know that Σ_1 or Σ_2 must be exceptional. It is absurd. Hence the case with $b = 1$ does not take place.

(7) *The case* γ'. By virtue of the shape of the graph of C, the same reasoning as in (1) and (2) are applicable to this case. Hence we can prove the statement of Theorem 1 in this case also.

(8) *The case* δ. We denote by $C_{i,j}$ and C_0 the irreducible components of C with the self-intersection number $n_{i,j}$ and n_0, respectively. By Lemma 1 (ii) and (i), we know that A has only one non-ordinary singular point p_1 and that $C_0 = B_1$. Suppose that A is irreducible. By Lemma 1 (v), the curve $\tilde{B}_1$ on $\tilde{S}$ intersects the proper transform $\tilde{A}$ of A by the mapping σ_0^{-1} at only one point. As was seen in § 2, $\tilde{B}_1$ intersects at most two other components of $\tilde{B}$. Since C_0 intersects the other components of C at three points, by Lemma 1 (iv), we know that $\sigma_1 = \mathrm{id.}$ and that the self-intersection number $(C_0^2) = -1$. Hence we obtain A from C by the successive contraction of exceptional curves of the first kind. But we obtain an irreducible algebraic curve only in the cases where $(q_1/l_1, q_2/l_2, q_3/l_3) = (1/2, 1/3, 1/6-k)$ ($k = 0, 1, 2, 3$) or $(q_1/l_1, q_2/l_2, q_3/l_3) = (1/2, 1/3, 2/5)$. In the former case, the self-intersection number of the curve so obtained equals k. In the latter case, it equals 7. Since the self-intersection number of a plane algebraic curve of degree n is n^2, it is a contradiction. Therefore, A must be reducible. But, if A is reducible, the same reasoning as in (6) leads to a contradiction. Hence the case δ does not take place.

(9) *The case* ε. Since every rational function on $\boldsymbol{P}^2$ has at least one indetermination point, it is impossible that $b = 1$ and $n_1 = 0$. Hence we can apply the same reasoning as in (2) to this case to obtain the same result as in (2).

Thus we have proved Theorem 1.

4. **Conclusions.** As was seen in Introduction, Theorem 1 implies

the following.

THEOREM 2. *Consider an algebraic curve A in the complex projective plane $\boldsymbol{P}^2$. If there exists a transcendental automorphism of the complement $\boldsymbol{P}^2 \setminus A$, then A must belong to one of the three classes of curves* (i), (ii) *and* (iii) *described in Theorem* 1.

If A belongs to the class (iii), A must contain irreducible components other than A_0. By Proposition in § 1, we obtain the following.

COROLLARY 1. *Under the same assumption as in Theorem* 1, *consider an irreducible component A' of A. If A' is not the curve A_0 in* (iii) *of Theorem* 1, *the universal covering of $A' \setminus \{$the singular points of A on $A'\}$ is biholomorphically equivalent to the Riemann sphere $\boldsymbol{P}^1$ or to the Gaussian plane $\boldsymbol{C}$.*

REMARK. By this corollary, we know that there exists an example of an algebraic curve A in $\boldsymbol{P}^2$ such that $\boldsymbol{P}^2 \setminus A$ has no transcendental automorphism and that the logarithmic Kodaira dimension $\bar{\kappa}(\boldsymbol{P}^2 \setminus A) \leqq 1$.

Since the degenerating locus of a birational transformation of $\boldsymbol{P}^2$ consists of rational curves, we obtain the following easily.

COROLLARY 2. *There exists an algebraic curve in $\boldsymbol{P}^2$ whose complement has no analytic automorphism besides the identity.*

We do not know whether the complement $\boldsymbol{P}^2 \setminus E$ of a non-singular cubic curve E in $\boldsymbol{P}^2$ has a transcendental automorphism.

Combining Theorem 2 with the result of Suzuki [6, Chap. III], we obtain the following theorem (see also [1]).

THEOREM 3. *Let A be an algebraic curve in the complex Euclidean plane $\boldsymbol{C}^2$. If $\boldsymbol{C}^2 \setminus A$ has a transcendental automorphism, then A can be transformed into an algebraic curve A_0 belonging to one of the following three classes of curves in $\boldsymbol{C}^2$ by an algebraic automorphism of $\boldsymbol{C}^2$:*

(i) *An algebraic curve $P(x) + Q(x) \cdot y = 0$ where $P(x)$ and $Q(x)$ are polynomials of x.*

(ii) *The sum of several prime curves of a rational function $x^m y^n$ where $m \in \boldsymbol{Z}^* = \boldsymbol{Z} \setminus \{0\}$ and $n \in \boldsymbol{Z}^*$.*

(iii) *The sum of several prime curves of a rational function $x^m[x^l y + P_{l-1}(x)]^n$, where $l \in \boldsymbol{Z}^+ = \{k \in \boldsymbol{Z}; k > 0\}$, $m \in \boldsymbol{Z}^*$ and $n \in \boldsymbol{Z}^*$ and where $P_{l-1}(x)$ is a polynomial of x of degree at most $l-1$ with $P_{l-1}(0) \neq 0$.*

Conversely, if an algebraic curve A in $\boldsymbol{C}^2$ belongs to one of the classes (i), (ii) *and* (iii), *then $\boldsymbol{C}^2 \setminus A$ has a transcendental automorphism* (*see* § 1).

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