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COMPACT TRANSFORMATION GROUPS ON RATIONAL COHOMOLOGY CAYLEY PROJECTIVE PLANES

KOICHI IWATA

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0. Introduction. Let M be a compact simply connected 16-dimensional differentiable manifold whose rational cohomology ring is isomorphic to that of the Cayley projective plane $P(\boldsymbol{Cay})$, that is,

$$H^*(M; \boldsymbol{Q}) \cong \boldsymbol{Q}[u]/(u^3) \qquad \deg u = 8 ,$$

and G be a compact connected Lie group which acts on M differentiably. We say that a pair (G, M) is isomorphic to (G', M'), if there exist a Lie group isomorphism $h: G \to G'$ and a diffeomorphism $f: M \to M'$ satisfying

$$f(gx) = h(g)f(x) ,$$

for every $g \in G$ and for every $x \in M$. A G-action on M induces an effective G/H-action on M, where H is the intersection of all isotropy groups. We say that (G, M) is essentially isomorphic to (G', M'), if there exists an isomorphism between the induced pairs with effective actions $(G/H, M)$ and $(G'/H', M')$. In this paper, we shall prove the following theorems.

THEOREM I. *Suppose that G acts on M with a codimension one orbit. Then, (G, M) is essentially isomorphic to*

$$(\boldsymbol{Spin}(9), \boldsymbol{F}_4/\boldsymbol{Spin}(9)) , \qquad (\boldsymbol{Sp}(3), \boldsymbol{F}_4/\boldsymbol{Spin}(9)) ,$$
$$(\boldsymbol{Sp}(3) \times \boldsymbol{U}(1), \boldsymbol{F}_4/\boldsymbol{Spin}(9)) \quad or \quad (\boldsymbol{Sp}(3) \times \boldsymbol{Sp}(1), \boldsymbol{F}_4/\boldsymbol{Spin}(9)) ,$$

described in §1, *Examples* 1 *and* 3.

THEOREM II. *Every G-action on M with codimension two principal orbits has at least two isolated singular orbits.*

In §1, Example 2, we give one more example of G-actions with codimension two principal orbits and three isolated singular orbits. We do not know any other examples of G-actions on M with codimension two principal orbits. After cohomological preliminaries in §2, we prove Theorem I in §3 and Theorem II in §4.

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1. Some group actions on Cayley projective planes. We observe some examples of group actions on Cayley projective planes. Let $\mathfrak{J}$ be the set of all 3×3 Hermitian matrices over the Cayley number field ***Cay***. It is a 27-dimensional $\boldsymbol{R}$-module with respect to the matrix sum and the scalar multiplication. A matrix $X \in \mathfrak{J}$ has the form

$$X = X(\xi, u) = \begin{pmatrix} \xi_1 & u_3 & \bar{u}_2 \\ \bar{u}_3 & \xi_2 & u_1 \\ u_2 & \bar{u}_1 & \xi_3 \end{pmatrix},$$

where $\xi_1, \xi_2, \xi_3 \in \boldsymbol{R}$ and $u_1, u_2, u_3 \in \boldsymbol{Cay}$. Let

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$F_1^u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & \bar{u} & 0 \end{pmatrix}, \quad F_2^u = \begin{pmatrix} 0 & 0 & \bar{u} \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}, \quad F_3^u = \begin{pmatrix} 0 & u & 0 \\ \bar{u} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, the set $\{E_1, E_2, E_3, F_1^{e_i}, F_2^{e_i}, F_3^{e_i},\ i = 0, 1, \cdots, 7\}$ constitutes an $\boldsymbol{R}$-basis of $\mathfrak{J}$. Here, $\{e_i, i = 0, 1, \cdots, 7\}$ is the standard basis of ***Cay***. The Jordan product $\circ$ is defined in $\mathfrak{J}$ by

$$X \circ Y = (XY + YX)/2\,, \qquad X, Y \in \mathfrak{J}\,.$$

An $\boldsymbol{R}$-isomorphism $x: \mathfrak{J} \to \mathfrak{J}$ is called an automorphism of $\mathfrak{J}$, when

$$x(X \circ Y) = xX \circ xY\,,$$

for all $X, Y \in \mathfrak{J}$. It is well known that the group of automorphisms of $\mathfrak{J}$ is the exceptional Lie group $\boldsymbol{F}_4$. The Cayley projective plane $P(\boldsymbol{Cay})$, defined by

$$\{X \in \mathfrak{J} \mid X \circ X = X,\ \text{trace}\, X = 1\}\,,$$

is identified with the left coset space $\boldsymbol{F}_4/\boldsymbol{Spin}(9)$, where

$$\boldsymbol{Spin}(9) = \{x \in \boldsymbol{F}_4 \mid xE_1 = E_1\}\,.$$

$\boldsymbol{Spin}(9)$ contains

$$\boldsymbol{Spin}(8) = \{x \in \boldsymbol{F}_4 \mid xE_i = E_i,\ i = 1, 2, 3\}$$

and $\boldsymbol{Spin}(8)$ contains

$$\boldsymbol{Spin}(7) = \{x \in \boldsymbol{Spin}(8) \mid xF_3^1 = F_3^1\}\,.$$

We can find detailed accounts on ***Cay***, the Lie group $\boldsymbol{F}_4$ and its subgroups in elaborate papers [6], [7].

EXAMPLE 1. *The natural* $\boldsymbol{Spin}(9)$*-action on* $P(\boldsymbol{Cay})$. Let

$$\mu\colon \boldsymbol{Spin}(9) \times P(\boldsymbol{Cay}) \to P(\boldsymbol{Cay})$$

be the natural $\boldsymbol{Spin}(9)$-action (that is, $\boldsymbol{Spin}(9)$-action through the inclusion $\boldsymbol{Spin}(9) \subset \boldsymbol{F}_4$) on $P(\boldsymbol{Cay})$. Define for a fixed s, $0 \leqq s \leqq 1$,

$$A_s = \{X(\xi, u) \in P(\boldsymbol{Cay}) \mid \xi_1 = s\} .$$

We can show that μ is transitive on A_s for any s and

(i) $A_1 = \{E_1\}$ is a fixed point,

(ii) A_0 is an 8-dimensional sphere. The isotropy group at $E_2 \in A_0$ is $\boldsymbol{Spin}(8)$.

(iii) For each s, $0<s<1$, A_s is a 15-dimensional sphere. The isotropy group at $(E_1 + E_2 + F_3^1)/2 \in A_{1/2}$ is $\boldsymbol{Spin}(7)$.

EXAMPLE 2. *The natural* $\boldsymbol{Spin}(8)$*-action on* $P(\boldsymbol{Cay})$. For any $x \in \boldsymbol{Spin}(8)$, there exists a triple

$$(x_1, x_2, x_3) \in \boldsymbol{SO}(8) \times \boldsymbol{SO}(8) \times \boldsymbol{SO}(8) ,$$

satisfying

$$x_1 u x_2 v = \overline{x_3 \overline{uv}} ,$$

for all $u, v \in \boldsymbol{Cay}$. In fact, x_i is determined by

$$x F_i^u = F_i^{x_i u} , \qquad u \in \boldsymbol{Cay} , \qquad i = 1, 2, 3 .$$

Then, the natural $\boldsymbol{Spin}(8)$-action $\mu' = \mu \mid \boldsymbol{Spin}(8) \times P(\boldsymbol{Cay})$ on $P(\boldsymbol{Cay})$ is given by

$$\mu'\left(x, \begin{pmatrix} \xi_1 & u_3 & \bar{u}_2 \\ \bar{u}_3 & \xi_2 & u_1 \\ u_2 & \bar{u}_1 & \xi_3 \end{pmatrix}\right) = \begin{pmatrix} \xi_1 & x_3 u_3 & \overline{x_2 u_2} \\ \overline{x_3 u_3} & \xi_2 & x_1 u_1 \\ x_2 u_2 & \overline{x_1 u_1} & \xi_3 \end{pmatrix} .$$

We can see easily the following:

(i) E_1, E_2 and E_3 are fixed points.

(ii) For each s, $0 < s < 1$, μ' is transitive on 7-spheres:

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & s & u_1 \\ 0 & \bar{u}_1 & 1-s \end{pmatrix} \,\middle|\, |u_1|^2 = s(1-s)\right\} ,$$

$$\left\{\begin{pmatrix} 1-s & 0 & \bar{u}_2 \\ 0 & 0 & 0 \\ u_2 & 0 & s \end{pmatrix} \,\middle|\, |u_2|^2 = s(1-s)\right\} ,$$

$$\left\{\begin{pmatrix} s & u_3 & 0 \\ \bar{u}_3 & 1-s & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, |u_3|^2 = s(1-s)\right\} .$$

(iii) For fixed s, t, $0 < s < 1$, $0 < t < 1$, $0 < 1 - s - t < 1$, μ' is transitive on

$$\left\{\begin{pmatrix} s & u_3 & \bar{u}_2 \\ \bar{u}_3 & t & u_1 \\ u_2 & \bar{u}_1 & 1-s-t \end{pmatrix}\right\} = S^7 \times S^7 .$$

The isotropy group at $(E_1 + E_2 + E_3 + F_1^1 + F_2^1 + F_3^1)/3$ is G_2.

EXAMPLE 3. $\boldsymbol{Sp}(3) \times \boldsymbol{Sp}(1)$*-action on* $P(\boldsymbol{Cay})$. By regarding the quaternion number field $\boldsymbol{H}$ as the subalgebra of $\boldsymbol{Cay}$ spanned by $\{e_0, e_1, e_2, e_3\}$, we can consider that any element of $\boldsymbol{Cay}$ has the form

$$a + be_4 \qquad a, b \in \boldsymbol{H} .$$

Then, every matrix $X \in \mathfrak{J}$ can be written as follows:

$$X = X_H + F(be_4) ,$$

where

$$X_H = \begin{pmatrix} \xi_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \xi_2 & a_1 \\ a_2 & \bar{a}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i \in \boldsymbol{R} , \qquad a_i \in \boldsymbol{H} ,$$

$$F(be_4) = \begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix} e_4 , \qquad b = (b_1, b_2, b_3) \in \boldsymbol{H} \times \boldsymbol{H} \times \boldsymbol{H} .$$

Yokota [7, § 4] shows that $\boldsymbol{Sp}(3) \times \boldsymbol{Sp}(1)/\boldsymbol{Z}_2$ is isomorphic to a compact subgroup of $\boldsymbol{F}_4$ by a map φ: $\boldsymbol{Sp}(3) \times \boldsymbol{Sp}(1) \to \boldsymbol{F}_4$, defined by

$$\varphi(A, p)(X_H + F(be_4)) = AX_H A^* + F((pbA^*)e_4) , \qquad A \in \boldsymbol{Sp}(3) , \quad p \in \boldsymbol{Sp}(1) .$$

Here, A^* denotes the transpose conjugate of A.

Now, observe the $\boldsymbol{Sp}(3) \times \boldsymbol{Sp}(1)$-action on $\boldsymbol{F}_4/\boldsymbol{Spin}(9)$ induced by φ. Let $X(t)$, $1/2 \leqq t \leqq 1$, be a matrix of $\mathfrak{J}$, given by

$$\begin{pmatrix} t & \{t(1-t)\}^{1/2}e_4 & 0 \\ -\{t(1-t)\}^{1/2}e_4 & 1-t & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} t & 0 & 0 \\ 0 & 1-t & 0 \\ 0 & 0 & 0 \end{pmatrix} + F((0, 0, \{t(1-t)\}^{1/2})e_4) .$$

We can see the following:

(i) The isotropy group at $X(1)$ is

$$\left\{\left(\begin{array}{c|c} a & \\ \hline & Y \end{array}\right) \times p \,\middle|\, Y \in \boldsymbol{Sp}(2),\ a, p \in \boldsymbol{Sp}(1)\right\} \cong \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(2) \times \boldsymbol{Sp}(1) .$$

The orbit through $X(1)$ is diffeomorphic to $P_2(\boldsymbol{H})$.

(ii) The isotropy group at $X(1/2)$ is

$$\left\{\left(\begin{array}{c|c} Y & \\ \hline & p \end{array}\right) \times p \,\middle|\, Y \in \boldsymbol{Sp}(2),\ p \in \boldsymbol{Sp}(1)\right\} \cong \boldsymbol{Sp}(2) \times \boldsymbol{Sp}(1)\ .$$

The orbit through $X(1/2)$ is diffeomorphic to S^{11}.

(iii) The isotropy group at $X(t)$, $1/2 < t < 1$, is

$$\left\{\begin{pmatrix} a & & \\ & b & \\ & & p \end{pmatrix} \times p \,\middle|\, a, b, p \in \boldsymbol{Sp}(1)\right\} \cong \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1)\ .$$

The orbit through $X(t)$ is 15-dimensional.

2. **Cohomology of orbits.** 2.1. Suppose that M is a compact simply connected 16-dimensional differentiable manifold, satisfying

$$H^*(M; \boldsymbol{Q}) \cong \boldsymbol{Q}[u]/(u^3)\ , \qquad \deg u = 8\ .$$

We call such a manifold a compact rational cohomology Cayley projective plane. Let M_1, M_2 be 16-dimensional compact connected differentiable submanifolds of M, such that

$$M_1 \cup M_2 = M \quad \text{and} \quad M_1 \cap M_2 = \partial M_1 = \partial M_2\ .$$

Let

$$f_s^*: H^*(M; \boldsymbol{Q}) \to H^*(M_s; \boldsymbol{Q}) \qquad (s = 1, 2)$$

be the homomorphism induced by the inclusion $f_s: M_s \subset M$. Considering the cohomology exact sequence of (M, M_s), we obtain

$$(1) \qquad P(M_{3-s}, \partial M_{3-s}; t) - tP(M_s; t) = P(\ker f_s^*; t) - tP(\operatorname{im} f_s^*; t)\ .$$

Using this and the Poincaré duality for M_s:

$$(2) \qquad P(M_s, \partial M_s; t) = t^{16} P(M_s; t^{-1})\ ,$$

we have the following lemma in the same way as in [2, Lemma 2.1.1].

LEMMA 1. *Let n_s be the non-negative integer, such that*

$$f_s^*(u^{n_s}) \neq 0 \quad \textit{and} \quad f_s^*(u^{n_s+1}) = 0\ .$$

Then we have $n_1 + n_2 = 1$.

Now, assume that a compact connected Lie group G acts on M differentiably with a codimension 1 orbit G/K. Then, by [2, Lemma 1.2.1], G/K is a principal orbit and there are just two singular orbits G/K_1, G/K_2. We can assume that $K \subset K_1 \cap K_2$ and that there is a closed invariant tubular neighborhood M_s of G/K_s in M, such that

$$M = M_1 \cup M_2 , \qquad M_1 \cap M_2 = \partial M_1 = \partial M_2 = G/K .$$

Let

$$k_s = 16 - \dim G/K_s \qquad (s = 1, 2) .$$

Then

$$2 \leqq k_s \leqq 16 - 8n_s$$

and we have:

LEMMA 2 ([2, Lemma 2.2.3]). *If $k_2 > 2$, then G/K_1 is simply connected and hence K_1 is connected.*

Our aim of this section is to prove:

PROPOSITION 1. *The two singular orbits G/K_1, G/K_2 are orientable and their Poincaré polynomials are either*

$$\begin{cases} P(G/K_s; t) = 1 + t^8 , \\ P(G/K_{3-s}; t) = 1 , \end{cases}$$

or

$$\begin{cases} P(G/K_s; t) = 1 + t^4 + t^8 , \\ P(G/K_{3-s}; t) = 1 + t^{11} , \end{cases}$$

for $s = 1, 2$.

2.2. PROOF OF PROPOSITION 1. Without loss of generality, we can assume that $n_1 = 1$ and $n_2 = 0$. Then, (1) turns to

$$P(M_1, \partial M_1; t) - tP(M_2; t) = t^8 + t^{16} - t , \tag{3}$$

$$P(M_2, \partial M_2; t) - tP(M_1; t) = t^{16} - t(1 + t^8) . \tag{4}$$

Note that if G/K_s is orientable, we have

$$P(M_s, \partial M_s; t) = t^{k_s} P(G/K_s; t) \tag{5}$$

by the Thom isomorphism.

(a) First, suppose that both G/K_1 and G/K_2 are orientable. Then, from the above formulas it follows that

$$(1 - t^{k_1+k_2-2})P(G/K_1; t) = t^{k_2-1}(1 - t^7 - t^{15}) + 1 + t^8 - t^{15} , \tag{6}$$

$$(1 - t^{k_1+k_2-2})P(G/K_2; t) = t^{k_1-1}(1 + t^8 - t^{15}) + 1 - t^7 - t^{15} . \tag{7}$$

(i) *The case $k_1 \equiv k_2 \bmod 2$.* By (6), k_2 is even and both sides of (6) are divisible by $1 - t^2$. Hence we have

$$\begin{aligned}(1 + t^2 + \cdots + t^{k_1+k_2-4})&P(G/K_1; t)\\ &= (1 + t + \cdots + t^{14})(1 - t + t^2 - t^3 + \cdots + t^{k_2-2})\\ &\qquad + t^8(1 + t^2 + \cdots + t^{k_2-4}) .\end{aligned}$$

Therefore $\chi(G/K_1) \neq 0$. This implies that $P(G/K_1; t)$ is an even function and $k_2 = 16$. Similarly, $k_1 = 8$ follows from (7). Thus we have

$$\begin{cases} P(G/K_1; t) = 1 + t^8 , \\ P(G/K_2; t) = 1 . \end{cases}$$

(ii) *The case* $k_1 \not\equiv k_2 \bmod 2$. 1°. If k_1 is even and k_2 is odd, then by (6), $\chi(G/K_1) = 3$ and hence $P(G/K_1; t)$ is an even function. Therefore, we obtain from (6)

$$\begin{cases} P(G/K_1; t) = t^{k_2-1} + 1 + t^8 , \\ t^{k_1+k_2-2}P(G/K_1; t) = t^{k_2-1}(t^7 + t^{15}) + t^{15} . \end{cases}$$

Since $k_2 \leqq 16$, it follows that $k_1 = 8$, $k_2 = 5$ by the Poincaré duality and hence

$$\begin{cases} P(G/K_1; t) = 1 + t^4 + t^8 , \\ P(G/K_2; t) = 1 + t^{11} . \end{cases}$$

2°. If k_1 is odd and k_2 is even, then in the same way as in 1°, we have

$$\begin{cases} P(G/K_2; t) = t^{k_1-1}(1 + t^8) + 1 , \\ t^{k_1+k_2-2}P(G/K_2; t) = t^{k_1+14} + t^7 + t^{15} . \end{cases}$$

This implies $k_1 = 9$ and $k_2 = 0$, which is contrary to $k_2 \geqq 2$. Hence, this case does not occur.

(b) Next, consider the case where one of the two singular orbits is orientable and the other is not.

Assume that G/K_1 is orientable and G/K_2 is not. Then by Lemma 2, we have $k_1 = 2$ and (3) turns to

$$t^{15}P(G/K_2; t^{-1}) = t^{14}P(G/K_1; t^{-1}) + t^{15} - t^8 - 1 .$$

By (2) and (5),

$$t^{14}P(G/K_1; t^{-1}) = P(G/K_1; t) .$$

Moreover, by the argument of [2, 2.4 $\sim$ 2.6], we have

$$t^{15}P(G/K_2; t^{-1}) = t^{2k_2-1}P(G/K_2; t) .$$

Therefore,

$$(1 - t^{2k_2})P(G/K_1; t) = (1 - t^{2k_2+6})(1 + t^4) + t^{2k_2-1} - t^{15} .$$

It follows that $P(G/K_1; t)$ is an even function and $k_2 = 8$. Hence, we have

$$(1 - t^{16})P(G/K_1; t) = (1 - t^{22})(1 + t^8) ,$$

which is impossible. Similarly, we can see that the case where G/K_1 is non-orientable and G/K_2 is orientable does not occur.

(c) If we suppose that G/K_1 and G/K_2 are non-orientable, we have $k_1 = k_2 = 2$ by Lemma 2. From [2, 2.4 $\sim$ 2.6] it follows that

$$(1 + t^3)P(G/K_2; t) = (1 + t^3)P(G/K_1; t) - (t^7 + t^8) ,$$

which is impossible. Thus, the proof of Proposition 1 is completed.

3. Actions with codimension one orbits. 3.1. As in the previous section, let M be a compact rational cohomology Cayley projective plane and G be a compact connected Lie group which acts on M differentiably with a codimension one principal orbit G/K. To classify (G, M) up to essential isomorphism, we can assume that G acts on M almost effectively. In this case, G acts on the principal orbit G/K almost effectively. Therefore, K does not contain any positive dimensional closed normal subgroup of G. There are just two singular orbits G/K_1 and G/K_2. We can assume $K \subset K_1$ and $K \subset K_2$. Each G/K_s has a closed invariant tubular neighborhood M_s, such that

$$M = M_1 \cup M_2 , \qquad M_1 \cap M_2 = \partial M_1 = \partial M_2 = G/K$$

and

$$M_s = G \underset{K_s}{\times} D^{k_s} , \qquad s = 1, 2 ,$$

as G-manifold. Here, K_s acts on a k_s-dimensional disk D^{k_s} via the slice representation

$$\sigma_s \colon K_s \to O(k_s) .$$

This K_s-action is transitive on the $(k_s - 1)$-sphere ∂D^{k_s}. M is formed from M_1 and M_2 by the identification of their boundaries under a G-equivariant diffeomorphism $f\colon \partial M_1 \to \partial M_2$. We denote such a manifold by $M(f)$. The following lemma of Uchida [2, Lemma 5.3.1] plays a fundamental rôle in our classification problem.

LEMMA 3. *Let $f, f'\colon \partial M_1 \to \partial M_2$ be G-equivariant diffeomorphisms. Then, $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if one of the following conditions is satisfied:*

(i) *f is G-diffeotopic to f',*

(ii) *$f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on M_1,*

(iii) $f'f^{-1}$ *is extendable to a* G-*equivariant diffeomorphism on* M_2.

Notice that the set of all G-equivariant diffeomorphisms $\partial M_1 \to \partial M_2$ is naturally identified with $N(K, G)/K$, where $N(K, G)$ denotes the normalizer of K in G.

We recall one more result on Lie group actions on compact rational cohomology Cayley projective planes, due to Chang and Skjelbred.

LEMMA 4 ([1, Theorem 2.2 and Proposition 3.8]). *Let* M *be a compact rational cohomology Cayley projective plane and let* G *be a compact connected Lie group acting almost effectively on* M. *Then,* $\operatorname{rank} G \leqq 4$. *Moreover,* $G_2 \times T^2$ *cannot act almost effectively on* M, *where* T^2 *is a 2-dimensional torus.*

3.2. We show:

PROPOSITION 2. *Assume that the Poincaré polynomials of two singular orbits* G/K_1, G/K_2 *are given by*

$$\begin{cases} P(G/K_1; t) = 1 + t^8 , \\ P(G/K_2; t) = 1 . \end{cases}$$

Then, (G, M) *is essentially isomorphic to* $(\boldsymbol{Spin}(9), \boldsymbol{F}_4/\boldsymbol{Spin}(9))$, *where* $\boldsymbol{Spin}(9)$ *acts naturally on* $\boldsymbol{F}_4/\boldsymbol{Spin}(9)$.

PROOF. Since $\dim G/K_2 = 0$, we have

$$K_2 = G , \qquad G/K = K_2/K = S^{15} .$$

It follows from Lemma 4 that

$$G = \boldsymbol{Spin}(9) , \qquad K \cong \boldsymbol{Spin}(7) .$$

By Lemma 2, G/K_1 is simply connected and K_1 is connected. Therefore,

$$K_1 = \boldsymbol{Spin}(8) .$$

Consider the slice representation

$$\sigma_1 \colon K_1 \to \boldsymbol{O}(8) .$$

The projections

$$p_i \colon \boldsymbol{Spin}(8) \to \boldsymbol{SO}(8) , \qquad i = 1, 2, 3 ,$$

defined by

$$p_i(x_1, x_2, x_3) = x_i ,$$

are irreducible and mutually different real 8-dimensional representations of $\boldsymbol{Spin}(8)$. Their complexifications are also irreducible and mutually different. On the other hand, it can be seen by Weyl's formula that there are just three 8-dimensional irreducible complex representations of $\boldsymbol{Spin}(8)$. Therefore, σ_1 is equivalent to some one among p_i's by [2, Lemma 5.5.1]. Since K_1 acts transitively on S^7 via σ_1 with the isotropy group K, we have $K = p_i^{-1}(\boldsymbol{SO}(7))$ for some i ($i = 1, 2, 3$). Put

$$K^{(i)} = p_i^{-1}(\boldsymbol{SO}(7)) , \qquad i = 1, 2, 3 .$$

Then, $\boldsymbol{Spin}(9)/K^{(1)} = \boldsymbol{SO}(9)/\boldsymbol{SO}(7)$ and $\boldsymbol{Spin}(9)/K^{(2)} = S^{15}$ by [6, Remark 6.3]. Define an $\boldsymbol{R}$-isomorphism $x\colon \mathfrak{J} \to \mathfrak{J}$ by

$$x\begin{pmatrix} \xi_1 & u_3 & \bar{u}_2 \\ \bar{u}_3 & \xi_2 & u_1 \\ u_2 & \bar{u}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \bar{u}_2 & -u_3 \\ u_2 & \xi_3 & -\bar{u}_1 \\ -\bar{u}_3 & -u_1 & \xi_2 \end{pmatrix} .$$

Then $x \in \boldsymbol{Spin}(9)$ by [6, Lemma 3.2], and $x^{-1}K^{(2)}x = K^{(3)}$. Therefore, we can assume that $K = K^{(3)}$ and $\sigma_1 = p_3$, because of $G/K = S^{15}$. The uniqueness of the slice representation

$$\sigma_2\colon K_2 \to \boldsymbol{O}(16)$$

is obvious. Moreover, since

$$N(K, G)/K = N(K, K_2)/K = N(\boldsymbol{Spin}(7), \boldsymbol{Spin}(9))/\boldsymbol{Spin}(7) \cong \boldsymbol{Z}_2$$

is generated by the class of the antipodal involution of $S^{15} = K_2/K$, we can see by Lemma 3 that (G, M) is uniquely determined up to essential isomorphism. On the other hand, we have seen in Example 1 that the pair $(\boldsymbol{Spin}(9), \boldsymbol{F}_4/\boldsymbol{Spin}(9))$ with the natural $\boldsymbol{Spin}(9)$-action is an example of (G, M) in our consideration. This completes the proof of Proposition 2.

3.3. PROPOSITION 3. *Suppose that the Poincaré polynomials of singular orbits are of the form*

$$\begin{cases} P(G/K_1; t) = 1 + t^4 + t^8 , \\ P(G/K_2; t) = 1 + t^{11} . \end{cases}$$

Then, (G, M) *is essentially isomorphic to* $(\boldsymbol{Sp}(3), \boldsymbol{F}_4/\boldsymbol{Spin}(9))$, $\boldsymbol{Sp}((3) \times \boldsymbol{U}(1), \boldsymbol{F}_4/\boldsymbol{Spin}(9))$ *or* $(\boldsymbol{Sp}(3) \times \boldsymbol{Sp}(1), \boldsymbol{F}_4/\boldsymbol{Spin}(9))$. *Here, in each case, the group acts on* $\boldsymbol{F}_4/\boldsymbol{Spin}(9)$ *through* φ *defined in* §1, *Example* 3.

PROOF. Since $k_2 = 5$, it follows from Lemma 2 that G/K_1 is simply connected and K_1 is connected. We can assume that

$$G = G' \times U ,$$

where G' is a compact simply connected Lie group which acts on G/K_1 almost effectively and U is a compact connected Lie group which acts on G/K_1 trivially. By our assumption, $\operatorname{rank} K_1 = \operatorname{rank} G$. Therefore,

$$K_1 = K_1' \times U\,,$$

where K_1' is a subgroup of G' and (G', K_1') is pairwise locally isomorphic to $(\boldsymbol{Sp}(3), \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(2))$ or $(\boldsymbol{G}_2, \boldsymbol{SO}(4))$. By an argument similar to that of [2, Lemma 9.2.2], we can show that K_1' acts on $K_1/K = S^7$ transitively. Therefore, (G', K_1') is pairwise locally isomorphic to $(\boldsymbol{Sp}(3), \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(2))$, because $\boldsymbol{SO}(4)$ cannot act transitively on S^7. Note that $\operatorname{rank} U \leqq 1$, by Lemma 4. First, we consider the case $U = \{1\}$; that is,

$$G = \boldsymbol{Sp}(3)\,, \qquad K_1 = \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(2)\,.$$

Then we have

$$K = \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1)\,, \qquad K_2 \cong \boldsymbol{Sp}(2)\,.$$

Since any representation $\boldsymbol{Sp}(2) \to \boldsymbol{Sp}(3)$ is reducible, we can assume that

$$K_2 = \left\{ \left(\begin{array}{c|c} Y & \\ \hline & 1 \end{array} \right) \middle|\, Y \in \boldsymbol{Sp}(2) \right\}$$

up to conjugation. The first factor $\boldsymbol{Sp}(1)$ of K_1 acts trivially on K_1/K. For, if $\boldsymbol{Sp}(1)$ acts on K_1/K almost effectively, then K has the form

$$\left\{ \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix} \middle|\, \alpha, \beta \in \boldsymbol{Sp}(1) \right\}.$$

This contradicts our assumption $K \subset K_2$. Hence we have

$$K = \boldsymbol{Sp}(1) \times H\,,$$

where $H \subset \boldsymbol{Sp}(2)$, $H \cong \boldsymbol{Sp}(1)$. The slice representations

$$\sigma_1\colon K_1 \to \boldsymbol{O}(8)\,, \qquad \sigma_2\colon K_2 \to \boldsymbol{O}(5)$$

are uniquely determined up to equivalence. Moreover, $N(K, G)/K = N(K, K_2)/K \cong \boldsymbol{Z}_2$ is generated by the class of the antipodal involution of $K_2/K = S^4$. Therefore, by Lemma 3, (G, M) is uniquely determined up to essential isomorphism. Next, consider the case

$$G = \boldsymbol{Sp}(3) \times U\,, \quad K_1 = \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(2) \times U\,, \quad U \neq \{1\}\,.$$

Since G acts on M almost effectively by our assumption, we may suppose that U acts on K_1/K non-trivially. Then,

$$K = \boldsymbol{Sp}(1) \times (V \times 1) \circ U\,,$$

where $V \subset \boldsymbol{Sp}(2)$, $V \cong \boldsymbol{Sp}(1)$. In this situation, note that $\operatorname{rank} K_2 = \operatorname{rank} G - 1$. We can show as in the case $U = \{1\}$

$$K_2 = (W \times 1) \circ U\,,$$

where $W \subset \boldsymbol{Sp}(3)$, $W \cong \boldsymbol{Sp}(2)$. The slice representations

$$\sigma_1: K_1 \to \boldsymbol{O}(8) , \qquad \sigma_2: K_2 \to \boldsymbol{O}(5)$$

are determined uniquely up to equivalence. Moreover, we have

$$N(K, G)^\circ = K \quad \text{and} \quad N(K, G)/K \cong \boldsymbol{Z}_2 \quad \text{in case} \quad U = \boldsymbol{Sp}(1) ,$$
$$N(K, G)^\circ/K \cong \boldsymbol{U}(1) \quad \text{and} \quad N(K, G)/N(K, G)^\circ \cong \boldsymbol{Z}_2 \quad \text{in case} \quad U = \boldsymbol{U}(1) .$$

Therefore, when $U = \boldsymbol{Sp}(1)$ or $\boldsymbol{U}(1)$, we can show by Lemma 3 that (G, M) is determined uniquely up to essential isomorphism. On the other hand, the $\boldsymbol{Sp}(3) \times \boldsymbol{Sp}(1)$-action φ on $P(\boldsymbol{Cay})$ of §1, Example 3 gives examples of (G, M) in our consideration, in case $G = \boldsymbol{Sp}(3) \times \boldsymbol{Sp}(1)$, $\boldsymbol{Sp}(3) \times \boldsymbol{U}(1)$ or $\boldsymbol{Sp}(3)$. Thus the proof of Proposition 3 is completed.

From Propositions 1, 2 and 3, Theorem I follows easily.

4. **Actions with codimension two principal orbits.** In this section, we shall prove Theorem II. As a simple consequence from [3, Theorem 0.1], we can see that there exists at least one isolated singular orbit. Therefore, from now on, we assume that a compact connected Lie group G acts differentiably and almost effectively on a compact rational cohomology Cayley projective plane M with codimension two principal orbit G/H and only one isolated singular orbit G/K. Then, we know that there exists a non-isolated singular orbit, say, G/L. Let

$$k = 16 - \dim G/K , \qquad l = 16 - \dim G/L .$$

Since $2 < l < k$, it follows that G/K, G/L are simply connected and K, L are connected. K acts on a $(k-1)$-sphere via the slice representation $K \to \boldsymbol{O}(k)$. This K-action has codimension one principal orbit K/H and two singular orbits K/L_1, K/L_2, where L_1, L_2 are conjugate to L in G.

As in §2, the following two cases are possible:

$$\text{(i)} \quad \begin{cases} P(G/K; t) = 1 , \\ P(G/L; t) = 1 + t^8 , \end{cases}$$

$$\text{(ii)} \quad \begin{cases} P(G/K; t) = 1 + t^4 + t^8 , \\ P(G/L; t) = 1 + t^{11} . \end{cases}$$

First, we show that the case (i) does not occur. Suppose that $G = K$ acts on M almost effectively and $G = G' \times U$, where G' is a connected semi-simple Lie group which acts almost effectively on G/L and U is a connected Lie group which acts trivially on G/L. Then, $L = L' \times U$, where L' is a compact subgroup of G'. Since G/L is indecomposable, G' is simple. Therefore, (G', L') is pairwise locally isomorphic to $(\boldsymbol{Spin}(9),$

$\boldsymbol{Spin}(8)$). If follows from Lemma 4 that $U = \{1\}$ and hence $L = L' = \boldsymbol{Spin}(8)$. On the other hand, $L/H = S^6$ by [5, (2.2)]. This is a contradiction, because $\boldsymbol{Spin}(8)$ cannot act transitively on S^6.

Now, consider the case (ii). Note that $k = 8$ and $l = 5$ in this case. Via the slice representation $K \to \boldsymbol{O}(8)$, K acts on S^7 with codimension one principal orbit K/H. Using [5, (5.2)], we can show that

$$(*)\qquad \begin{aligned} &K/L = S^3\,, \\ &K/H = K/L_1 \times K/L_2 = S^3 \times S^3\,, \\ &H = L_1 \cap L_2\,. \end{aligned}$$

Let $G = G' \times U$, where G' is a compact connected Lie group which acts on G/K almost effectively and U is a compact connected Lie group which acts on G/K trivially. Then, $K = K' \times U$, where K' is a compact subgroup of G', and (G', K') is pairwise locally isomorphic to $(\boldsymbol{Sp}(3), \boldsymbol{Sp}(2) \times \boldsymbol{Sp}(1))$ or $(\boldsymbol{G}_2, \boldsymbol{SO}(4))$. We shall show that both of these are impossible. Note that rank $U \leqq 1$ by Lemma 4.

(a) Suppose that (G', K') is pairwise locally isomorphic to $(\boldsymbol{Sp}(3), \boldsymbol{Sp}(2) \times \boldsymbol{Sp}(1))$. If U acts on K/L_1 trivially, then by the conjugacy of L_2 with L_1 it acts on K/L_2 trivially. It follows that $L_1 = L_2$ and therefore $K/H = K/L_1$ This is a contradiction. Consequently, the U-action on K/L_1 is not trivial. Now suppose that U acts on K/L_1 non-trivially. If $U \cong \boldsymbol{Sp}(1)$, then $U/U \cap L_1 = S^3$ and therefore $U \cap L_1 = \{1\}$. So we can assume $L_1 = \boldsymbol{Sp}(2) \times V$, where $V \subset \boldsymbol{Sp}(1) \times U$, $V \cong \boldsymbol{Sp}(1)$. Since L_2 is conjugate to L_1 in G, it follows that $L_2 = \boldsymbol{Sp}(2) \times V'$, where $V' \subset \boldsymbol{Sp}(1) \times U$, $V' \cong \boldsymbol{Sp}(1)$. It is easy to see that $H = L_1 \cap L_2$ contains a maximal torus of V and therefore $\dim H \geqq \dim \boldsymbol{Sp}(2) + 1 = 11$. This is a contradiction, because $\dim K = 16$ and $\dim K/H = 6$. If we suppose that $U \cong \boldsymbol{U}(1)$ acts on K/L_1 non-trivially, then in the same way as above we can show that $H = L_1 \cap L_2$ is isomorphic to $\boldsymbol{Sp}(2) \times \boldsymbol{U}(1)$ or H has two connected components. This leads us to a contradiction. Thus we have shown that (G', K') cannot be pairwise locally isomorphic to $(\boldsymbol{Sp}(3), \boldsymbol{Sp}(2) \times \boldsymbol{Sp}(1))$.

(b) Next, suppose that (G', K') is pairwise locally isomorphic to $(\boldsymbol{G}_2, \boldsymbol{SO}(4))$. If U acts on K/L_1 trivially, then by the conjugacy of L_2 with L_1 in G, $H = L_1 \cap L_2$ contains U as a normal subgroup. Since G acts on M almost effectively by our assumption, it follows from $(*)$ that $U = \{1\}$ and $\dim H = 0$. Since $\pi_1(K/H) = 0$ by $(*)$, we have $\pi_1(K) = 0$. This contradicts $\pi_1(K) = \pi_1(\boldsymbol{SO}(4)) = \boldsymbol{Z}_2$. Hence the U-action on K/L_1 is not trivial. Now assume that U acts on K/L_1 non-trivially. If $U \cong \boldsymbol{U}(1)$, then $\dim H = 1$. Since K/H is 2-connected, H is connected and therefore $H \cong U(1)$. This is a contradiction, because $\pi_1(H) = \pi_1(K) = \pi_1(\boldsymbol{SO}(4) \times U) = \boldsymbol{Z}_2 + \boldsymbol{Z}$. So we suppose that $U \cong \boldsymbol{Sp}(1)$ acts on K/L_1 non-trivially. Then U acts on G/L_1 non-trivially. Since $\operatorname{rank} L_1 = \operatorname{rank} G - 1 = 2$ and $\dim G - \dim L_1 = 11$, it follows from [4, Proposition 2] that $L_1 = (\boldsymbol{SU}(2) \times 1) \circ V$, where $V \cong \boldsymbol{Sp}(1)$, $\boldsymbol{SU}(2) \subset \boldsymbol{G}_2$. The inclusions $\boldsymbol{SU}(2) \subset \boldsymbol{SO}(4) \subset \boldsymbol{G}_2$ are given as follows ([8, §3.3]). Identify $\boldsymbol{SU}(2)$ with $\{g \in \boldsymbol{G}_2 \mid ge_i = e_i,\ i = 1, 2\}$ and let A be the identity component of the centralizer of $\boldsymbol{SU}(2)$ in $\boldsymbol{G}_2$. Then we can see that A is isomorphic to $\boldsymbol{Sp}(1)$, $\boldsymbol{SU}(2) \cap A \cong \boldsymbol{Z}_2$ and $\boldsymbol{SO}(4)$ is identified with the subgroup $\boldsymbol{SU}(2) \cdot A/\boldsymbol{Z}_2$ of $\boldsymbol{G}_2$. Moreover, $\boldsymbol{SO}(4) = N(\boldsymbol{SU}(2), \boldsymbol{G}_2)$. Since L_2 is conjugate to L_1, we can write $L_1 \cap L_2 \cong \boldsymbol{SU}(2) \times \{y \in A \mid hyh^{-1} = y\}$, for some fixed $h \in A$. The second factor on the right hand side contains the maximal torus of A through h. Therefore, $\dim H = \dim (L_1 \cap L_2) \geqq 4$. This is a contradiction, because $\dim K/H = 6$ and $\dim K = 9$. Hence, (G', K') is not pairwise locally isomorphic to $(\boldsymbol{G}_2, \boldsymbol{SO}(4))$.

The proof of Theorem II is thus completed.

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Department of Mathematics
College of General Education
Tohoku University
Sendai, 980
Japan