Tôhoku Math. Journ.
35 (1983), 557-572.

AUGMENTED SCHOTTKY SPACES AND A UNIFORMIZATION OF RIEMANN SURFACES

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(Received August 5, 1982)

0. Introduction. In this paper, we will consider the following problem. Let S be a compact Riemann surface with nodes. Does there exist a point in an augmented Schottky space representing the surface S? We will give a complete answer to the problem. An answer to special cases of the problem has been obtained by Sato [5]. Let $\langle G_0\rangle$ be a fixed marked Schottky group and $\tilde{\Sigma}_0$ a fixed basic system of Jordan curves for $\langle G_0\rangle$ (see § 1 for the definition). Let S be a compact Riemann surface with nodes and Σ a basic system of loops and nodes satisfying the following assumption: The set Σ' of Jordan curves and points induced from Σ is compatible with $\tilde{\Sigma}_0$ and the pair (S, Σ) has Property (A) (see § 1 for the definitions). Under the assumption, there exists a point representing S in the augmented Schottky space associated with $\tilde{\Sigma}_0$. In this paper, we will consider the problem in the general case without the above assumption. The answer to the problem is affirmative, and is stated in Theorem 3.

In § 1, we will list notations and terminologies. In § 2, we will introduce the interchange operator which plays as essential role in studying the question stated above, and in § 3, we will explain illustratively the operator by some examples. In § 4, we will treat the problem stated above. We will consider another problem in a forthcoming paper. We give a point τ in an augmented Schottky space, which represents a compact Riemann surface S with nodes. Then for any sequence of points $\{\tau_n\}$ in the Schottky space tending to the point τ, does the Riemann surface $S(\tau_n)$ represented by τ_n converge to S as $n\to\infty$?

1. Notations and terminologies. 1-1. In order to elliminate trouble and expense in printing, we use alternatives to some notations in the previous papers [4], [5]; for example, we replace $\tilde{\gamma}_j$ and $\tilde{\gamma}(i_1, i_2, \cdots, i_\mu)$ by C_{2g+j} and $C(i_1, i_2, \cdots, i_\mu)$, respectively. Throughout this paper, we let $\langle G_0\rangle$ be a fixed marked Schottky group of genus $g\geqq 2$ generated by

Partly supported by the Grants-in-Aid for Scientific and Co-operative Research, the Ministry of Education, Science and Culture, Japan.

$A_{0,1}, A_{0,2}, \cdots, A_{0,g}$: $\langle G_0\rangle = \langle A_{0,1}, A_{0,2}, \cdots, A_{0,g}\rangle$. Let $C_{0,1}, C_{0,g+1}; C_{0,2}, C_{0,g+2}; \cdots; C_{0,g}, C_{0,2g}$ be defining curves of $A_{0,1}, A_{0,2}, \cdots, A_{0,g}$, respectively. Namely they are mutually disjoint Jordan curves on the Riemann sphere $\hat{C} = C \cup \{\infty\}$ which comprise the boundary of $2g$-ply connected region ω_0, and $A_{0,j}$ maps $C_{0,j}$ onto $C_{0,g+j}$ and $A_{0,j}(\omega_0) \cap \omega_0 = \emptyset$ for each $j = 1, 2, \cdots, g$. If mutually disjoint Jordan curves $C_{0,1}, C_{0,2}, \cdots, C_{0,2g}, C_{0,2g+1}, C_{0,2g+2}, \cdots, C_{0,4g-3}$ on $\hat{C}$ have the following properties (i) and (ii), then we call $\tilde{\Sigma}_0 = \{C_{0,1}, \cdots, C_{0,2g}; C_{0,2g+1}, \cdots, C_{0,4g-3}\}$ a *basic system of Jordan curves for* $\langle G_0\rangle$: (i) $C_{0,2g+j}$ $(j = 1, 2, \cdots, 2g-3)$ lie in ω_0. (ii) Each component of $\omega_0 \setminus \bigcup_{j=1}^{2g-3} C_{0,2g+j}$ is a triply connected planer domain. In particular, if a basic system of Jordan curves $\tilde{\Sigma}_0$ has the following property (iii), we call $\tilde{\Sigma}_0$ a *standard system of Jordan curves for* $\langle G_0\rangle$: (iii) For each $i = 1, 2, \cdots, g$ and $j = 1, 2, \cdots, 2g-3$, $C_{0,i}$ and $C_{0,g+i}$ lie on the same side of $C_{0,2g+j}$.

We let $C_{0,i(1)}, C_{0,i(2)}, \cdots, C_{0,i(k)}, C_{0,g+i'(1)}, \cdots, C_{0,g+i'(l)}$ and $C_{0,j(1)}, C_{0,j(2)}, \cdots, C_{0,j(m)}, C_{0,g+j'(1)}, \cdots, C_{0,g+j'(n)}$ be the defining curves in $\tilde{\Sigma}_0$ in the interior and to the exterior to $C_{0,2g+j}$, respectively, where $i(1) < \cdots < i(k) \leqq g$, $i'(1) < \cdots < i'(l) \leqq g$; $j(1) < \cdots < j(m) \leqq g$, $j'(1) < \cdots < j'(n) \leqq g$. Then we say that the curve $C_{0,2g+j}$ gives a *partition* $\{i(1), \cdots, i(k), g+i'(1), \cdots, g+i'(l)\} \cup \{j(1), \cdots, j(m), g+j'(1), \cdots, g+j'(n)\}$ of the set $\{1, 2, \cdots, 2g\}$.

1-2. Let $\alpha_{0,i}(i = 1, 2, \cdots, g)$ and $\gamma_{0,j}(j = 1, 2, \cdots, 2g-3)$ be the images of $C_{0,i}$ and $C_{0,2g+j}$, respectively, under the natural projection Π_0: $\Omega(G_0) \to \Omega(G_0)/G_0 = S_0$, where $\Omega(G_0)$ is the region of discontinuity of G_0. Then $\Sigma_0 = \{\alpha_{0,1}, \cdots, \alpha_{0,g}; \gamma_{0,1}, \cdots, \gamma_{0,2g-3}\}$ is a basic system of loops (resp. a standard system of loops) if $\tilde{\Sigma}_0$ is a basic system of Jordan curves (resp. a standard system of Jordan curves) (see [4, pp. 155, 156] for the definitions). We call Σ_0 the *projection* of $\tilde{\Sigma}_0$ onto S_0.

Cut the Riemann surface S_0 along the loops $\alpha_{0,i}(i = 1, 2, \cdots, g)$. We denote by $\alpha'_{0,i}$ and $\alpha'_{0,g+i}$ the resulting two topological circles. We call $\Sigma'_0 = \{\alpha'_{0,1}, \cdots, \alpha'_{0,2g}; \gamma_{0,1}, \cdots, \gamma_{0,2g-3}\}$ the *set of Jordan curves induced from* Σ_0, or simply the *induced set* from Σ_0. Each $\gamma_{0,j}$ divides the set $\{\alpha'_{0,1}, \cdots, \alpha'_{0,g}, \alpha'_{0,g+1}, \cdots, \alpha'_{0,2g}\}$ into two parts $\{\alpha'_{0,i(1)}, \cdots, \alpha'_{0,i(k)}, \alpha'_{0,g+i'(1)}, \cdots, \alpha'_{0,g+i'(l)}\}$ and $\{\alpha'_{0,j(1)}, \cdots, \alpha'_{0,j(m)}, \alpha'_{0,g+j'(1)}, \cdots, \alpha'_{0,g+j'(n)}\}$, where $i(1) < \cdots < i(k) \leqq g$, $i'(1) < \cdots < i'(l) \leqq g$; $j(1) < \cdots < j(m) \leqq g$, $j'(1) < \cdots < j'(n) \leqq g$. Then we say that $\gamma_{0,j}$ gives a *partition* $\{i(1), \cdots, i(k), g+i'(1), \cdots, g+i'(l)\} \cup \{j(1), \cdots, j(m), g+j'(1), \cdots, g+j'(n)\}$ of the set $\{1, 2, \cdots, 2g\}$. If each $\gamma_{0,j}(j = 1, 2, \cdots, 2g-3)$ gives the same partition as $C_{0,2g+j}$, we say Σ'_0 is *compatible* with $\tilde{\Sigma}_0$ and denote the fact by $\Sigma'_0 \sim \tilde{\Sigma}_0$.

1-3. We drop the suffices "0" of $C_{0,i}, C_{0,2g+j}, \cdots$, for simplicity if there is no confusion. In [5], we defined the cycle corresponding to α_i

as follows. Let $C_{2g+i(1)}, C_{2g+i(2)}, \cdots, C_{2g+i(l)}$ be a sequence of C_{2g+j} in $\tilde{\Sigma}_0$ each of which separates p_i and p_{g+i}, and which are arranged from p_i to p_{g+i}, where p_i and p_{g+i} are the repelling and the attracting fixed points of $A_{0,i}$. Then the sequence $(\alpha_i; \gamma_{i(1)}, \gamma_{i(2)}, \cdots, \gamma_{i(l)})$ of elements in Σ_0 was called the cycle corresponding to α_i in [5].

Here we will introduce cycles with direction, which are called ordered cycles. As in [5], we construct the tree of $\tilde{\Sigma}_0$. We represent C_i and C_{2g+j} by using multi-suffices as in [5]. We first define the direction $\varepsilon(=+1$ or $-1)$ of C_{2g+j} from C_1 as follows. Let $C_{2g+j} = C(i_1, i_2, \cdots, i_\mu)$ (we wrote $\tilde{\gamma}(i_1, i_2, \cdots, i_\mu)$ for $C(i_1, i_2, \cdots, i_\mu)$ in [5]). If C_{2g+j} is passed through from $C(i_1, i_2, \cdots, i_{\mu-1})$ to $C(i_1, i_2, \cdots, i_\mu, i_{\mu+1})$ on the tree of $\tilde{\Sigma}_0$, then we say that C_{2g+j} is passed through in the *positive direction*, and we denote C_{2g+j} by C^{+1}_{2g+j}. We write γ_j^{+1} for the projection of C^{+1}_{2g+j} onto S_0. If C_{2g+j} is passed through in the direction opposite to the above, then we say that C_{2g+j} is passed in the *negative direction*, and we denote C_{2g+j} by C^{-1}_{2g+j}. We write γ_j^{-1} for the projection of C^{-1}_{2g+j}.

DEFINITION. We say $C_{2g+j} = C(i_1, i_2, \cdots, i_\mu)$ (resp. $C_i = C(j_1, j_2, \cdots, j_\sigma)$) is *behind* $C_{2g+l} = C(i'_1, i'_2, \cdots, i'_\nu)$ if $\nu < \mu$ and $i_k = i'_k (k = 1, 2, \cdots, \nu)$ (resp. $\nu < \sigma$ and $j_k = i'_k (k = 1, 2, \cdots, \nu)$), and denote the fact by $C_{2g+l} < C_{2g+j}$ (resp. $C_{2g+l} < C_i$). Otherwise, we say that C_{2g+j} (resp. C_i) is *not behind* C_{2g+l} and we denote the fact by $C_{2g+l} \nless C_{2g+j}$ (resp. $C_{2g+l} \nless C_i$).

REMARK 1. γ_j is a dividing loop if and only if either $C_{2g+j} < C_i$ and $C_{2g+j} < C_{g+i}$, or $C_{2g+j} \nless C_i$ and $C_{2g+j} \nless C_{g+i}$ for each $i = 1, 2, \cdots, g$.

REMARK 2. $\tilde{\Sigma}_0$ is a standard system of Jordan curves if and only if either $C_{2g+j} < C_i$ and $C_{2g+j} < C_{g+i}$, or $C_{2g+j} \nless C_i$ and $C_{2g+j} \nless C_{g+i}$ for each $j = 1, 2, \cdots, 2g-3$ and for each $i = 1, 2, \cdots, g$.

We define the ordered cycle corresponding to α_i as follows. We denote the shortest path from C_i to C_{g+i} on the tree of $\tilde{\Sigma}_0$ by

$$(1) \qquad C_i, C^{\delta(1)}_{2g+i(1)}, C^{\delta(2)}_{2g+i(2)}, \cdots, C^{\delta(k)}_{2g+i(k)}, C_{g+i}\ .$$

Here $\delta(l) (l = 1, 2, \cdots, k)$ are determined by $\delta(l) = +1$ or $\delta(l) = -1$ according as $C_{2g+l} < C_{g+i}$ or $C_{2g+l} < C_i$. The projection

$$(2) \qquad (\alpha_i; \gamma^{\delta(1)}_{i(1)}, \gamma^{\delta(2)}_{i(2)}, \cdots, \gamma^{\delta(k)}_{i(k)})$$

of (1) onto S_0 is called the *ordered cycle* corresponding to α_i, and is denoted by $L_{0,i}$.

1-4. Example. We write (a, b, c, d) for a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $A_{0,1} = (-3, 9\exp(4\pi i/3) - 1, 1, -\exp(4\pi i/3))$, $A_{0,2} = (3\exp(\pi i/3), 8, 1, -3\exp(2\pi i/3))$ and $A_{0,3} = (3\exp(5\pi i/3), -9\exp(5\pi i/3)-1, 1, -3)$. Then $\langle G_0\rangle = \langle A_{0,1}, A_{0,2}, A_{0,3}\rangle$ is a marked Schottky group. Let $C_1\colon |z - 3\exp(4\pi i/3)| = 1$, $C_2\colon |z - 3\exp(2\pi i/3)| = 1$, $C_3\colon |z - 3| = 1$, $C_4\colon |z + 3| = 1$, $C_5\colon |z - 3\exp(\pi i/3)| = 1$, $C_6\colon |z - 3\exp(5\pi i/3)| = 1$, and let $C_{0,7}, C_{0,8}, C_{0,9}$ be as in Fig. 1. Then $\tilde{\Sigma}_0 = \{C_{0,1}, \cdots, C_{0,6}; C_{0,7}, C_{0,8}, C_{0,9}\}$ is a basic system of Jordan curves for $\langle G_0\rangle$. We have a Riemann surface $S_0 = \Omega(G_0)/G_0$ and loops $\alpha_{0,1}, \alpha_{0,2}, \alpha_{0,3}, \gamma_{0,1}, \gamma_{0,2}, \gamma_{0,3}$ on S_0 as in Fig. 2. The tree of $\tilde{\Sigma}_0$ is as in Fig. 3. Identifying $C_{0,i}$ and $C_{0,3+i} (i = 1, 2, 3)$ as in Fig. 3, we obtain Fig. 4. We have three ordered cycles $L_{0,1} = (\alpha_{0,1}; \gamma_{0,1}, \gamma_{0,2})$, $L_{0,2} = (\alpha_{0,2}; \gamma_{0,2}^{-1}, \gamma_{0,3})$ and $L_{0,3} = (\alpha_{0,3}; \gamma_{0,3}^{-1}, \gamma_{0,1}^{-1})$, which correspond to $A_{0,1}$, $A_{0,2}$ and $A_{0,3}$, respectively, where we write $\gamma_{0,j}$ for $\gamma_{0,j}^{+1}$ for simplicity.

1-5. Let I be a subset of $\{1, 2, \cdots, g\}$ and J a subset of $\{1, 2, \cdots,$

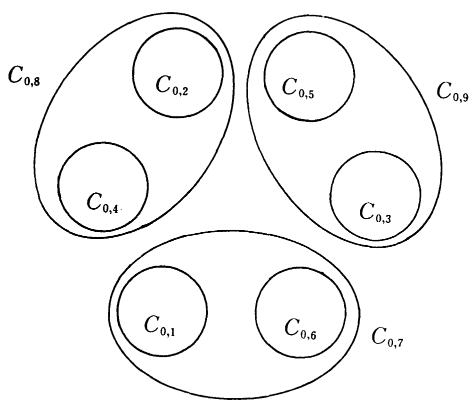

FIGURE 1

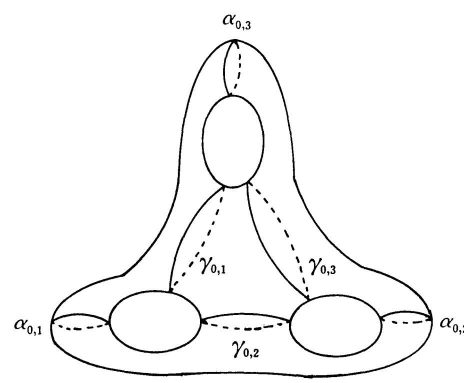

FIGURE 2

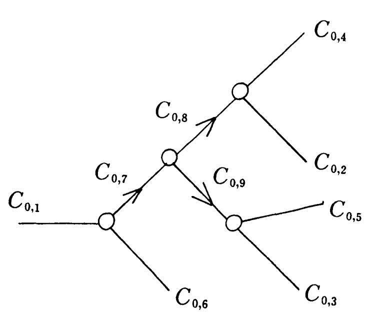

FIGURE 3

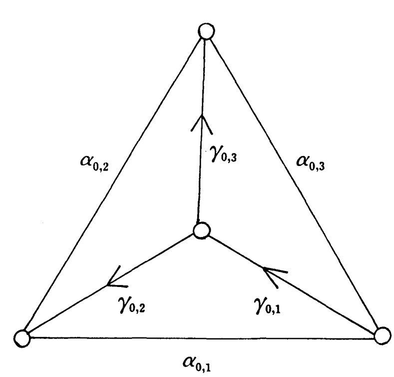

FIGURE 4

$2g-3\}$. We denote by $|I|$ and $|J|$ the cardinality of I and J, respectively. Let $L_{0,j(1)}, L_{0,j(2)}, \cdots, L_{0,j(t)}$ be the complete list of cycles containing γ_j^δ, and let $\alpha_{0,k}$ be the "α-loops" contained in $L_{0,k}(1 \leqq k \leqq t)$, where $t = t(j)$ depends on j. We define the subset $I(J)$ of $\{1, 2, \cdots, g\}$ by

$$I(J) = \{i \in \{1, 2, \cdots, g\} | \alpha_{0,i} \text{ is contained in } L_{0,j(k)} \text{ for some } k(1 \leqq k \leqq t(j)) \text{ and for some } j \in J\} .$$

In this paper we assume that $I \supset I(J)$. As in [5], we define sets $\mathfrak{S}_g(\tilde{\Sigma}_0)$, $\delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$, $\mathfrak{S}_g^{I,J}(\tilde{\Sigma}_0)$ and $\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$. We call the set $\mathfrak{S}_g(\tilde{\Sigma}_0)$ and $\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$ the *Schottky space with respect to* $\tilde{\Sigma}_0$ and the *augmented Schottky space associated with* $\tilde{\Sigma}_0$, respectively.

1-6. Let S be a compact Riemann surface without (resp. with) nodes. We call the set $\Sigma = \{\alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3}\}$ of loops (resp. loops and nodes) on S having the following property *a basic system of loops* (resp. *a basic system of loops and nodes*). Each component of $S - \bigcup_{i=1}^g \alpha_i - \bigcup_{j=1}^{2g-3} \gamma_j$ is a planar and triply connected region of type [3, 0] (resp. [3, 0], [2, 1], [1, 2] or [0, 3]), where a surface of type $[m, n]$ means the sphere with m disks removed and n points deleted.

In the same way as in §1-2, we can define the following: The set $\Sigma' = \{\alpha_1', \cdots, \alpha_{2g}'; \gamma_1, \cdots, \gamma_{2g-3}\}$ *of Jordan curves and points induced from* Σ which is simply called *the set induced from* Σ; the partition by γ_j; compatibility of Σ' and $\tilde{\Sigma}_0$, which is denoted by $\Sigma' \sim \tilde{\Sigma}_0$.

If the pair (S, Σ) with $\Sigma' \sim \tilde{\Sigma}_0$ has the following property, we say that (S, Σ) has *Property* (A) *with respect to* $\tilde{\Sigma}_0$ (or simply Property (A)): If $\gamma_j \in \Sigma$ is a node, then $\alpha_i \in \Sigma$ are nodes for all $i \in I(\{j\})$, where $I(\{j\})$ is the set defined in §1-5 with respect to $\tilde{\Sigma}_0$.

2. The interchange operator.

2-1. Let $\langle G_0 \rangle$, $\tilde{\Sigma}_0$, Σ_0 and Σ_0' be as in §1. Assume that $\Sigma_0' \sim \tilde{\Sigma}_0$. In this section except in §§2-6, 2-7, we drop the suffices "0" of $A_{0,i}$, $C_{0,i}$, $C_{0,2g+j}$, $\cdots$, if there is no confusion.

Let $\gamma_j \in \Sigma_0$. Let $I(\{j\}) = \{j(1), j(2), \cdots, j(t)\} (j(1) < j(2) < \cdots < j(t))$, where $t = t(j)$ depends on j. Then $L_{0,j(1)}, L_{0,j(2)}, \cdots, L_{0,j(t)}$ is the complete list of ordered cycles containing γ_j:

$$\begin{aligned} L_{0,j(1)} &= (\alpha_{j(1)}; \gamma_{j(1),1}^\delta, \cdots, \gamma_{j(1),n(1)}^\delta) \\ L_{0,j(2)} &= (\alpha_{j(2)}; \gamma_{j(2),1}^\delta, \cdots, \gamma_{j(2),n(2)}^\delta) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ L_{0,j(t)} &= (\alpha_{j(t)}; \gamma_{j(t),1}^\delta, \cdots, \gamma_{j(t),n(t)}^\delta) , \end{aligned}$$

where $\gamma_{j(k),m(k)} = \gamma_j (k = 1, 2, \cdots, t)$ and δ of $\gamma_{i,l}^\delta$ represent the directions of $\gamma_{i,l}$ in $L_{0,i}$ (cf. §1-3).

We define the interchange operator $I_g(\alpha_{j(k)}, \gamma_j)$ of $\alpha_{j(k)}$ and γ_j acting on $\tilde{\Sigma}_0$. We denote by $\tilde{\Sigma}^* = \{C_1^*, \cdots, C_{2g}^*; C_{2g+1}^*, \cdots, C_{4g-3}^*\}$ the image of $\tilde{\Sigma}_0$ under $I_g(\alpha_{j(k)}, \gamma_j)$, where C_i^* and C_{2g+l}^* are defined as follows. Let $\delta(j(k), m(k))$ be the direction of $\gamma_{j(k),m(k)}$ in $L_{0,j(k)}$.

CASE I. $j(k) \neq 1$, and $\delta(j(k), m(k)) = +1$ (cf. Example 4 in §3). In this case, we have always $C_{2g+j} < C_{g+j(k)}$. C_i^* and C_{2g+l}^* are defined as follows.

(i) $C_{2g+j}^* = C_{j(k)}$, $C_{g+j(k)}^* = C_{2g+j}$ and $C_{j(k)}^* = A_{j(k)}^{-1}(C_{2g+j})$.

(ii) $C_i^* = A_{j(k)}^{-1}(C_i)(i \neq g + j(k))$ (resp. $C_{2g+l}^* = A_{j(k)}^{-1}(C_{2g+l})$) in the case of $C_{2g+j} < C_i$ (resp. $C_{2g+j} < C_{2g+l}$).

(iii) $C_i^* = C_i$ $(i \neq j(k))$ (resp. $C_{2g+l}^* = C_{2g+l}$) in the case of $C_{2g+j} \nless C_i$ (resp. $C_{2g+j} \nless C_{2g+l}$).

CASE II. $j(k) \neq 1$, and $\delta(j(k), m(k)) = -1$ (cf. Example 2 in §3). In this case, we have $C_{2g+j} < C_{j(k)}$. We define C_i^* and C_{2g+l}^* as follows.

(i) $C_{2g+j}^* = C_{g+j(k)}$, $C_{g+j(k)}^* = C_{2g+j}$ and $C_{j(k)}^* = A_{j(k)}(C_{2g+j})$.

(ii) $C_i^* = A_{j(k)}(C_i)$ $(i \neq j(k))$ (resp. $C_{2g+l}^* = A_{j(k)}(C_{2g+l})$) in the case of $C_{2g+j} < C_i$ (resp. $C_{2g+j} < C_{2g+l}$).

(iii) $C_i^* = C_i(i \neq g + j(k))$ (resp. $C_{2g+l}^* = C_{2g+l}$) in the case of $C_{2g+j} \nless C_i$ (resp. $C_{2g+j} \nless C_{2g+l}$).

CASE III. $j(k) = 1$ (cf. Example 1 in §3). In this case, $C_{2g+j} < C_{g+j(k)}(=C_{g+1})$. $\delta(j(k), m(k))$ is always equal to $+1$. C_i^* and C_{2g+l}^* are defined as follows:

(i) $C_{2g+j}^* = C_1$, $C_{g+1}^* = C_{2g+j}$ and $C_i^* = A_1^{-1}(C_{2g+j})$.

(ii) $C_i^* = A_1^{-1}(C_i)(i \neq g + 1)$ (resp. $C_{2g+l}^* = A_1^{-1}(C_{2g+l})$) in the case of $C_{2g+j} < C_i$(resp. $C_{2g+j} < C_{2g+l}$).

(iii) $C_i^* = C_i(i\neq 1)$(resp. $C_{2g+l}^* = C_{2g+l}$) in the case of $C_{2g+j} \nless C_i(i \neq 1)$ (resp. $C_{2g+j} \nless C_{2g+l}$).

2-2. We determine the direction $\varepsilon^*(2g + l)$ of C_{2g+l}^* from C_1^* in the image $\tilde{\Sigma}_0^*$ of $\tilde{\Sigma}_0$ under $I_g(\alpha_{j(k)}, \gamma_j)$ as follows.

Case I in §2-1. (i) $\varepsilon^*(2g + j)$ is equal to -1.

(ii) $\varepsilon^*(2g + l)$ are equal to -1 for l such that γ_l^0 are contained in $L_{0,j(k)}$ (we denote the fact by $\gamma_l^0 \in L_{0,j(k)}$) and $C_{2g+j} < C_{2g+l}$.

(iii) Otherwise, $\varepsilon^*(2g + l)$ are equal to $+1$.

Case II in §2-1. (i) $\varepsilon^*(2g + j)$ is equal to $+1$.

(ii) $\varepsilon^*(2g + l)$ are equal to -1 for l such that $\gamma_l^0 \in L_{0,j(k)}$ and $C_{2g+j} < C_{2g+l}$.

(iii) Otherwise, $\varepsilon^*(2g + l)$ are equal to $+1$.

Case III in §2-1. $\varepsilon^*(2g + l)(l = 1, 2, \cdots, 2g - 3)$ are equal to $+1$.

2-3. The following cases may occur for $\widetilde{\Sigma}_0^*$ when interchange operators are applied. Here for simplicity, we write C_i and C_{2g+l} for elements C_i^* and C_{2g+l}^* in $\widetilde{\Sigma}_0^*$, respectively.

Let $\delta(j(k), m(k))$ be the direction of $\gamma_j = \gamma_{j(k),m(k)}$ in $L_{j(k)}$.

Case I′. $j(k) \neq 1$, $C_{2g+j} < C_{j(k)}$, and $\delta(j(k), m(k)) = +1$ (cf. Example 3 in §3).

Case II′. $j(k) \neq 1$, $C_{2g+j} < C_{g+j(k)}$, and $\delta(j(k), m(k)) = -1$ (cf. Example 5 in §3).

Case III′. $j(k) = 1$ and $\delta(j(k), m(k)) = -1$ (cf. Example 6 in §3).

For these cases, $I_g(\alpha_{j(k)}, \gamma_j)$ are defined as follows. Namely if we set $I_g(\alpha_{j(k)}, \gamma_j)(\widetilde{\Sigma}_0) = \widetilde{\Sigma}_0^* = C_1^*, \cdots, C_{2g}^*; C_{2g+1}^*, \cdots, C_{4g-3}^*\}$, C_i^* and C_{2g+l}^* are defined as follows.

Case I′. (i) $C_{2g+j}^* = C_{g+j(k)}$, $C_{j(k)}^* = C_{2g+j}$ and $C_{g+j(k)}^* = A_{j(k)}(C_{2g+j})$.

(ii) $C_i^* = A_{j(k)}(C_i)(i \neq j(k))$ (resp. $C_{2g+l}^* = A_{j(k)}(C_{2g+l})$) in the case of $C_{2g+j} < C_i$(resp. $C_{2g+j} < C_{2g+l}$).

(iii) $C_i^* = C_i(i \neq g + j(k))$ (resp. $C_{2g+l}^* = C_{2g+l}$) in the case of $C_{2g+j} \not< C_i$ (resp. $C_{2g+j} \not< C_{2g+l}$).

Case II′. (i) $C_{2g+j}^* = C_{j(k)}$, $C_{j(k)}^* = C_{2g+j}$ and $C_{g+j(k)}^* = A_{j(k)}^{-1}(C_{2g+j})$.

(ii) $C_i^* = A_{j(k)}^{-1}(C_i)(i \neq g + j(k))$ (resp. $C_{2g+l}^* = A_{j(k)}^{-1}(C_{2g+l})$) in the case of $C_{2g+j} < C_i$(resp. $C_{2g+j} < C_{2g+l}$).

(iii) $C_i^* = C_i(i \neq j(k))$ (resp. $C_{2j+l}^* = C_{2g+l}$) in the case of $C_{2g+j} \not< C_i$ (resp. $C_{2g+j} \not< C_{2g+l}$).

Case III′. (i) $C_{2g+j}^* = C_1$, $C_1^* = C_{2g+j}$ and $C_{g+1}^* = A_1^{-1}(C_{2g+j})$.

(ii) $C_i^* = A_1^{-1}(C_i)(i \neq g + 1)$ (resp. $C_{2g+l}^* = A_1^{-1}(C_{2g+l})$) in the case of $C_{2g+j} < C_i$ (resp. $C_{2g+j} < C_{2g+l}$).

(iii) $C_i^* = C_i(i \neq 1)$ (resp. $C_{2g+l}^* = C_{2g+l}$) in the case of $C_{2g+j} \not< C_i$ (resp. $C_{2g+j} \not< C_{2g+l}$).

In the above cases, we determine the direction $\varepsilon^*(2g + l)$ of C_{2g+l}^* from C_1^* in $\widetilde{\Sigma}_0^*$ as follows. Let $\varepsilon(2g + l)$ be the direction of C_{2g+l} from C_1 in $\widetilde{\Sigma}_0$.

Case I′. (i) $\varepsilon^*(2g + j)$ is equal to $+1$.

(ii) $\varepsilon^*(2g + l)$ are equal to $-\varepsilon(2g + l)$ for l such that $\gamma_l^\delta \in L_{0,j(k)}$ and $C_{2g+j} < C_{2g+l}$.

(iii) Otherwise, $\varepsilon^*(2g + l)$ are equal to $\varepsilon(2g + l)$.

Case II′. (i) $\varepsilon^*(2g + j)$ is equal to -1.

(ii) $\varepsilon^*(2g + l)$ are equal to $-\varepsilon(2g + l)$ for l such that $\gamma_l^\delta \in L_{0,j(k)}$ and $C_{2g+j} < C_{2g+l}$.

(iii) Otherwise, $\varepsilon^*(2g + l)$ are equal to $\varepsilon(2g + l)$.

Case III′. (i) $\varepsilon^*(2g + j)$ is equal to -1.

(ii) $\varepsilon^*(2g + l)$ are equal to $-\varepsilon(2g + l)$ for l such that $\gamma_l^\delta \in L_{0,j(k)}$.

(iii) Otherwise, $\varepsilon^*(2g+l)$ are equal to $\varepsilon(2g+l)$.

The following cases may occur for $\tilde{\Sigma}_0^*$ when interchange operators are applied. We write again C_i and C_{2g+l} for elements C_i^* and C_{2g+l}^* in $\tilde{\Sigma}_0^*$, respectively. Let $\delta(j(k), m(k))$ be the direction of $\gamma_{j(k),m(k)}$ in $L_{j(k)}$.

Case I″. $j(k) \neq 1$, $C_{2g+j} < C_{g+j(k)}$, and $\delta(j(k), m(k)) = +1$ (cf. Example 4 in §3).

Case II″. $j(k) \neq 1$, $C_{2g+j} < C_{j(k)}$ and $\delta(j(k), m(k)) = -1$ (cf. Example 2 in §3).

Case III″. $j(k) = 1$ and $\delta(j(k), m(k)) = +1$ (cf. Example 1 in §3).

We note that Cases I, II, and III in §2-1 are contained in Cases I″, II″, and III″, respectively. For these cases, $I_g(\alpha_{j(k)}, \gamma_j)$ are defined in the same method as in §2-1. The direction $\varepsilon^*(2g+l)$ of C_{2g+l}^* from C_1^* in the image $\tilde{\Sigma}_0^*$ are similarly determined as in §2-2. Namely we determine $\varepsilon^*(2g+j)$ as the same one as that in §2-2, and $\varepsilon^*(2g+l)(l \neq j)$ by replacing $+1$ (resp. -1) in §2-2 by $+\varepsilon$(resp. $-\varepsilon$) if the direction of C_{2g+l} from C_1 in $\tilde{\Sigma}_0$ is ε. From now on, we write Cases I, II and III for Cases I″, II″ and III″, respectively.

2-4. We define the interchange operator acting on Σ_0 and Σ_0'. Let α_i^* and γ_j^* be the images of C_i^* and C_{2g+j}^*, respectively, under the natural projection $\Pi_0: \Omega(G_0) \to S_0$. We define the image Σ_0^*(resp. $\Sigma_0^{*\prime}$) of Σ_0 (resp. Σ_0') under $I_g(\alpha_{j(k)}, \gamma_j)$ by $\Sigma_0^* = \{\alpha_1^{*\prime}, \cdots, \alpha_{2g}^{*\prime}; \gamma_1^*, \cdots, \gamma_{2g-3}^*\}$(resp. as the set induced from Σ_0^* with $\Sigma_0^{*\prime} \sim \tilde{\Sigma}_0^*$). Furthermore ordered cycles $L_{0,i}^*$, which are the images of $L_{0,i}$ under $I_g(\alpha_{j(k)}, \gamma_j)$, are defined for the tree of $\tilde{\Sigma}_0^*$ with the direction determined in §§2-2, 2-3 in the same method as in §1-3.

Let $L_{0,j(h)} = (\alpha_{j(h)}; \gamma_{j(h),1}, \cdots, \gamma_{j(h),n(h)})(h = 1, 2, \cdots, t)$. We denote by $\delta(i, l)$ and $\delta^*(i, l)$ the direction of $\gamma_{i,l}$ in $L_{0,j(h)}$ and of $\gamma_{i,l}^*$ in $L_{0,j(h)}^*$, respectively. Then we easily see the following.

THEOREM 1-1. (i) *If $\delta(j(k), m(k))$ is equal to $+1$ (Cases* I, I′ *and* III *in* §§2-1, 2-3), *then*

$$L_{0,j(k)}^* = (\alpha_{j(k)}^*; \gamma_{j(k),m(k)+1}^{*\delta}, \cdots, \gamma_{j(k),n(k)}^{*\delta}, \gamma_{j(k),m(k)}^{*+1}, \gamma_{j(k),1}^{*\delta}, \gamma_{j(k),2}^{*\delta}, \cdots, \gamma_{j(k),m(k)-1}^{*\delta}),$$

where $\delta^*(j(k), i) = \delta(j(k), i)(i \neq m(k))$.

(ii) *If $\delta(j(k), m(k))$ is equal to -1* (Cases II, II′ *and* III′ *in* §§2-1, 2-3), *then*

$$L_{0,j(k)}^* = (\alpha_{j(k)}^*; \gamma_{j(k),m(k)-1}^{*-\delta}, \cdots, \gamma_{j(k),1}^{*-\delta}, \gamma_{j(k),m(k)}^{*-1}, \gamma_{j(k),n(k)}^{*-\delta}, \gamma_{j(k),n(k)-1}^{*-\delta}, \gamma_{j(k),n(k)-2}^{*-\delta}, \cdots, \gamma_{j(k),m(k)+1}^{*-\delta}),$$

where $\delta^*(j(k), i) = \delta(j(k), i)(i \neq m(k))$.

THEOREM 1-2. *For* $h \neq k$, (i) *if* $\delta(j(k), m(k)) = +1$ (*Cases* I, I′ *and* III *in* §§2-1, 2-3), *then*

$$\begin{aligned} L^*_{0,j(h)} = (\alpha^*_{j(h)};\ & \gamma^{*^\delta}_{j(h),1}, \cdots, \gamma^{*^\delta}_{j(h),m(h)-1}, \gamma^{*^{-\delta}}_{j(k),m(k)-1}, \\ & \cdots, \gamma^{*^{-\delta}}_{j(k),1}, \gamma^{*^{-1}}_{j(k),m(k)}, \gamma^{*^{-\delta}}_{j(k),n(k)}, \cdots, \gamma^{*^{-\delta}}_{j(k),m(k)+1}, \\ & \gamma^{*^\delta}_{j(h),m(h)+1}, \cdots, \gamma^{*^\delta}_{j(h),n(h)}), \end{aligned}$$

where $\delta^*(i, l) = \delta(i, l)((i, l) \neq (j(k), m(k)))$.

(ii) *If* $\delta(j(k), m(k)) = -1$ (*Cases* II, II′ *and* III′ *in* §§2-1, 2-3), *then*

$$\begin{aligned} L^*_{0,j(h)} = (\alpha^*_{j(h)};\ & \gamma^{*^\delta}_{j(h),1}, \cdots, \gamma^{*^\delta}_{j(h),m(h)-1}, \gamma^{*^\delta}_{j(k),m(k)+1}, \\ & \cdots, \gamma^{*^\delta}_{j(k),n(k)}, \gamma^{*^{+1}}_{j(k),m(k)}, \gamma^{*^\delta}_{j(k),1}, \cdots, \gamma^{*^\delta}_{j(k),m(k)-1}, \\ & \gamma^{*^\delta}_{j(k),m(h)+1}, \cdots, \gamma^{*^\delta}_{j(h),n(h)}), \end{aligned}$$

where $\delta^*(i, l) = \delta(i, l)((i, l) \neq (j(k), m(k)))$.

REMARK. (i) In Theorem 1-2, (i), $L^*_{0,j(h)}$ is obtained with $\gamma^\delta_{j(h),m(h)}$ in $L_{0,j(h)}$ replaced by the sequence

$$\gamma^{-\delta}_{j(k),m(k)-1}, \cdots, \gamma^{-\delta}_{j(k),1}, \gamma^{-1}_{j(k),m(k)}, \gamma^{-\delta}_{j(k),n(k)}, \gamma^{-\delta}_{j(k),n(k)-1}, \cdots, \gamma^{-\delta}_{j(k),m(k)+1},$$

and then with an asterisk attached to every γ^δ_l.

(ii) In Theorem 1-2, (ii), $L^*_{0,j(h)}$ is obtained with $\gamma^\delta_{j(h),m(h)}$ in $L_{0,j(h)}$ replaced by the sequence

$$\gamma^\delta_{j(k),m(k)+1}, \cdots, \gamma^\delta_{j(k),n(k)}, \gamma^{+1}_{j(k),m(k)}, \gamma^\delta_{j(k),1}, \cdots, \gamma^\delta_{j(k),m(k)-1}$$

and then with an asterisk attached to every γ^δ_l.

CONVENTION. For $\cdots \gamma^\delta_j \gamma^\delta_h \gamma^{-\delta}_h \gamma^\delta_l \cdots$, we write $\cdots \gamma^\delta_j \gamma^\delta_l \cdots$. Namely we eliminate $\gamma^\delta_h \gamma^{-\delta}_h$ from the sequence.

THEOREM 1-3. *For an ordered cycle* $L_{0,i} = (\alpha_i; \gamma^\delta_{i(1)}, \cdots, \gamma^\delta_{i(n)})$ *which does not contain* γ^δ_j, *the image* $L^*_{0,i}$ *under* $I_g(\alpha_{j(k)}, \gamma_j)$ *is* $(\alpha^*_i; \gamma^{*^\delta}_{i(1)}, \cdots, \gamma^{*^\delta}_{i(n)})$, *where* $\delta^*(i(l)) = \delta(i(l))(l = 1, 2, \cdots, n)$.

REMARK. In Theorems 1-1 and 1-2, $\gamma^*_{i,l} = \gamma_{i,l}((i, l) \neq (j(k), m(k)))$, $\gamma^*_{j(k),m(k)} = \alpha_{j(k)}$, $\alpha^*_i = \alpha_i(i \neq j(k))$ and $\alpha^*_{j(k)} = \gamma_{j(k),m(k)} = \gamma_j$. In Theorem 1-3, $\alpha^*_i = \alpha_i$ and $\gamma^*_{i(l)} = \gamma_{i(l)}(l = 1, 2, \cdots, n)$, and so $I_g(\alpha_{j(k)}, \gamma_j)(L_{0,i}) = L_{0,i}$.

2-5. We study the images $A^*_{0,1}, \cdots, A^*_{0,g}$ of $A_{0,1}, \cdots, A_{0,g}$, respectively, under $I_g(\alpha_{j(k)}, \gamma_j)$. $A^*_{0,i}$ is defined as the word in $A_{0,1}, \cdots, A_{0,g}$ which maps $C^*_{0,i}$ onto $C^*_{0,g+i}$, that is, $A^*_{0,i}(C^*_{0,i}) = C^*_{0,g+i}$. We easily see the following from §2-1.

THEOREM 2. *Let* $A^*_{0,i}(i = 1, 2, \cdots, g)$ *be the images of* $A_{0,i}$ *under* $I_g(\alpha_{j(k)}, \gamma_j)$. *Then*

(1) $A^*_{0,i} = A_{0,i}$ *in the case of* $C_{2g+j} \not< C_i$ *and* $C_{2g+j} \not< C_{g+i}$.

(2) *In the case of* $C_{2g+j} < C_i$ *and* $C_{2g+j} < C_{g+i}$,

(i) $A^*_{0,i} = A^{-1}_{0,j(k)} A_{0,i} A_{0,j(k)}$ *if* $C_{2g+j} < C_{g+j(k)}$,

(ii) $A^*_{0,i} = A_{0,j(k)} A_{0,i} A^{-1}_{0,j(k)}$ *if* $C_{2g+j} < C_{j(k)}$.

(3) *In the case of* $C_{2g+j} < C_i$ *and* $C_{2g+j} \not< C_{g+i}$,

(i) $A^*_{0,i} = A_{0,i} A_{0,j(k)}$ *if* $C_{2g+j} < C_{g+j(k)}$,

(ii)
$$A^*_{0,i} = \begin{cases} A_{0,i} A^{-1}_{0,j(k)} & (i \neq j(k)) \\ A_{0,j(k)} & (i = j(k),\ \delta(j(k), m(k)) = +1) \\ A^{-1}_{0,j(k)} & (i = j(k),\ \delta(j(k), m(k)) = -1) \end{cases}$$

if $C_{2g+j} < C_{j(k)}$.

(4) *In the case of* $C_{2g+j} < C_{g+i}$ *and* $C_{2g+j} \not< C_i$,

(i)
$$A^*_{0,i} = \begin{cases} A^{-1}_{0,j(k)} A_{0,i} & (i \neq j(k)) \\ A_{0,j(k)} & (i = j(k),\ \delta(j(k), m(k)) = +1) \\ A^{-1}_{0,j(k)} & (i = j(k),\ \delta(j(k), m(k)) = -1) \end{cases}$$

if $C_{2g+j} < C_{g+j(k)}$, *and*

(ii) $A^*_{0,i} = A_{0,j(k)} A_{0,i}$ *if* $C_{2g+j} < C_{j(k)}$.

We denote by $\langle G^*_0 \rangle$ the marked Schottky group generated by $A^*_{0,1}$, $A^*_{0,2}, \cdots, A^*_{0,g}$: $\langle G^*_0 \rangle = \langle A^*_{0,1}, A^*_{0,2}, \cdots, A^*_{0,g} \rangle$. Let $\tau_0 = (t_{0,1}, \cdots, t_{0,g}, \rho_{0,1}, \cdots, \rho_{0,2g-3})$ and $\tau^*_0 = (t^*_{0,1}, \cdots, t^*_{0,g}, \rho^*_{0,1}, \cdots, \rho^*_{0,2g-3})$ be the points in $\mathfrak{S}_g(\tilde{\Sigma}_0)$ and in $\mathfrak{S}_g(\tilde{\Sigma}^*_0)$ corresponding to $\langle G_0 \rangle$ and $\langle G^*_0 \rangle$, respectively. We define $I_g(\alpha_{j(k)}, \gamma_j)(\tau)$ by $I_g(\alpha_{j(k)}, \gamma_j)(\tau) = \tau^*$. We denote by $\mathrm{mult}(A_{0,l})$ the multiplier $\lambda_{0,l}(|\lambda_{0,l}| > 1)$ of $A_{0,l}$.

Corollary. (1) $t^*_{0,i} = t_{0,i}$ *in the case of Theorem* 2, (1) *and* (2).

(2) $t^*_{0,i} = 1/\mathrm{mult}(A_{0,i} A_{0,j(k)})$ *in the case of Theorem* 2, (3) (i), *and*

$$t^*_{0,i} = \begin{cases} t_{0,i} & (i = j(k)) \\ 1/\mathrm{mult}(A_{0,i} A^{-l}_{0,j(k)}) & (i \neq j(k)) \end{cases}$$

in the case of Theorem 2, (3)(ii).

(3)
$$t^*_{0,i} = \begin{cases} t_{0,i} & (i = j(k)) \\ 1/\mathrm{mult}(A^{-1}_{0,j(k)} A_{0,i}) & (i \neq j(k)) \end{cases}$$

in the case of Theorem 2, (4)(i), *and*

$$t^*_{0,i} = 1/\mathrm{mult}(A_{0,j(k)} A_{0,i})$$

in the case of Theorem 2, (4)(ii).

2-6. Thus far we defined the interchange operator $I_g(\alpha_{j(k)}, \gamma_j)$ acting only on the center of the Schottky space $\mathfrak{S}_g(\tilde{\Sigma}_0)$. Here we extend the operator to the whole space $\mathfrak{S}_g(\tilde{\Sigma}_0)$.

Let $\langle G\rangle = \langle A_1, \cdots, A_g\rangle$ be a marked Schottky group. From now on, we write $I_g(j(k), j)$ for $I_g(\alpha_{j(k)}, \gamma_j)$. We define the operator $I_g(j(k), j)$ on $\langle G\rangle$ as follows. Suppose that

$$I_g(j(k), j)(\langle G_0\rangle) = \langle A^*_{0,1}, \cdots, A^*_{0,g}\rangle = \langle G^*_0\rangle ,$$

where

$$A^*_{0,i} = W_i(A_{0,1}, \cdots, A_{0,g}) \quad (i = 1, 2, \cdots, g) \tag{3}$$

are words in $A_{0,1}, \cdots, A_{0,g}$. Then we define $I_g(j(k), j)$ acting on $\langle G\rangle$ by

$$I_g(j(k), j)(\langle G\rangle) = \langle A^*_1, \cdots, A^*_g\rangle = \langle G^*\rangle ,$$

where $A^*_i = W_i(A_1, \cdots, A_g)(i = 1, 2, \cdots, g)$ are the words obtained with $A_{0,l}(l = 1, 2, \cdots, g)$ in (3) replaced by A_l.

Let τ and τ^* be the points in $\mathfrak{S}_g(\tilde{\Sigma}_0)$ and in $\mathfrak{S}_g(\tilde{\Sigma}^*_0)$ corresponding to $\langle G\rangle$ and $\langle G^*\rangle$, respectively. We define the operator $I_g(j(k), j)$ acting on $\mathfrak{S}_g(\tilde{\Sigma}_0)$ by $I_g(j(k), j)(\tau) = \tau^*$.

2-7. We give a compact Riemann surface S of genus g with or without nodes, and a basic system of loops and nodes $\Sigma = \{\alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3}\}$ on S such that one of the sets $\Sigma' = \{\alpha'_1, \cdots, \alpha'_{2g}; \gamma_1, \cdots, \gamma_{2g-3}\}$ induced from Σ is compatible with $\tilde{\Sigma}_0$. We define an ordered cycle L_i with respect to $\tilde{\Sigma}$ by replacing $\alpha_{0,i}$ and $\gamma^\delta_{0,l}$ in the cycle $L_{0,i}$ by α_i and γ^δ_l, respectively.

Suppose that $I(\{j\}) = \{j(1), j(2), \cdots, j(t(j))\}$ and that $\gamma_{j(k),m(k)} = \gamma_j$ for each $k = 1, 2, \cdots, t(j)$. We define the operator $I_g(j(k), j)$ acting on Σ by

$$I_g(j(k), j)(\Sigma) = \{\alpha^*_1, \cdots, \alpha^*_g; \gamma^*_1, \cdots, \gamma^*_{2g-3}\} = \Sigma^* ,$$

where α^*_i and γ^*_l are defined as follows:

(1) In the case where α^*_i and γ^*_l are contained in $L^*_{j(k)}(k = 1, 2, \cdots, t(j))$, $\gamma^*_{i,l} = \gamma_{i,l}((i, l) \neq (j(k), m(k)))$, $\gamma^*_{j(k),m(k)} = \alpha_{j(k)}$, $\alpha^*_i = \alpha_i(i \neq j(k))$ and $\alpha^*_{j(k)} = \gamma_{j(k),m(k)} = \gamma_j$.

(2) Otherwise, $\alpha^*_i = \alpha_i$ and $\gamma^*_l = \gamma_l$.

Let $\Sigma^{*\prime} = \{\alpha^{*\prime}_1, \cdots, \alpha^{*\prime}_{2g}; \gamma^*_1, \cdots, \gamma^*_{2g-3}\}$ be the set induced from Σ^* such that $\Sigma^{*\prime} \sim \tilde{\Sigma}^*_0$. Then we define the operator $I_g(j(k), j)$ acting on Σ' by $I_g(j(k), j)(\Sigma') = \Sigma^{*\prime}$.

Let $L_i = (\alpha_i; \gamma^\delta_{i(1)}, \cdots, \gamma^\delta_{i(t(i))})$. Suppose

$$I_g(j(k), j)(L_{0,i}) = (\alpha^*_{0,i}; \gamma^{*\delta}_{0,i}, \cdots, \gamma^{*\delta}_{0,i(t(i))}) = L^*_{0,i} .$$

Then we define $I_g(j(k), j)(L_i)$ by

$$I_g(j(k), j)(L_i) = (\alpha^*_i; \gamma^{*\delta}_1, \cdots, \gamma^{*\delta}_{i,t(i)}) ,$$

where for each $l = 1, 2, \cdots, t(i)$, δ of $\gamma^{*\delta}_l$ is equal to δ of $\gamma^{*\delta}_{0,l}$ in $L^*_{0,i}$.

3. Examples. In this section, we explain the interchange operators introduced in the previous section by some illustrative examples. We take the example in §1-4. In this section, we drop the superscript "+1" of γ_i^{+1} for simplicity.

EXAMPLE 1. Let $\tilde{\Sigma}_1 = \{C_{11}, C_{12}, \cdots, C_{16}; C_{17}, C_{18}, C_{19}\}$ be the image of $\tilde{\Sigma}_0$ under the interchange operator $I_g(\alpha_{0,1}, \gamma_{0,2})$. We set $\Phi_1 = I_g(\alpha_{0,1}, \gamma_{0,2})$. Let $\Sigma_1 = \{\alpha_{11}, \alpha_{12}, \alpha_{13}; \gamma_{11}, \gamma_{12}, \gamma_{13}\}$ be the projection of $\tilde{\Sigma}_1$. Then we have $C_{11} = A_{0,1}^{-1}(C_{0,8})$, $C_{12} = A_{0,1}^{-1}(C_{0,2})$, $C_{13} = C_{0,3}$, $C_{14} = C_{0,8}$, $C_{15} = C_{0,5}$, $C_{16} = C_{0,6}$, $C_{17} = C_{0,7}$, $C_{18} = C_{0,1}$ and $C_{19} = C_{0,9}$. Furthermore, we have $L_{11} = \Phi_1(L_{0,1}) = (\alpha_{11}; \gamma_{12}, \gamma_{11})$, $L_{12} = \Phi_1(L_{0,2}) = (\alpha_{12}; \gamma_{12}, \gamma_{11}, \gamma_{13})$, and $L_{13} = \Phi_1(L_{0,3}) = (\alpha_{13}; \gamma_{13}^{-1}, \gamma_{11}^{-1})$. The tree of $\tilde{\Sigma}_1$ and the curves of $\tilde{\Sigma}_1$ on $\hat{C}$ are as in Fig. 5 and Fig. 6, respectively. Setting $A_{1i} = \Phi_1(A_{0,i})(i = 1, 2, 3)$, we have $A_{11} = A_{0,1}$, $A_{12} = A_{0,2}A_{0,1}$ and $A_{13} = A_{0,3}$.

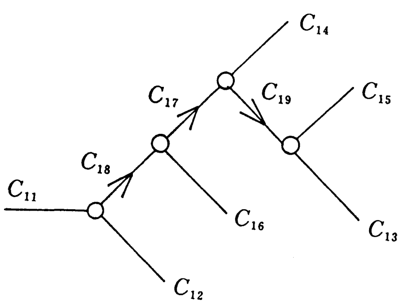

FIGURE 5

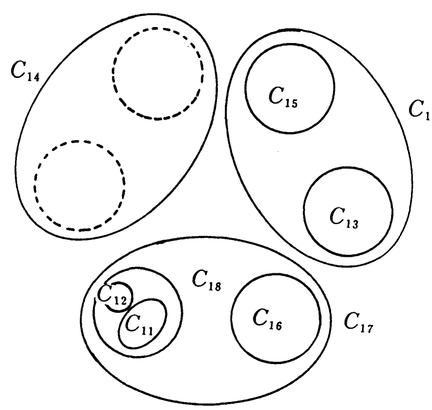

FIGURE 6

EXAMPLE 2. Let $\tilde{\Sigma}_2 = \{C_{21}, C_{22}, \cdots, C_{26}; C_{27}, C_{28}, C_{29}\}$ be the image of $\tilde{\Sigma}_1$ under $I_g(\alpha_{13}, \gamma_{11})$. We set $\Phi_2 = I_g(\alpha_{11}, \gamma_{11})$. Let $\Sigma_2 = \{\alpha_{21}, \alpha_{22}, \alpha_{23}; \gamma_{21}, \gamma_{22}, \gamma_{23}\}$ be the projection of $\tilde{\Sigma}_2$. Then we have $C_{21} = C_{11}$, $C_{22} = C_{12}$, $C_{23} = A_{13}(C_{17})$, $C_{24} = A_{13}(C_{14})$, $C_{25} = A_{13}(C_{15})$, $C_{26} = C_{17}$, $C_{27} = C_{16}$, $C_{28} = C_{18}$, $C_{29} = A_{13}(C_{19})$. Furthermore, we have $L_{23} = \Phi_2(L_{13}) = (\alpha_{23}^{-1}; \gamma_{21}, \gamma_{23}^{-1}) = (\alpha_{23}; \gamma_{23}, \gamma_{21}^{-1})$, $L_{21} = \Phi_2(L_{11}) = (\alpha_{21}; \gamma_{22}, \gamma_{21}, \gamma_{23}^{-1})$ and $L_{22} = \Phi_2(L_{12}) = (\alpha_{22}; \gamma_{22}, \gamma_{21}, \gamma_{22}^{-1}, \gamma_{23}) = (\alpha_{22}; \gamma_{22}, \gamma_{21})$. The tree of $\tilde{\Sigma}_2$ is as in Fig. 7. Setting $A_{2i} = \Phi_2(A_{1i})(i = 1, 2, 3)$, we have $A_{21} = A_{13}A_{11}$, $A_{22} = A_{13}A_{12}$ and $A_{23} = A_{13}^{-1}$.

EXAMPLE 3. Let $\tilde{\Sigma}_3 = \{C_{31}, C_{32}, \cdots, C_{36}; C_{37}, C_{38}, C_{39}\}$ be the image of $\tilde{\Sigma}_2$ under $I_g(\alpha_{23}, \gamma_{23})$. We set $\Phi_3 = I_g(\alpha_{23}, \gamma_{23})$. Let $\Sigma_3 = \{\alpha_{31}, \alpha_{32}, \alpha_{33}; \gamma_{31}, \gamma_{32}, \gamma_{33}\}$ be the projection of $\tilde{\Sigma}_3$. Then we have $C_{31} = C_{21}$, $C_{32} = C_{22}$, $C_{33} = C_{29}$, $C_{34} = A_{23}(C_{24})$, $C_{35} = C_{25}$, $C_{36} = A_{23}(C_{29})$, $C_{37} = C_{27}$, $C_{38} = C_{28}$ and $C_{39} = C_{26}$. Furthermore, we have $L_{33} = \Phi_3(L_{23}) = (\alpha_{33}; \gamma_{31}^{-1}, \gamma_{33})$, $L_{31} = \Phi_3(L_{21}) = (\alpha_{31}; \gamma_{32}, \gamma_{31}, \gamma_{31}^{-1},$

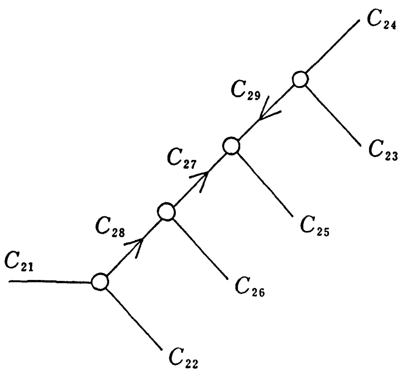

FIGURE 7

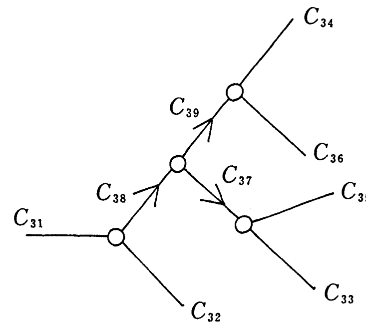

FIGURE 8

$\gamma_{33}) = (\alpha_{31}; \gamma_{32}, \gamma_{33})$, and $L_{32} = \Phi_3(L_{22}) = (\alpha_{32}; \gamma_{32}, \gamma_{31})$. The tree of $\tilde{\Sigma}_3$ is as in Fig. 8. Setting $A_{3i} = \Phi_3(A_{2i})(i = 1, 2, 3)$, we have $A_{31} = A_{23}A_{21}$, $A_{32} = A_{22}$ and $A_{33} = A_{23}$.

EXAMPLE 4. Let $\tilde{\Sigma}_4 = \{C_{41}, C_{42}, \cdots, C_{46}; C_{47}, C_{48}, C_{49}\}$ be the image of $\tilde{\Sigma}_3$ under $I_g(\alpha_{32}, \gamma_{32})$. We set $\Phi_4 = I_g(\alpha_{32}, \gamma_{32})$. Let $\Sigma_4 = \{\alpha_{41}, \alpha_{42}, \alpha_{43}; \gamma_{41}, \gamma_{42}, \gamma_{43}\}$ be the projection of $\tilde{\Sigma}_4$. Then we have $C_{41} = C_{31}$, $C_{42} = A_{32}^{-1}(C_{38})$, $C_{43} = A_{32}^{-1}(C_{33})$, $C_{44} = A_{32}^{-1}(C_{34})$, $C_{45} = C_{38}$, $C_{46} = A_{32}^{-1}(C_{36})$, $C_{47} = A_{32}^{-1}(C_{37})$, $C_{48} = C_{32}$ and $C_{49} = A_{32}^{-1}(C_{39})$. Furthermore, we have $L_{42} = \Phi_4(L_{32}) = (\alpha_{42}; \gamma_{41}, \gamma_{42})$, $L_{41} = \Phi_4(L_{31}) = (\alpha_{41}; \gamma_{42}^{-1}, \gamma_{41}^{-1}, \gamma_{43})$ and $L_{43} = \Phi_4(L_{33}) = (\alpha_{43}; \gamma_{41}^{-1}, \gamma_{43})$. The tree of $\tilde{\Sigma}_4$ is as in Fig. 9. Setting $A_{4i} = \Phi_4(A_{3i})(i = 1, 2, 3)$, we have $A_{41} = A_{32}^{-1}A_{31}$, $A_{42} = A_{32}$ and $A_{43} = A_{32}^{-1}A_{33}A_{32}$.

EXAMPLE 5. Let $\tilde{\Sigma}_5 = \{C_{51}, C_{52}, \cdots, C_{56}; C_{57}, C_{58}, C_{59}\}$ be the image of $\tilde{\Sigma}_4$ under $I_g(\alpha_{43}, \gamma_{41})$ and let $\Sigma_5 = \{\alpha_{51}, \alpha_{52}, \alpha_{53}; \gamma_{51}, \gamma_{52}, \gamma_{53}\}$ be the projection of $\tilde{\Sigma}_5$. We set $\Phi_5 = I_g(\alpha_{43}, \gamma_{41})$. Then we have $C_{51} = C_{41}$, $C_{52} = A_{43}^{-1}(C_{42})$, $C_{53} =$

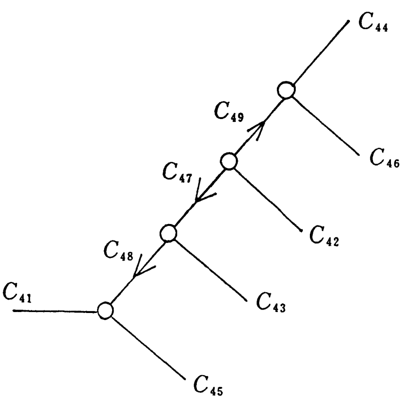

FIGURE 9

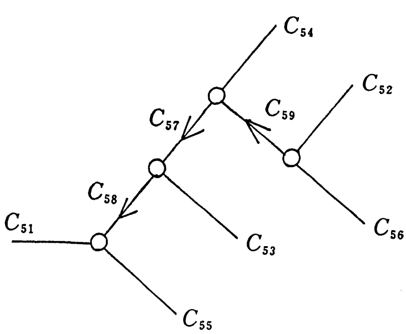

FIGURE 10

C_{47}, $C_{54}=A_{43}^{-1}(C_{44})$, $C_{55}=C_{45}$, $C_{56}=A_{43}^{-1}(C_{47})$, $C_{57}=C_{43}$, $C_{58}=C_{48}$ and $C_{59}=A_{43}^{-1}(C_{49})$. Furthermore, we have $L_{53}=\Phi_5(L_{43})=(\alpha_{53}^{-1};\gamma_{53},\gamma_{51})=(\alpha_{53};\gamma_{51}^{-1},\gamma_{53}^{-1})$, $L_{51}=\Phi_5(L_{41})=(\alpha_{51};\gamma_{52}^{-1},\gamma_{51}^{-1},\gamma_{53}^{-1},\gamma_{53})=(\alpha_{51};\gamma_{52}^{-1},\gamma_{51}^{-1})$, $L_{52}=\Phi_5(L_{42})=(\alpha_{52};\gamma_{53},\gamma_{51},\gamma_{52})$. The tree of $\tilde{\Sigma}_5$ is as in Fig. 10. Setting $A_{5i}=\Phi_5(A_{4i})(i=1,2,3)$, we have $A_{51}=A_{43}^{-1}A_{41}$, $A_{52}=A_{42}A_{43}$ and $A_{53}=A_{43}^{-1}$.

EXAMPLE 6. Let $\tilde{\Sigma}_6=\{C_{61},C_{62},\cdots,C_{66};C_{67},C_{68},C_{69}\}$ be the image of $\tilde{\Sigma}_5$ under $I_g(\alpha_{51},\gamma_{51})$ and let $\Sigma_6=\{\alpha_{61},\alpha_{62},\alpha_{63};\gamma_{61},\gamma_{62},\gamma_{63}\}$ be the projection of $\tilde{\Sigma}_6$. We set $\Phi_6=I_g(\alpha_{51},\gamma_{51})$. Then we have $C_{61}=C_{57}$, $C_{62}=A_{51}^{-1}(C_{52})$, $C_{63}=C_{53}$, $C_{64}=A_{51}^{-1}(C_{57})$, $C_{65}=C_{55}$, $C_{66}=A_{51}^{-1}(C_{56})$, $C_{67}=C_{51}$, $C_{68}=C_{58}$ and $C_{69}=A_{51}^{-1}(C_{59})$. Furthermore, we have $\Phi_6(L_{51})=(\alpha_{61}^{-1};\gamma_{61},\gamma_{62}^{-1})=(\alpha_{61};\gamma_{62},\gamma_{61}^{-1})$, $\Phi_6(L_{52})=(\alpha_{62};\gamma_{62},\gamma_{61},\gamma_{62}^{-1},\gamma_{62})=(\alpha_{62};\gamma_{63},\gamma_{61})$ and $\Phi_6(L_{53})=(\alpha_{63};\gamma_{62},\gamma_{61}^{-1},\gamma_{63}^{-1})$. The tree of $\tilde{\Sigma}_6$ is as in Fig. 11. Setting $A_{6i}=\Phi_6(A_{5i})$ $(i=1,2,3)$, we have $A_{61}=A_{51}^{-1}$, $A_{62}=A_{52}A_{51}$ and $A_{63}=A_{51}^{-1}A_{53}$.

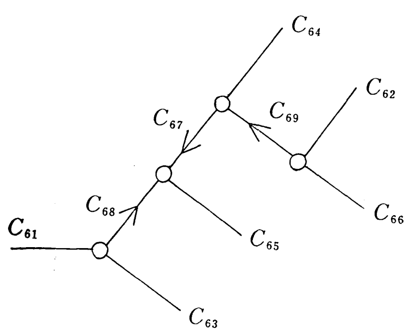

FIGURE 11

4. **Uniformization.** 4-1. Let $\langle G_0\rangle$, $\tilde{\Sigma}_0$ and $S_0=\Omega(G_0)/G_0$ be a fixed marked Schottky group, a fixed basic system of Jordan curves, and the compact Riemann surface of genus g without nodes, respectively, as in §1. Let S be a compact Riemann surface of genus g with nodes and let $\Sigma=\{\alpha_1,\cdots,\alpha_g;\gamma_1,\cdots,\gamma_{2g-3}\}$ be a basic system of loops and nodes on S such that all nodes on S are elements of Σ. Let $\Sigma'=\{\alpha_1',\cdots,\alpha_{2g}';\gamma_1,\cdots,\gamma_{2g-3}\}$ be one of the sets induced from Σ. We choose a basic system of loops $\Sigma_1=\{\alpha_{11},\cdots,\alpha_{1g};\gamma_{11},\cdots,\gamma_{1,2g-3}\}$ with $\alpha_{1i}=\alpha_{0i}(i=1,2,\cdots,g)$ on S_0 such that one of the sets $\Sigma_1'=\{\alpha_{11}',\cdots,\alpha_{1,2g}';\gamma_{11},\cdots,\gamma_{1,2g-3}\}$ induced from Σ_1 is compatible with Σ', which is similarly defined as in §1. Then we easily see the following.

PROPOSITION 1. *There is a basic system of Jordan curves* $\tilde{\Sigma}_1=\{C_{0,1},C_{0,2},\cdots,C_{0,2g}:C_{1,2g+1},\cdots,C_{1,4g-3}\}$ *for* $\langle G_0\rangle$ *satisfying the following conditions* (i)-(ii): (i) $\tilde{\Sigma}_1\sim\Sigma_1'$, *and* (ii) $\Pi(C_{1,2g+j})=\gamma_{1,j}(j=1,2,\cdots,2g-3)$, *where* Π *is the natural projection from* $\Omega(G_0)$ *onto* S_0.

4-2. Let $L_{j,1}, L_{j,2}, \cdots, L_{j,t(j)}$ be the complete list of ordered cycles containing γ_j^{+1} or γ_j^{-1} with $\gamma_j \in \Sigma$. We let $\gamma_{j(1)}, \gamma_{j(2)}, \cdots, \gamma_{j(r)}$ be the complete list of nodes $\{\gamma_l\}$ in Σ.

Step 1. (1) Suppose that there are $i \in \{1, 2, \cdots, t(j(k))\}$ and $k \in \{1, 2, \cdots, r\}$ such that $\alpha_{j(k),i}$ in $L_{j(k),i}$ is a loop. For example, let $\alpha_{j(1),m(1)}$ be the loop, which is identical with $\alpha_{i(1)}$ for some $i(1) \in \{1, 2, \cdots, g\}$. We denote the images of Σ, Σ' and $\widetilde{\Sigma}_1$ under $I_g(i(1), j(1))$ by Σ_{11}, Σ'_{11} and $\widetilde{\Sigma}_{11}$, respectively. Then we note that $I_g(i(1), j(1))(\gamma_{j(1)})$ is a loop and $I_g(i(1), j(1))(\alpha_{i(1)})$ is a node.

(2) Suppose that all $\alpha_{j(k),i}(i = 1, 2, \cdots, t(j(k)); k = 1, 2, \cdots, r)$ are nodes. Then the pair (S, Σ) has Property (A).

Step 2. In the case of Step 1, (1), the complete list of nodes $\{\gamma_l\}$ with $\gamma_l \in \Sigma_{11}$ is $\gamma_{j(2)}, \gamma_{j(3)}, \cdots, \gamma_{j(r)}$. Let $L_{j(k),i}(i = 1, 2, \cdots, t(j(k), 1))$ be ordered cycles containing $\gamma_{j(k)}^{\delta}$ with respect to Σ_{11} for $k = 2, 3, \cdots, r$.

(1) Suppose that there are $i \in \{1, 2, \cdots, t(j(k), 1)\}$ and $k \in \{2, 3, \cdots, r\}$ such that $\alpha_{j(k),i}$ is a loop. For example, let $\alpha_{(j)(2),m(2)}$ be the loop, which is identical with $\alpha_{i(2)}$ for some $i(2) \in \{1, 2, \cdots, g\} \setminus \{i(1)\}$. We denote the images of Σ_{11}, Σ'_{11} and $\widetilde{\Sigma}_{11}$ under $I_g(i(2), j(2))$ by Σ_{12}, Σ'_{12} and $\widetilde{\Sigma}_{12}$, respectively. We note that $I_g(i(2), j(2))(\gamma_{j(2)})$ is a loop and $I_g(i(2), j(2))(\alpha_{i(2)})$ is a node.

(2) Suppose that all $\alpha_{j(k),i}(i = 1, 2, \cdots, t(j(k), 1)); k = 2, 3, \cdots, r)$ are nodes. Then the pair (S, Σ_{11}) has Property (A).

Step 3. We continue the same procedure as above, and finally we find a number $s(\leqq r)$ such that the following (i) and (ii) are satisfied: (i) Let Σ_{1l}, Σ'_{1l} and $\widetilde{\Sigma}_{1l}(l = 1, 2, \cdots, s)$ be the images of $\Sigma_{1,l-1}$, $\Sigma'_{1,l-1}$ and $\widetilde{\Sigma}_{1,l-1}$, respectively, under the interchange operator $I_g(i(l), j(l))$, where $\Sigma_{1,l-1} = \Sigma$, $\Sigma'_{1,l-1} = \Sigma'$ and $\widetilde{\Sigma}_{1,l-1} = \Sigma_1$ for $l = 1$. Then for each $l = 1, 2, \cdots, s$, $I_g(i(l), j(l))(\gamma_{j(l)})$ is a loop. (ii) For any $j(k)$ $(k = s + 1, s + 2, \cdots, r)$, $\alpha_{j(k),u}$ $(u = 1, 2, \cdots, t(j(k), s + 1))$ in $L_{j(k),u}$ are nodes, where $L_{j(k),u}$ are cycles containing $\gamma_{j(k)}^{\delta}$ with respect to Σ_s.

We write Σ^* and $\widetilde{\Sigma}_0^*$ for Σ_{1s} and $\widetilde{\Sigma}_{1s}$, respectively. Then the pair (S, Σ^*) has Property (A). From Steps 1 through 3, we have the following.

PROPOSITION 2. *Given a compact Riemann surface S of genus $g(g \geqq 2)$ with nodes and a basic system of loops and nodes Σ on S such that all nodes on S are contained in Σ, then a finite number of interchange operators can be applied to Σ so that the resulting basic system of loops and nodes Σ^* is such that the pair (S, Σ^*) has Property* (A).

4-3. From Propositions 1 and 2 above, and [5, Theorem 2], we have the following.

THEOREM 3. *Let S and Σ be as in Proposition 2. Let $\widetilde{\Sigma}_1$ be a basic* *system of Jordan curves raised from Σ as in Proposition* 1. *Let $\tilde{\Sigma}_0^*$ be the basic system of Jordan curves raised from $\tilde{\Sigma}_1$ by application of the interchange operators in Proposition* 2. *Then there exists a point in the augmented Schottky space with respect to $\tilde{\Sigma}_0^*$ which represents the Riemann surface S.*

REFERENCES

[1] L. BERS, Spaces of degenerating Riemann surfaces, Ann of Math. Studies 79 (1974), 43-55.

[2] L. BERS, Automorphic forms for Schottky groups, Advances in Math. 16 (1975), 332-361.

[3] J. S. BIRMAN, The algebraic structure of surface mapping class groups, in Discrete groups and automorphic functions, 1977, (W. J. Harvey, ed.), Academic press, London-New York-San Francisco, 163-198.

[4] H. SATO, On augmented Schottky spaces and automorphic forms, I, Nagoya Math. J. 75 (1979), 151-175.

[5] H. SATO, Introduction of new coordinates to the Schottky space—The general case-, J. Math. Soc. Japan 35 (1983), 23-35.

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