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# SECOND ORDER DIFFERENTIAL OPERATORS AND DIRICHLET INTEGRALS WITH SINGULAR COEFFICIENTS:

# I. FUNCTIONAL CALCULUS OF ONE-DIMENSIONAL OPERATORS

BERNARD GAVEAU(*), MASAMI OKADA(**) AND TATSUYA OKADA(***)

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**Introduction.** In this series of works, we try to develop a constructive theory mainly on special examples of elliptic second order operators (and also, sometimes, hyperbolic operators) with very irregular coefficients (for example, there can be Dirac measures along hypersurfaces in the second order terms). Our aim is to compute as explicitly as possible, examples of fundamental solutions and to show new phenomena which occur in such situations. Our motivations come from various areas: first in mathematical physics it is more important to have explicit models than general theory; for example in this work, we have "explicit" formulas for transmission of waves or of heat in one dimensional medium with discontinuous indices; in the second paper of this series, we shall also examine higher dimensional situations related to interface problems. The second motivation is more mathematical; recently, Fukushima [2] has developped a remarkable theory of Dirichlet integrals allowing rather general coefficients and he constructed in the abstract manner stochastic processes associated to them; unfortunately very few examples were given apart from the usual Brownian motion although many natural examples come from mathematical physics, engineering problems, analysis in several complex variables, and even in algebraic topology. Our work will give some examples in these various areas.

This first part studies the one-dimensional case; we first give general motivation (coming from physics) to study operators of the type $c^{-2}(x)d/dx(a^{-2}(x)d/dx)$ and we also give two general methods of solution: the spectral method in the self-adjoint case and the method of Green functions in the general case. It is quite surprising that both methods lead to very concrete results: we can write an explicit form of the spectral measure as a series (*which is not a perturbation series*), provided that $c/a$ has a finite number of accumulation points of the set of discontinuities and $\log(c/a)$ is of bounded variation. The method is to reduce everything to an infinite product of $2\times 2$ matrices which can be done explicitly; Chapter II gives example with piecewise constant coefficients and Chapter III gives the formula for the infinite product.

In Chapter IV, we introduce, on a simple example, a new kind of singular perturbation problem and we show that a limit of operators with irregular coefficients is a rather subtle phenomenon. Finally, Chapter V gives the same kind of formulas as in Chapter III but for radial 3-dimensional problems.

## CHAPTER I. Definition of operators with general coefficients and their applications.

The purposes of this introductory chapter are to give a motivation for the introduction of operators with irregular coefficients arising in several problems of mathematical physics, to give a mathematical definition of these operators and finally to fix certain notations concerning spectral resolution and Titchmarsh-Kodaira-Yosida theory.

**1. Motivation coming from mathematical physics problems.**

(a) *Heat transfer in a general medium.* We consider here the heat transfer in a general medium in $\boldsymbol{R}^n$ ($n = 1, 2, 3$). The material constituting the medium is characterized at each point $x$ by two coefficients: the first is the specific heat $c^2(x)$; its meaning is that when the temperature at $x$ increases by 1 degree then the heat in the material at that point increases by 1 Joule. If $T(x)$ is the temperature at $x$ and $Q(x)$ is the heat at $x$, then

$$Q(x) = c^2(x)T(x) .$$

The second coefficient is the diffusion coefficient denoted by $a^{-2}(x)$; its meaning is that, at each point $x$, the flux of heat $\boldsymbol{J}$ is given by

$$\boldsymbol{J}(x) = \frac{1}{a^2(x)}\nabla T(x) .$$

If $V$ is a fixed volume with boundary $S$, and if there are no internal sources of heat inside $V$, the variation in time $dt$ of the quantity of heat inside $V$ is $d_t\int_V Q(x, t)dx$ and it is equal to the heat flux through $S$ in time $dt$

$$\left(\int_S \boldsymbol{J}(x, t)\cdot \boldsymbol{n}(x)dS\right)dt$$

and we obtain the law of heat diffusion (Fourier's law)

$$\frac{d}{dt}\int_V c^2(x)T(x, t)dx = \int_S \frac{1}{a^2(x)}\nabla T(x, t)\cdot \boldsymbol{n}dS$$

($\boldsymbol{n}$ is the external normal to $S$, $dS$ is the area element) and so we obtain

$$\frac{d}{dt}\int_V c^2(x)T(x, t)dx = \int_V \operatorname{div}\left(\frac{1}{a^2(x)}\nabla T(x, t)\right)dx . \tag{1.1}$$

To derive this law (1.1) we have not assumed that $a$ and $c$ are continuous coefficients; they may be discontinuous.

We shall suppose that the coefficients $a$ and $c$ are $C^1$ and $C^0$ functions respectively on subdomains of the domain of definition but they can be discontinuous across a finite set of hypersurfaces in $\boldsymbol{R}^n$ and their jump across these hypersurfaces are finite jumps. Let $D_i$, $D_j$ be domains of the total domain where $a$ and $c$ are $C^1$ and $C^0$ functions, respectively. Taking for $V$ a small domain contained in $D_i$ or contained in $D_j$ and denoting

$$a_i = a|_{D_i} \quad c_i = c|_{D_i}$$

we obtain that $T_i = T|_{D_i}$ satisfies the usual heat equation

$$\frac{\partial T_i(x, t)}{\partial t} = \frac{1}{c_i^2(x)} \operatorname{div}\Big(\frac{1}{a_i^2(x)} \nabla T_i(x, t)\Big) \quad \text{in} \quad D_i . \tag{1.2}$$

Let $S_{ij}$ be the hypersurface separating $D_i$ from $D_j$; take for $V$ a small domain cutting $S_{ij}$. Then (1.1) becomes

$$\begin{aligned} &\frac{d}{dt}\Big(\int_{V\cap D_i} c_i^2(x) T_i(x, t)dx + \int_{V\cap D_j} c_j^2(x) T_j(x, t)dx\Big) \\ &\quad = \lim_{\varepsilon\to 0}\Big(\int_{V_\varepsilon \cap D_i} \operatorname{div}\Big(\frac{1}{a_i^2(x)}\nabla T_i(x, t)\Big)dx + \int_{V_\varepsilon \cap D_j} \operatorname{div}\Big(\frac{1}{a_j^2(x)}\nabla T_j(x, t)\Big)dx \\ &\qquad + \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} \operatorname{div}\Big(\frac{1}{a_i^2(x)}\nabla T(x, t)\Big)dx , \end{aligned}$$

where $V_\varepsilon = V - \Gamma_\varepsilon$ and $\Gamma_\varepsilon$ is a tubular neighborhood of thickness $\varepsilon$ around $S_{ij}$. If we integrate by parts the second member of this last equation and if we take into account the equation (1.2) in each domain $D_i$, $D_j$ we obtain the boundary condition

$$0 = \frac{1}{a_i^2(x)}(\nabla T_i(x, t) \cdot \boldsymbol{n}_i) + \frac{1}{a_j^2(x)}(\nabla T_j(x, t) \cdot \boldsymbol{n}_j) , \tag{1.3}$$

where $\boldsymbol{n}_i$ and $\boldsymbol{n}_j$ are the external normal of $S_{ij}$ pointing outwards $D_i$ and $D_j$, respectively.

Moreover, we impose that $T(x, t)$ is continuous everywhere.

(b) *Wave transmission in a general medium.* In wave transmission we consider the equation

$$\frac{\partial^2 u}{\partial t^2} = \operatorname{div}\Big(\frac{1}{a^2(x)}\nabla u(x)\Big)$$

where $1/a$ is the velocity of the waves and we take $c = 1$. (But this is not necessary in general).

(c) *Schrödinger equation with variable effective mass.* The Schrödinger equation is

$$\frac{1}{i}\frac{\partial u}{\partial t} = \operatorname{div}\Big(\frac{1}{2m^*(x)}\nabla u(x)\Big) + Vu$$

where $V$ is a potential function and $m^*(x)$ is the effective mass of the particle at point $x$; this effective mass can vary from point to point if the particle travels in different media (for example in a crystal the mass of the electron is not its usual mass).

**2. Relation with the general theory of Dirichlet integrals.** In the case when $c \equiv 1$, we can also consider the Dirichlet integral

$$(1.4) \qquad I(u, v) = \int \frac{1}{a^2(x)} \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} dx \equiv \sum_i \int_{D_i} \frac{1}{a_i^2(x)} \sum_{k=1}^{n} \frac{\partial u_i}{\partial x_k} \frac{\partial v_i}{\partial x_k} dx .$$

This is a particular case of the theory of Dirichlet integrals with discontinuous coefficients [2]. The operator associated to this integral is defined by

$$Lu = \operatorname{div}\left(\frac{1}{a(x)^2} \nabla u\right)$$

and with the boundary condition (1.3) on $\bar{D}_i \cap \bar{D}_j$. But the problem considered in $n^\circ 1$ is more general than the one associated to a Dirichlet integral, because it is not self-adjoint.

**3. Definition of the operator $L$.** We are looking for the solutions of the Cauchy problem

$$(1.5) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = Lu \\ u|_{t=0} = u_0 \end{cases}$$

where the notation $L$ means

$$(1.6) \qquad (Lu)(x) = \frac{1}{c^2(x)} \operatorname{div}\left(\frac{1}{a^2(x)} \nabla u(x)\right)$$

with the boundary conditions

(1°) $u(x)$ is continuous everywhere

(2°)

$$(1.7) \qquad \frac{1}{a_i^2(x)}(\nabla u_i \cdot \boldsymbol{n}_i) + \frac{1}{a_j^2(x)}(\nabla u_j \cdot \boldsymbol{n}_j) = 0$$

on the surface of separation of $D_i$ and $D_j$. We also have to specify certain boundary condition on the surface of the domain of definition of $u$ or at infinity but they can be specified in $L$ as a condition of type (2°) or more general mixed conditions.

**4. The one dimensional case: methods of solution.** In the sequel of this work, we shall mainly be interested in the one-dimensional case. The notations introduced in this section will be used throughout our work. The real line is divided in intervals

$$l_0 = -\infty < l_1 < l_2 < \cdots < l_{N-1} < l_N = \infty .$$

In each interval $[l_{i-1}, l_i] = I_i$ we define $a_i$ and $c_i$ which are $C^1$ and $C^0$ functions, respectively, but they may have discontinuity at points $l_i$. The operator $L$ is defined by

$$Lu = \frac{1}{c_i^2}\frac{\partial}{\partial x}\left(\frac{1}{a_i^2}\frac{\partial u}{\partial x}\right) \quad \text{in} \quad I_i \tag{1.8}$$

with boundary conditions

$$u(l_i^-) = u(l_i^+) \tag{1.9}$$

$$\frac{1}{a_i^2(l_i^-)}\frac{\partial u}{\partial x}(l_i^-) = \frac{1}{a_{i+1}^2(l_i^+)}\frac{\partial u}{\partial x}(l_i^+) . \tag{1.10}$$

We see that we must find the kernel of $F(L)$ for a function $F$ of a real variable, for example

$$F(\xi) = \exp(-t\xi) , \quad \exp(\pm it\, \xi^{1/2}) \quad \text{or} \quad \exp(it\xi) .$$

If we pose the problems as in Section 1. We have two methods to do this.

*First method: the functional calculus for a self-adjoint $L$.* Let us suppose that $c = 1$ so that $L$ is self-adjoint with respect to the Lebesgue measure; $L$ becomes a negative operator; let $-k^2$ and $u(x, \pm k)$ be respectively a generalized eigenvalue and the corresponding generalized eigenfunctions. By von Neumann theory, there exists a $2\times 2$ matrix $\rho_{\varepsilon\varepsilon'}(k)$ so that

$$\delta(x-y) = \int_0^\infty dk \sum_{\varepsilon,\varepsilon'=\pm 1} u(x, \varepsilon k)u^*(y, \varepsilon' k)\rho_{\varepsilon\varepsilon'}(k) .$$

$\rho_{\varepsilon\varepsilon'}(k)$ is the spectral matrix; it is hermitian and can be diagonalized; by considering special linear combinations we can reduce $\rho_{\varepsilon\varepsilon'}$ to be $\delta_{\varepsilon\varepsilon'}$; then

$$\delta(x-y) = \int_{-\infty}^\infty u(x, k)u^*(y, k)dk , \tag{1.11}$$

and

$$F(L)(x, y) = \int_{-\infty}^\infty F(-k^2)u(x, k)u^*(y, k)dk . \tag{1.12}$$

We want to find explicit expansion for the $u(x, \pm k)$.

*Second method: method of Titchmarsh-Kodaira-Yosida for a general $L$.* This method applies for $c \not\equiv 1$; let us assume that there exist $m$ and $M$ such that

$$0 < m \leqq a\,, \quad c \leqq M < \infty\,.$$

For $\lambda$ in $\boldsymbol{C} - \boldsymbol{R}^-$ we consider the problems

$$(1.13) \qquad (P_\pm) \qquad (\lambda - L)u(x, \lambda) = 0 \qquad \text{if} \quad x \to \pm\infty\,.$$

Call $u_\pm(x, \lambda)$ the solution (supposed to be unique modulo constants); the Green function is

$$(1.14) \qquad G(x, y; \lambda) = \begin{cases} -\dfrac{a^2(x_0)c^2(y)u_-(x, \lambda)u_+(y, \lambda)}{W(u_-, u_+)(x_0)} & (x \leqq y) \\ -\dfrac{a^2(x_0)c^2(y)u_-(y, \lambda)u_+(x, \lambda)}{W(u_-, u_+)(x_0)} & (x > y) \end{cases}$$

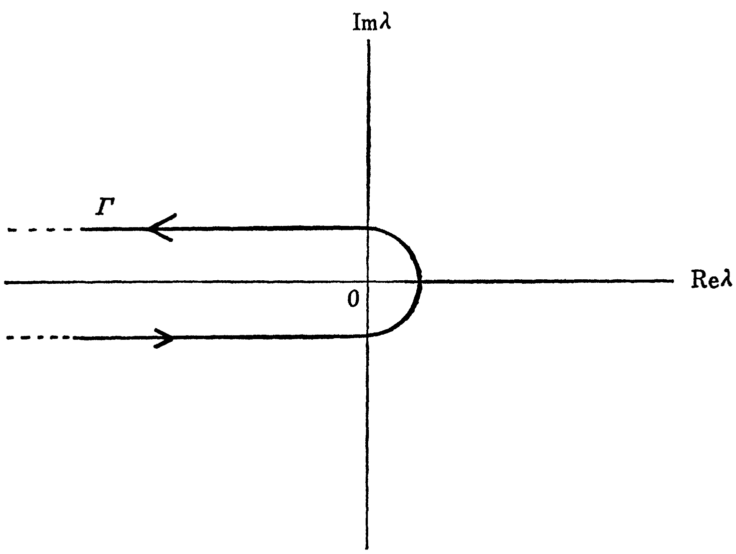
where $W(u_-, u_+)(x_0) = (u_-u_+' - u_-'u_+)_{x=x_0}$ is the Wronskian of the two solutions, and $x_0$ is any point on $\boldsymbol{R}$. Then for $\lambda \in \boldsymbol{C} - \boldsymbol{R}_-$ and $f \in L^2(\boldsymbol{R}) \cap C^0(\boldsymbol{R})$, we can prove that

$$u = (\lambda - L)^{-1}f = \int_{-\infty}^{\infty} G(x, y; \lambda)f(y)dy$$

satisfies $(\lambda - L)u = f$ and $u \to 0$ if $x \to \pm\infty$.
The heat kernel $p_t(x, y)$ is given by

$$(1.15) \qquad p_t(x, y) = \frac{1}{2i\pi}\int_\Gamma e^{\lambda t}G(x, y; \lambda)d\lambda\,,$$

where $\Gamma$ is a contour in the complex $\lambda$ plane around the negative real axis (as in the figure).



REMARKS. 1. Neither $G(x, y, \lambda)$ nor $p_t(x, y)$ are continuous in $y$ in general if the coefficients of the operators are not continuous.

2. The computations involved in the spectral resolution or in the Titchmarsh-Kodaira method are very similar; we shall do them using a statistical mechanics method (transfer matrix).

## CHAPTER II. The case of piecewise constant coefficients.

**1. Hypothesis and general formulas for the transfer matrix.** We shall assume the situation of Chapter I, n°6: namely $l_0 = -\infty < l_1 = 0 < l_2 < \cdots < l_{N-1} < l_N = \infty$ and on each interval $I_i = [l_{i-1}, l_i]$, we suppose that $c_i$ and $a_i$ are constants. We can always reduce ourselves to the case $l_1 = 0$ and we can also assume that

$$l_j = (j-1)l$$

by refining the partition by the $l_j$'s. We denote also $u_j = u|_{I_j}$. The two eigenfunctions on $I_j$ are $\exp(\pm ika_jc_jx)$ associated to the eigenvalue $-k^2$ or $\exp(\pm\lambda^{1/2}a_jc_jx)$ associated to $\lambda \in \boldsymbol{C} - \boldsymbol{R}^-$ (determination $\lambda^{1/2} > 0$ if $\lambda > 0$). We shall do the computation in the first case (it does not really matter which case we take). We look for an eigenfunction $u(x, k)$ such that

$$u_j(x, k) = A_j(k)\exp(ika_jc_jx) + B_j(k)\exp(-ika_jc_jx)\ . \tag{2.1}$$

The boundary conditions at $l_j$ can be written as

$$\begin{aligned} A_{j+1}\exp(ika_{j+1}c_{j+1}l_j) &+ B_{j+1}\exp(-ika_{j+1}c_{j+1}l_j) \\ &= A_j\exp(ika_jc_jl_j) + B_j\exp(-ika_jc_jl_j) \end{aligned}$$

$$\begin{aligned} \frac{c_{j+1}}{a_{j+1}}&(A_{j+1}\exp(ika_{j+1}c_{j+1}l_j) - B_{j+1}\exp(-ika_{j+1}c_{j+1}l_j)) \\ &= \frac{c_j}{a_j}(A_j\exp(ika_jc_jl_j) - B_j\exp(-ika_jc_jl_j)) \end{aligned}$$

which can be rewritten as

$$\begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix} = T_j(k)\begin{pmatrix} A_j \\ B_j \end{pmatrix} \tag{2.2}$$

where $T_j(k)$ is the $2\times 2$ matrix:

$$T_j = \frac{1}{2a_jc_{j+1}}\begin{pmatrix} *1 & *2 \\ *3 & *4 \end{pmatrix} \tag{2.3}$$

where

$$\begin{aligned} *1 &= (a_jc_{j+1} + a_{j+1}c_j)\exp(ik(a_jc_j - a_{j+1}c_{j+1})l_j) \\ *2 &= (a_jc_{j+1} - a_{j+1}c_j)\exp(-ik(a_jc_j + a_{j+1}c_{j+1})l_j) \\ *3 &= (a_jc_{j+1} - a_{j+1}c_j)\exp(ik(a_jc_j + a_{j+1}c_{j+1})l_j) \\ *4 &= (a_jc_{j+1} + a_{j+1}c_j)\exp(-ik(a_jc_j - a_{j+1}c_{j+1})l_j)\ . \end{aligned}$$

DEFINITION. $T_j(k)$ is the transfer matrix for momentum $k$.

**2. The self-adjoint case.** Referring to formulas (1.11) and (1.12) we need to compute integrals such as

$$K(x, y) = \int_{-\infty}^{\infty} F(k^2)u(x, k)u^*(y, k)dk \,, \tag{2.4}$$

where $F(k^2)$ denotes an even function of $k^2$ for $x$, $y$ in $I_j$ and $I_l$, respectively. Replacing $u_j$ and $u_l$ by their values (2.1) and using the fact that $F$ is even, we have

$$\begin{aligned} K(x, y) = \int_{-\infty}^{\infty} F(k^2)dk(C_{jl}^{(-)}(k)&\exp(ik(a_jx - a_ly)) \\ &+ C_{jl}^{(+)}(k)\exp(ik(a_jx + a_ly))) \end{aligned} \tag{2.5}$$

for $x \in I_j$, $y \in I_l$ where $C_{jl}^{(\pm)}(k)$ are called *spectral coefficients* and are

$$\begin{aligned} C_{jl}^{(-)}(k) &= A_j(k)A_l^*(k) + B_j(-k)B_l^*(-k) \\ C_{jl}^{(+)}(k) &= A_j(k)B_l^*(k) + B_j(-k)A_l^*(-k) \,. \end{aligned} \tag{2.6}$$

Now we write the condition of spectral resolution (1.11), i.e., we take $F \equiv 1$. If $x$, $y$ are in $I_1$, $x - y$ can take any real value $z$ and we must have from (2.5) with $j = l = 1$ and $F \equiv 1$

$$\delta(z) = \int_{-\infty}^{\infty} dk(C_{11}^{(-)}(k)\exp(ika_1(x - y)) + C_{11}^{(+)}(k)\exp(ika_1(x + y)))$$

so that

$$C_{11}^{(-)}(k) = \frac{a_1}{2\pi} \,. \tag{2.7}$$

If now $x$ is in $I_1$ and $y$ is in $I_N$, we must have

$$0 = \int_{-\infty}^{\infty} dk[C_{1N}^{(-)}(k)\exp(ik(a_1x - a_Ny)) + C_{1N}^{(+)}(k)\exp(ik(a_1x + a_Ny))]$$

and because $a_1x + a_Ny$ can take any real value, we deduce

$$C_{1N}^{(+)}(k) = 0 \,. \tag{2.8}$$

Let us now define the following matrix

$$U_j(k) = \begin{pmatrix} A_j(k) & B_j(k) \\ B_j(-k) & A_j(-k) \end{pmatrix} \tag{2.9}$$

so that using (2.6)

$$U_l(k)^* U_j(k) = \begin{pmatrix} C_{jl}^{(-)}(k) & C_{jl}^{(+)}(-k) \\ C_{jl}^{(+)}(k) & C_{jl}^{(-)}(-k) \end{pmatrix} \tag{2.10}$$

and also using (2.2), we obtain

$$U_{j+1}(k) = U_j(k)\,{}^tT_j(k) = U_1(k)\,{}^tT_1(k)\,{}^tT_2(k) \cdots {}^tT_j(k) .$$

In particular,

$$U_j(k)^* U_l(k) = (\bar{T}_{j-1}(k) \cdots \bar{T}_1(k))(U_1^*(k)U_1(k))({}^tT_1(k) \cdots {}^tT_{l-1}(k)) . \tag{2.11}$$

If we take in this formula $N = j$ and $l = 1$ and if we take into account the relations (2.7) and (2.8) giving $C_{11}^{(-)}(k)$ and $C_{1N}^{(+)}(k)$, we obtain from (2.11) and (2.10)

$$\begin{pmatrix} C_{1N}^{(-)}(k) & 0 \\ 0 & C_{1N}^{(-)}(-k) \end{pmatrix} = \bar{T}_{N-1}(k) \cdots \bar{T}_1(k) \begin{pmatrix} a_1/2\pi & C_{11}^{(+)}(-k) \\ C_{11}^{(+)}(k) & a_1/2\pi \end{pmatrix} . \tag{2.12}$$

This system of equations gives $C_{1N}^{(-)}(k)$ and $C_{11}^{(+)}(k)$. In particular, $U_1^*(k)U_1(k)$ is known and from (2.11) and (2.10), we know all the other spectral coefficients, provided that we can perform the product of the matrices $T_{j-1} \cdots T_1$.

In the self-adjoint case, $c_j \equiv 1$ for any $j$ and it is clear that

$$\det T_j(k) = \frac{a_{j+1}}{a_j} \tag{2.13}$$

so that $T_j(k) = \tilde{T}_j(k)(a_{j+1}/a_j)^{1/2}$ with $\det \tilde{T}_j = 1$. Denote

$$\tilde{T}_{j-1}(k) \cdots \tilde{T}_1(k) = \begin{pmatrix} M_j & N_j \\ N_j^* & M_j^* \end{pmatrix} \tag{2.14}$$

$$|M_j|^2 - |N_j|^2 = 1 .$$

Then

$$T_{j-1}(k) \cdots T_1(k) = (a_j/a_1)^{1/2} \begin{pmatrix} M_j & N_j \\ N_j^* & M_j^* \end{pmatrix} \tag{2.15}$$

and for $j = N$ we deduce $C$ from (2.10) as

$$C_{11}^{(+)}(k) = -\frac{a_1}{2\pi}\frac{N_N}{M_N} = C_{11}^{(+)}(-k)^*$$

$$C_{1N}^{(-)}(k) = \frac{(a_1a_N)^{1/2}}{2\pi}\frac{1}{M_N} = C_{1N}^{(-)}(-k)^*$$

$$C_{1j}^{(-)}(k) = \frac{(a_1a_j)^{1/2}}{2\pi}\frac{M_j^*M_N - N_j^*N_N}{M_N} = C_{1j}^{(-)}(-k)^*$$

$$C_{1j}^{(+)}(k) = \frac{(a_1a_j)^{1/2}}{2\pi}\frac{N_jM_N - N_NM_j}{M_N} = C_{1j}^{(+)}(-k)^* .$$

**3. The non self-adjoint case.** In the non self-adjoint case, we define

$$u_{-,1}(x, \lambda) = \exp(\lambda^{1/2} a_1 c_1 x)$$

so that $u_{-,1} \to 0$ if $x \to -\infty$

$$u_{+,N}(x, \lambda) = \exp(-\lambda^{1/2} a_N c_N x)$$

so that $u_{+,N} \to 0$ if $x \to \infty$.

Then, we have again to compute $u_{-,j}(x, \lambda)$ and $u_{+,j}(x, \lambda)$

$$u_{-,j}(x, \lambda) = A_j(\lambda)\exp(\lambda^{1/2} a_j c_j x) + B_j(\lambda)\exp(-\lambda^{1/2} a_j c_j x)$$

and so for $j > 1$,

$$\begin{pmatrix} A_j(\lambda) \\ B_j(\lambda) \end{pmatrix} = T_{j-1}(\lambda^{1/2}) \cdots T_1(\lambda^{1/2}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{2.16}$$

In particular, if we compute the Wronskian at $\infty$, we have

$$W(u_-, u_+)(\infty) = -2 a_N c_N \lambda^{1/2} A_N .$$

If $x \in I_j$, $y \in I_N$, we have by (1.14)

$$\begin{aligned} G(x, y, \lambda) = a_N c_N (A_j(\lambda)&\exp(\lambda^{1/2} a_j c_j x) + B_j(\lambda)\exp(-\lambda^{1/2} a_j c_j x)) \\ &\times \exp(-\lambda^{1/2} a_N c_N y)/(2\lambda^{1/2} A_N(\lambda)) . \end{aligned}$$

**4. The particular cases $N = 2$ or $3$: the self-adjoint case.** (a) These cases can be explicitly treated. We shall give the details only in the self-adjoint case (all $c_j = 1$) and just give the result for the general case. Also, we shall treat the case $N = 3$; we have $l_1 = 0$, and define $l_2 = l$.

(b) We want to compute $C_{ij}^{(+)}(k)$ for $1 \leqq i, j \leqq 3$. First we have

$$T_1(k) = (a_2/a_1)^{1/2} \begin{pmatrix} M_2 & N_2 \\ N_2^* & M_2^* \end{pmatrix} ,$$

where $M_2 = 2^{-1}(a_1 a_2)^{-1/2}(a_1 + a_2)$, $N_2 = (a_1 - a_2)$.

Then we have

$$T_2(k) T_1(k) = (a_3/a_1)^{1/2} \begin{pmatrix} M_3 & N_3 \\ N_3^* & M_3^* \end{pmatrix} ,$$

where

$$\begin{aligned} M_3 = \frac{1}{4(a_1 a_2^2 a_3)^{1/2}} [&(a_1 + a_2)(a_2 + a_3)\exp(ikl(a_2 - a_3)) \\ &+ (a_1 - a_2)(a_2 - a_3)\exp(-ikl(a_2 + a_3))] \end{aligned}$$

$$N_3 = \frac{1}{4(a_1 a_2^2 a_3)^{1/2}} [(a_1 - a_2)(a_2 + a_3)\exp(ikl(a_2 - a_3))$$

$$+ (a_1 + a_2)(a_2 - a_3)\exp(-ikl(a_2 + a_3))]$$

and we have $C_{11}^{(-)}(k) = a_1/2\pi$,

$$C_{11}^{(+)}(k) = -\frac{a_1}{2\pi}\frac{N_3}{M_3} = -\frac{a_1}{2\pi}\frac{(a_1 - a_2)(a_2 + a_3) + (a_1 + a_2)(a_2 - a_3)\exp(-2ikla_2)}{(a_1 + a_2)(a_2 + a_3) + (a_1 - a_2)(a_2 - a_3)\exp(-2ikla_2)}\,.$$

Then, we need $C_{12}^{(+)}(k)$ given by $U_2(k)^* U_1(k)$

$$U_2^*(k)U_1(k) = \bar{T}_1(k)U_1^*(k)U_1(k) = (a_2/a_1)^{1/2}\begin{pmatrix} M_2^* & N_2^* \\ N_2 & M_2 \end{pmatrix}\begin{pmatrix} a_1/2\pi & C_{11}^{(+)}(-k) \\ C_{11}^{(+)}(k) & a_1/2\pi \end{pmatrix}$$

$$C_{12}^{(-)}(k) = \frac{(a_1a_2)^{1/2}}{2\pi}\Big(M_2^* - N_2^*\frac{N_3}{M_3}\Big)\,, \qquad C_{12}^{(+)}(k) = \frac{(a_1a_2)^{1/2}}{2\pi}\Big(N_2 - M_2\frac{N_3}{M_3}\Big)$$

so that

$$C_{12}^{(\pm)}(k) = \frac{\mp 1}{\pi}\,\frac{a_1a_2(a_2 \mp a_3)}{(a_1 \mp a_2)(a_2 \mp a_3) + (a_1 \pm a_2)(a_2 \pm a_3)\exp(\pm 2ikla_2)}\,.$$

Then we have also

$$C_{13}^{(+)}(k) = 0\,,$$

$$C_{13}^{(-)}(k) = \frac{(a_1a_3)^{1/2}}{2\pi M_3} = \frac{2a_1a_2a_3}{\pi}[(a_1 + a_2)(a_2 + a_3)\exp(ikl(a_2 - a_3))$$
$$+ (a_1 - a_2)(a_2 - a_3)\exp(-ikl(a_2 + a_2))]^{-1}\,.$$

We compute $C_{22}^{(\pm)}(k)$ by using

$$U_2^*(k)U_2(k) = U_2^*U_1{}^tT_1 = \begin{pmatrix} C_{12}^{(-)}(k) & C_{12}^{(+)}(-k) \\ C_{12}^{(+)}(k) & C_{12}^{(-)}(-k) \end{pmatrix}\begin{pmatrix} M_2 & N_2^* \\ N_2 & M_2^* \end{pmatrix}(a_2/a_1)^{1/2}$$

so that

$$C_{22}^{(\pm)}(k) = \frac{1}{2a_1\pi}\Bigg[\frac{\mp(a_1 + a_2)a_1a_2(a_2 \mp a_3)}{(a_1 \mp a_2)(a_2 \mp a_3) + (a_1 \pm a_2)(a_2 \pm a_3)\exp(\pm 2ikla_2)}$$
$$\pm \frac{(a_1 - a_2)a_1a_2(a_2 \pm a_3)}{(a_1 \pm a_2)(a_2 \pm a_3) + (a_1 \mp a_2)(a_2 \mp a_3)\exp(\pm 2ikla_2)}\Bigg]\,.$$

Then we also obtain

$$C_{23}^{(\pm)}(k) = \frac{a_2a_3}{\pi}(a_1 \mp a_2)[(a_1 + a_2)(a_2 + a_3)\exp(\mp ikl(a_2 - a_3))$$
$$+ (a_1 - a_2)(a_2 - a_3)\exp(\pm ikl(a_2 + a_3))]^{-1}$$

and finally $C_{33}^{(+)}(k)$ is computed by

$$U_3^*(k)U_3(k) = U_3^*(k)U_1(k)\,{}^tT_1(k)\,{}^tT_2(k)$$
$$= \begin{pmatrix} C_{13}^{(-)}(k) & 0 \\ 0 & C_{13}^{(-)}(-k) \end{pmatrix}\begin{pmatrix} M_3 & N_3^* \\ N_3 & M_3^* \end{pmatrix}(a_3/a_1)^{1/2}$$

$$C_{33}^{(+)}(k) = \frac{a_3}{2\pi}\exp(-2ikla_3)\frac{(a_1 - a_2)(a_2 + a_3) + (a_1 + a_2)(a_2 - a_3)\exp(-2ikla_2)}{(a_1 - a_2)(a_2 - a_3) + (a_1 + a_2)(a_2 + a_3)\exp(-2ikla_2)}$$

$$C_{33}^{(-)}(k) = \frac{a_3}{2\pi} .$$

(c) Now we can compute the heat kernel using (1.12) or (2.5). We introduce the function

$$h(t, \xi, C, \alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2t}e^{ik\xi}(1 + Ce^{ik\alpha})^{-1}dk \tag{2.17}$$

well-defined for $|C| \neq 1$. It is a kind of $\theta$-function. Denote

$$p_t^{(jl)}(x, y) = p_t(x, y)|_{x\in I_j, y\in I_l}$$

for $j, l = 1, 2, 3$, and also recall the usual formula

$$g(t, \xi) = (4\pi t)^{-1/2}\exp(-\xi^2/(4t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2t}e^{ik\xi}dk .$$

Using (2.5) and the preceding values for the spectral coefficients we obtain

$$\begin{aligned}
p_t^{(1,1)}(x, y) &= a_1 g(t, a_1(x - y)) - a_1\frac{a_1 - a_2}{a_1 + a_2}h(t, a_1(x + y), K, -L) \\
&\quad - a_1\frac{a_2 - a_3}{a_2 + a_3}h(t, a_1(x + y) - L, K, -L) , \\
p_t^{(1,2)}(x, y) &= \frac{2a_1a_2}{a_1 + a_2}h(t, a_1x - a_2y, K, -L) \\
&\quad - \frac{2a_1a_2(a_2 - a_3)}{(a_1 + a_2)(a_2 + a_3)}h(t, a_1x + a_2y - L, K, -L) , \\
p_t^{(1,3)}(x, y) &= \frac{4a_1a_2a_3}{(a_1 + a_2)(a_2 + a_3)}h(t, a_1x - a_3y - l(a_2 - a_3), K, -L) , \\
p_2^{(2,2)}(x, y) &= a_2h(t, a_2(x - y), K, -L) \\
&\quad - a_2\frac{(a_1 - a_2)(a_2 - a_3)}{(a_1 + a_2)(a_2 + a_3)}h(t, a_2(x - y) + L, K, +L) \\
&\quad - a_2\frac{(a_2 - a_3)}{(a_2 + a_3)}h(t, a_2(x + y) - L, K, -L) \\
&\quad + a_2\frac{a_1 - a_2}{a_1 + a_2}h(t, a_2(x + y), K, +L) , \\
p_t^{(2,3)}(x, y) &= \frac{2a_2a_3}{a_2 + a_3}h(t, a_2x - a_3y - l(a_2 - a_3), K, -L)
\end{aligned} \tag{2.18}$$

$$+\frac{2a_2a_3(a_1-a_2)}{(a_1+a_2)(a_2+a_3)}h(t,\ a_2x+a_3y+l(a_2-a_3),\ K,\ +L)\ ,$$

$$p_t^{(3,3)}(x,\ y)=a_3g(t,\ a_3(x-y))+a_3\frac{(a_1-a_2)}{(a_1+a_2)}h(t,\ a_3(x+y)+2l(a_2-a_3),\ K,\ +L)$$
$$+\ a_3\frac{(a_2-a_3)}{(a_2+a_3)}h(t,\ a_3(x+y)-2la_3,\ K,\ -L)\ ,$$

and here $K=(a_1-a_2)(a_2-a_3)/(a_1+a_2)(a_2+a_3)$ and $L=+2la_2$.

(d) We now obtain the case $N=2$ which is special case of $N=3$ for $l=0$, $a_2=a_3$. In that case

$$h(t,\ \xi,\ C,\ L=0)=g(t,\ \xi)$$

and the heat kernel is simply

$$p_t^{(1,1)}(x,\ y)=a_1g(t,\ a_1(x-y))-\frac{a_1(a_1-a_2)}{a_1+a_2}g(t,\ a_1(x+y))\ ,$$

$$p_t^{(1,2)}(x,\ y)=\frac{2a_1a_2}{a_1+a_2}g(t,\ a_1x-a_2y)\ ,$$

$$p_t^{(2,2)}(x,\ y)=a_2g(t,\ a_2(x-y))+a_2\frac{(a_1-a_2)}{(a_1+a_2)}g(t,\ a_2(x+y))\ .$$

**5. The particular cases $N=3$ and 2: non self-adjoint cases.** We consider here the case $N=3$ but with the $c_i$'s not necessarily 1, i.e., the operator

$$L=\Big(\frac{1}{c_1^2}\mathbb{I}_{[x<0]}+\frac{1}{c_2^2}\mathbb{I}_{[0<x<l]}+\frac{1}{c_3^2}\mathbb{I}_{[l<x]}\Big)\frac{d}{dx}\Big(\Big(\frac{1}{a_1^2}\mathbb{I}_{[x<0]}+\frac{1}{a_2^2}\mathbb{I}_{[0<x<l]}+\frac{1}{a_3^2}\mathbb{I}_{[l<x]}\Big)\frac{d}{dx}\Big)\ . \tag{2.19}$$

Then

$$u_-(x,\ \lambda)=\exp(\lambda^{1/2}c_1a_1x)\qquad \text{for}\quad x<0$$
$$u_+(x,\ \lambda)=\exp(-\lambda^{1/2}c_3a_3x)\qquad \text{for}\quad x>l\ .$$

We define

$$u_-(x,\ \lambda)=\begin{cases}A\exp(\lambda^{1/2}c_2a_2x)+B\exp(-\lambda^{1/2}c_2a_2x) & 0<x<l\\ C\exp(\lambda^{1/2}c_3a_3x)+D\exp(-\lambda^{1/2}c_3a_3x) & x>l\end{cases}$$

$$u_+(x,\ \lambda)=\begin{cases}G\exp(\lambda^{1/2}c_1a_1x)+H\exp(-\lambda^{1/2}c_1a_1x) & x<0\\ E\exp(\lambda^{1/2}c_2a_2x)+F\exp(-\lambda^{1/2}c_2a_2x) & 0<x<l\ .\end{cases}$$

We write the boundary condition at 0 and $l$ for $u_\pm(x,\ \lambda)$ and we obtain

$$
(2.20)\qquad
\begin{aligned}
E &= \frac{1}{2}\,\frac{-a_2c_3 + a_3c_2}{a_3c_2}\exp(-\lambda^{1/2}l(a_2c_2 + a_3c_3)) \\
F &= \frac{1}{2}\,\frac{a_3c_2 + a_2c_3}{a_3c_2}\exp(\lambda^{1/2}l(a_2c_2 - a_3c_3))
\end{aligned}
$$

$$
(2.21)\qquad
\begin{aligned}
\frac{a_1}{2\lambda^{1/2}c_1H} = \frac{2a_1a_2a_3c_2}{\lambda^{1/2}}\exp(\lambda^{1/2}la_3c_3)[(a_2c_1 - a_1c_2)(a_3c_2 - a_2c_3) \\
\times\exp(-\lambda^{1/2}la_2c_2) + (a_2c_1 + a_1c_2)(a_2c_3 + c_2a_3)\exp(\lambda^{1/2}la_2c_2)]^{-1}
\end{aligned}
$$

Then

$$
(2.22)\qquad G(x, y, \lambda) = \begin{cases} \dfrac{a_1}{2c_1H\lambda^{1/2}}c(y)^2u_-(x, \lambda)u_+(y, \lambda) & \text{if} \quad x \leqq y \\ \dfrac{a_1}{2c_1H\lambda^{1/2}}c(y)^2u_-(y, \lambda)u_+(x, \lambda) & \text{if} \quad x \geqq y\,. \end{cases}
$$

For example, if we compute $G(x, y, \lambda)$ for $x < y < 0$ we have from (2.22)

$$
\begin{aligned}
G(x, y, \lambda) &= (c_1^2a_1/(2c_1\lambda^{1/2}H))\exp(\lambda^{1/2}a_1c_1x)\{G\exp(\lambda^{1/2}a_1c_1y) \\
&\quad + H\exp(-\lambda^{1/2}a_1c_1y)\} \\
&= (a_1c_1/(2\lambda^{1/2}))\exp(\lambda^{1/2}a_1c_1(x - y)) \\
&\quad + a_1c_12^{-1}\lambda^{-1/2}(G/H)\exp(\lambda^{1/2}a_1c_1(x + y))\,.
\end{aligned}
$$

But from the transmission conditions and (2.20), (2.21)

$$
G = \frac{1}{2}\left(E + F + \frac{a_1c_2}{a_2c_1}(E - F)\right)
$$

and so

$$
(2.23)\qquad \frac{G}{H} = \frac{\dfrac{a_2c_1 - a_1c_2}{a_2c_1 + a_1c_2} + \dfrac{-a_2c_3 + a_3c_2}{a_2c_3 + a_3c_2}\exp(-2\lambda^{1/2}la_2c_2)}{1 + \left(\dfrac{a_2c_1 - a_1c_2}{a_2c_1 + a_1c_2}\right)\left(\dfrac{-a_2c_3 + a_3c_2}{a_2c_3 + a_3c_2}\right)\exp(-2\lambda^{1/2}la_2c_2)}\,.
$$

Now using the contour integral (1.16) and performing the integral we obtain

$$
p_t^{(1,1)}(x, y) = a_1c_1g(t, a_1c_1(x - y)) + \frac{a_1c_1}{2}\int_{-\infty}^{\infty} e^{-\xi^2t}e^{i\xi a_1c_1(x+y)}\frac{G}{H}d\xi\,,
$$

where we replace $\lambda^{1/2}$ by $+i\xi$ in $G/H$ given by (2.23). With the same function $h$ given by (2.16) we obtain

$$
(2.24)\qquad
\begin{aligned}
p_t^{(1,1)}(x, y) &= a_1c_1g(t, a_1c_1(x - y)) \\
&\quad + \frac{a_1c_1(a_2c_1 - a_1c_2)}{(a_1c_2 + a_2c_1)}h(t, a_1c_1(x + y), K, -L)
\end{aligned}
$$

$$+\frac{a_1c_1(-a_2c_3+a_3c_2)}{a_2c_3+a_3c_2}h(t,\,a_1c_1(x+y)-L,\,K,\,-L)$$

In the same manner we also obtain for $x<0<l<y$ (case (1, 3))

$$\begin{aligned}p_t^{(1,3)}(x,\,y)=\frac{4a_1a_2a_3c_2c_3^2}{(a_2c_1+a_1c_2)(a_3c_2+a_2c_3)}&h(t,\,a_1c_1x\\ &-a_3c_3y+l(a_3c_3-a_2c_2),\,K,\,-L)\,.\end{aligned}\tag{2.25}$$

Here

$$L=+2la_2c_2$$
$$K=\left(\frac{a_2c_1-a_1c_2}{a_2c_1+a_1c_2}\right)\left(\frac{a_3c_2-a_2c_3}{a_2c_3+a_3c_2}\right)$$

In the case $N=2$, we obtain

$$p_t^{(1,1)}(x,\,y)=a_1c_1g(t,\,a_1c_1(x-y))+\frac{a_1c_1(a_2c_1-a_1c_2)}{a_1c_2+a_2c_1}g(t,\,a_1c_1(x+y))\tag{2.26}$$

$$p_t^{(1,2)}(x,\,y)=\frac{2a_1a_2c_2^2}{a_1c_2+a_2c_1}g(t,\,a_1c_1x-a_2c_2y)\tag{2.27}$$

$$p_t^{(2,2)}(x,\,y)=a_2c_2g(t,\,a_2c_2(x-y))+\frac{a_2c_2(a_1c_2-a_2c_1)}{a_1c_2+a_2c_1}g(t,\,a_2c_2(x+y))\tag{2.28}$$

$$p_t^{(2,1)}(x,\,y)=\frac{2a_1a_2c_1^2}{a_1c_2+a_2c_1}g(t,\,a_1c_1y-a_2c_2x)\,.\tag{2.29}$$

REMARK. Compare $p_t^{(2,1)}$ and $p_t^{(1,2)}$; here they differ by the exchange of $x$ and $y$ and *also* by the exchange of $c_1^2$ and $c_2^2$ in the coefficient in front of $g$ due to the non self-adjointness of the operator. Moreover they are not continuous (for example at 0): for example fix $x<0$; then $y\to p_t(x,\,y)$ is not continuous at 0 because

$$p_t(x,\,0^-)=g(t,\,a_1c_1)\frac{2a_2a_1c_1^2}{a_1c_2+a_2c_1}$$
$$p_t(x,\,0^+)=g(t,\,a_1c_1)\frac{2a_2a_1c_2^2}{a_1c_2+a_2c_1}$$

but if we fix $y>0$, then $x\to p_t(x,\,y)$ is continuous at $x=0$.

## CHAPTER III. The operator with general irregular coefficients.

**1. Computing a finite product of transfer matrices.** In Chapter II, we defined the transfer matrix $T_j(k)$ by formula (2.3) rewritten as

$$(3.1)\qquad T_j(k) = \frac{1}{2a_jc_{j+1}}\begin{pmatrix}\alpha_j \exp(ikl_j\theta_j) & \beta_j \exp(-ikl_j\sigma_j)\\ \beta_j \exp(ikl_j\sigma_j) & \alpha_j \exp(-ikl_j\theta_j)\end{pmatrix} \equiv \frac{1}{2a_jc_{j+1}}\hat{T}_j(k)$$

where $l_1 = 0$, $l_j = (j-1)l$ and

$$(3.2)\qquad \begin{cases}\alpha_j = a_jc_{j+1} + a_{j+1}c_j\\ \beta_j = a_jc_{j+1} - a_{j+1}c_j\\ \theta_j = a_jc_j - a_{j+1}c_{j+1}\\ \sigma_j = a_jc_j + a_{j+1}c_{j+1}\end{cases}$$

and $\det T_j = (a_{j+1}c_j/a_jc_{j+1})$; we rewrite (3.1) in the form

$$(3.3)\qquad T_j = \left(\frac{a_{j+1}c_j}{a_jc_{j+1}}\right)^{1/2}\frac{1}{2(a_jc_ja_{j+1}c_{j+1})^{1/2}}\hat{T}_j(k)\ .$$

We have seen in Chapter II, n°2 that the most important object is the product

$$T_NT_{N-1}\cdots T_1$$

of $N$ matrices $T_j$. Let

$$(3.4)\qquad \hat{T}_j = \hat{T}_j(k)$$

Then we have

$$(3.5)\qquad T_NT_{N-1}\cdots T_1 = \left(\frac{a_{N+1}c_1}{a_1c_{N+1}}\right)^{1/2}\frac{1}{2^N(a_{N+1}c_{N+1}(a_Nc_N\cdots a_2c_2)^2a_1c_1)^{1/2}}\hat{T}_N\hat{T}_{N-1}\cdots\hat{T}_1\ .$$

It is clear from (3.4) that we can write

$$(3.6)\qquad \hat{T}_N\cdots\hat{T}_1 = \begin{pmatrix}A_{N+1,1} & B_{N+1,1}\\ B^*_{N+1,1} & A^*_{N+1,1}\end{pmatrix}$$

and

$$\begin{pmatrix}A_{N+1,1}\\ B^*_{N+1,1}\end{pmatrix} = \hat{T}_N\begin{pmatrix}A_{N,1}\\ B^*_{N,1}\end{pmatrix}$$

which means

$$(3.7)\qquad \begin{aligned}A_{N+1,1} &= \alpha_N \exp(ikl_N\theta_N)A_{N,1} + \beta_N \exp(-ikl_N\sigma_N)B^*_{N,1}\\ B^*_{N+1,1} &= \beta_N \exp(ikl_N\sigma_N)A_{N,1} + \alpha_N \exp(-ikl_N\theta_N)B^*_{N,1}\ .\end{aligned}$$

Define

$$(3.8)\qquad \begin{aligned}A_{N+1,1} &= \alpha_1\cdots\alpha_NC_{N+1}\\ B^*_{N+1,1} &= \alpha_1\cdots\alpha_ND^*_{N+1}\end{aligned}$$

$$\gamma_j = \frac{\beta_j}{\alpha_j} \tag{3.9}$$

so that (3.7) becomes (recalling the definition (3.3) of $\theta_j$, $\sigma_j$)

$$\begin{aligned} C_{N+1} &= \exp(-ikl(N-1)a_{N+1}c_{N+1})\{\exp(ikl(N-1)a_Nc_N)C_N \\ &\quad + \gamma_N \exp(-ikl(N-1)a_Nc_N)D_N^*\} \\ D_{N+1}^* &= \exp(ikl(N-1)a_{N+1}c_{N+1})\{\gamma_N \exp(ikl(N-1)a_Nc_N)C_N \\ &\quad + \exp(-ikl(N-1)a_Nc_N)D_N^*\}\,. \end{aligned} \tag{3.10}$$

Now define

$$\begin{aligned} E_{N+1} &= \exp(-ikl(a_2c_2 + \cdots + a_Nc_N))C_{N+1} \\ F_{N+1}^* &= \exp(ikl(a_2c_2 + \cdots + a_Nc_N))D_{N+1}^* \end{aligned} \tag{3.11}$$

$$\begin{aligned} E_{N+1} &= \exp(-ikl(N-1)a_{N+1}c_{N+1})\{\exp(ikl(N-2)a_Nc_N)E_N \\ &\quad + \gamma_N \exp(-iklNa_Nc_N)\exp(-2ikl(a_2c_2 + \cdots + a_{N-1}c_{N-1}))F_N^*\} \\ F_{N+1}^* &= \exp(ikl(N-1)a_{N+1}c_{N+1})\{\gamma_N \exp(iklNa_Nc_N) \\ &\quad \times \exp(2ikl(a_2c_2 + \cdots + a_{N-1}c_{N-1}))E_N \\ &\quad + \exp(-ikl(N-2)a_Nc_N)F_N^*\}\,. \end{aligned} \tag{3.12}$$

On this form, it is almost obvious to perform the product of the matrices in a systematic way. The answer is that for $N \geqq 2$

$$\begin{aligned} E_N &= \exp(-ikl(N-2)a_Nc_N)\Big[\sum_{n\geqq 0} \sum_{1\leqq i_1<\cdots<i_{2n}\leqq N-1} \gamma_{i_{2n}}\gamma_{i_{2n-1}}\cdots\gamma_{i_1} \\ &\quad \times \exp\Big(-2ikl\Big\{\sum_1^{i_{2n}} a_rc_r - \sum_1^{i_{2n-1}} a_rc_r + \cdots - \sum_1^{i_1} a_rc_r\Big\}\Big)\Big] \\ F_N^* &= \exp(ikl(N-2)a_Nc_N)\Big[\sum_{n\geqq 0} \sum_{1\leqq i_1<\cdots<i_{2n+1}\leqq N-1} \gamma_{i_{2n+1}}\gamma_{i_{2n}}\cdots\gamma_{i_1} \\ &\quad \times \exp\Big(2ikl\Big\{\sum_1^{i_{2n+1}} a_rc_r - \sum_1^{i_{2n}} a_rc_r + \cdots + \sum_1^{i_1} a_rc_r - a_1c_1\Big\}\Big)\Big]\,. \end{aligned} \tag{3.13}$$

We can check this formula by replacing $E_N$ and $F_N^*$ given by (3.13) in (3.12); we obtain

$$\begin{aligned} E_{N+1} &= \exp(-ikl(N-1)a_{N+1}c_{N+1})\Big[\sum_{n\geqq 0} \sum_{1\leqq i_1<\cdots<i_{2n}\leqq N-1} \gamma_{i_{2n}}\cdots\gamma_{i_1} \\ &\quad \times \exp\Big(-2ikl\Big\{\sum_1^{i_{2n}} a_rc_r - \sum_1^{i_{2n-1}} a_rc_r + \cdots - \sum_1^{i_1} a_rc_r\Big\}\Big) \\ &\quad + \gamma_N \exp(-2ikl(a_2c_2 + \cdots + a_Nc_N)) \sum_{n\geqq 0} \sum_{1\leqq i_1<\cdots<i_{2n+1}\leqq N-1} \gamma_{i_{2n+1}}\cdots\gamma_{i_1} \\ &\quad \times \exp\Big(2ikl\Big\{\sum_1^{i_{2n+1}} a_rc_r - \cdots + \sum_1^{i_1} a_rc_r - a_1c_1\Big\}\Big)\Big] \end{aligned}$$

but this is obviously of the type given by formula (3.13) for $N+1$ instead of $N$ and $1 \leqq i_1 < \cdots < i_{2n} \leqq N$. In the same way, we also have

$$\begin{aligned} F^*_{N+1} = \exp(ikl(N-1)a_{N+1}c_{N+1})\Big[ & \gamma_N \exp(2ikl(a_2c_2 + \cdots + a_Nc_N)) \\ & \times \sum_{n\geqq 0} \sum_{1\leqq i_1<\cdots<i_{2n}\leqq N-1} \gamma_{i_{2n}} \cdots \gamma_{i_1} \exp\Big(-2ikl\Big\{\sum_1^{i_{2n}} a_rc_r - \cdots - \sum_1^{i_1} a_rc_r\Big\}\Big) \\ & + \sum_{n\geqq 0} \sum_{1\leqq i_1<\cdots<i_{2n+1}\leqq N-1} \gamma_{i_{2n+1}} \cdots \gamma_{i_1} \\ & \times \exp\Big(2ikl\Big\{\sum_1^{i_{2n+1}} a_rc_r - \sum_1^{i_{2n}} a_rc_r + \cdots + \sum_1^{i_1} a_rc_r - a_1c_1\Big\}\Big)\Big] \end{aligned}$$

which is again of the form (3.13) for $N+1$ instead of $N$ and $1 \leqq i_1 < \cdots < i_{2n+1} \leqq N$.

Coming back to the definition of $A_{N+1}$, $B_{N+1}$, we see by (3.8) and (3.11) that we have

$$\begin{aligned} A_{N,1} = {} & \alpha_1 \cdots \alpha_{N-1} \exp(ikl(a_2c_2 + \cdots + a_{N-1}c_{N-1})) \\ & \times \exp(-ikl(N-2)a_Nc_N)\Big[\sum_{n\geqq 0} \sum_{1\leqq i_1<\cdots<i_{2n}\leqq N-1} \gamma_{i_{2n}}\gamma_{i_{2n-1}} \cdots \gamma_{i_1} \\ & \times \exp\Big(-2ikl\Big(\sum_1^{i_{2n}} a_rc_r - \sum_1^{i_{2n-1}} a_rc_r + \cdots - \sum_1^{i_1} a_rc_r\Big)\Big)\Big] \\ B^*_{N,1} = {} & \alpha_1 \cdots \alpha_{N-1} \exp(-ikl(a_2c_2 + \cdots + a_{N-1}c_{N-1})) \\ & \times \exp(ikl(N-2)a_Nc_N)\Big[\sum_{n\geqq 0} \sum_{1\leqq i_1<\cdots<i_{2n+1}\leqq N-1} \gamma_{i_{2n+1}} \cdots \gamma_{i_1} \\ & \times \exp\Big(2ikl\Big(\sum_1^{i_{2n+1}} a_rc_r - \sum_1^{i_{2n}} a_rc_r + \cdots + \sum_1^{i_1} a_rc_r - a_1c_1\Big)\Big)\Big]. \end{aligned} \tag{3.14}$$

We also have the same algebraic formula for $ik$ replaced by $\lambda^{1/2}$.

**2. The heat kernel for a general finite $N$.** We write for $x \in I_j$

$$\begin{aligned} u_{-,j}(x,\lambda) &= A_j(\lambda)\exp(\lambda^{1/2}a_jc_jx) + B_j(\lambda)\exp(-\lambda^{1/2}a_jc_jx) \\ u_{-,1}(x,\lambda) &= \exp(\lambda^{1/2}a_jc_jx) \\ u_{+,j}(x,\lambda) &= D_j(\lambda)\exp(\lambda^{1/2}a_jc_jx) + E_j(\lambda)\exp(-\lambda^{1/2}a_jc_jx) \\ u_{+,N}(x,\lambda) &= \exp(-\lambda^{1/2}a_Nc_Nx)\,. \end{aligned} \tag{3.15}$$

The for $j > 1$, by (2.16) (with $ik \equiv \lambda^{1/2}$)

$$\begin{pmatrix} A_j(\lambda) \\ B_j(\lambda) \end{pmatrix} = T_{j-1} \cdots T_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and so

$$\begin{aligned} A_j(\lambda) &= (a_jc_1/a_1c_j)^{1/2}2^{-j+1}(a_jc_j(a_{j-1}c_{j-1} \cdots a_2c_2)^2a_1c_1)^{-1/2}A_{j,1} \\ B_j(\lambda) &= (a_jc_1/a_1c_j)^{1/2}2^{-j+1}(a_jc_j(a_{j-1}c_{j-1} \cdots a_2c_2)^2a_1c_1)^{-1/2}B_{j,1} \end{aligned} \tag{3.16}$$

and then for $j < N$

$$\begin{pmatrix} D_j(\lambda) \\ E_j(\lambda) \end{pmatrix} = T_j^{-1} \cdots T_{N-1}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} D_N \\ E_N \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$T_j^{-1} = \frac{1}{2a_{j+1}c_j} \begin{pmatrix} \alpha_j \exp(-ikl_j\theta_j) & -\beta_j \exp(-ikl_j\sigma_j) \\ -\beta_j \exp(ikl_j\sigma_j) & \alpha_j \exp(ikl_j\theta_j) \end{pmatrix}$$

so that we have to compute a backward product of the same type as before.

If $j < N$, we have

$$\begin{aligned} G^{(j,N)}(x, y, \lambda) = &-\frac{a_N c_N^2 2^{N-j-1}}{\lambda^{1/2} c_j A_{N,1}} (a_{N-1}c_{N-1}a_{N-2}c_{N-2} \cdots a_j c_j) \exp(-\lambda^{1/2} a_N c_N y) \\ &\times \{A_{j,1} \exp(\lambda^{1/2} a_j c_j x) + B_{j,1}^* \exp(-\lambda^{1/2} a_j c_j x)\} \end{aligned} \tag{3.17}$$

(rcall that in this notation $x \in I_j$, $y \in I_N$)

$$\begin{aligned} G^{(N,N)}(x, y, \lambda) = &-\frac{a_N c_N}{2\lambda^{1/2} A_{N,1}} \exp(-\lambda^{1/2} a_N c_N y) \{A_{N,1} \exp(\lambda^{1/2} a_N c_N x) \\ &+ B_{N,1}^* \exp(-\lambda^{1/2} a_N c_N x)\} \qquad (\text{for } x < y) \end{aligned}$$

and the heat kernel is given by

$$p_t^{(j,N)}(x, y) = \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} G^{(j,N)}(x, y, \lambda) d\lambda .$$

REMARK 1. All $A_{j,1}$ and $B_{j,1}^*$ are computed by (3.14) with $ik$ changed into $\lambda^{1/2}$.

REMARK 2. For practical purposes these kernels are sufficient; somehow, we have a source of heat at $y \in I_N$ and an observer somewhere at $x$; it is reasonable to have sources outside the medium.

**3. Going to the continuum limit: the case of continuous coefficients.** We suppose now that $a^2(x)$ and $c^2(x)$ are functions which are constant for $x < 0$ and for $x > L$. We denote these constants $a_{-\infty}$, $c_{-\infty}$ and $a_\infty$, $c_\infty$, respectively. We discretize the segment $[0, L]$ into $N$ subsegments of length $L/N = l$ and denote as usual

$$I_1 = ]-\infty, 0[, \cdots, I_j = ](j-2)l, (j-1)l[, \cdots, I_{N+2} = ]L, \infty[ .$$

We shall also assume that $a$ and $c$ are continuous functions with bounded variation. We replace $a$ and $c$ in $I_j$ by constant values $a_j$ and $c_j$.

Fix $x \in I_j$. We want to study the limiting behaviour of $A_j(\lambda)$ and $B_j(\lambda)$ given by (3.16) when $N \to \infty$, for $j$ tending also to infinity such that $x \in I_j$. Let us consider $A_j(\lambda)$ first; since $A_{j,1}$ is given by (3.14), we see that $A_j(\lambda)$ is the product of three factors:

$$(a_j c_1/a_1 c_j)^{1/2} 2^{-j+1}(a_j c_j (a_{j-1}c_{j-1} \cdots a_2 c_2)^2 a_1 c_1)^{-1/2} \alpha_1 \cdots \alpha_{j-1} \,. \tag{3.18}$$

$$\exp(ikl(a_2 c_2 + \cdots + a_{j-1}c_{j-1}))\exp(-ikl(j-2)a_j c_j) \,. \tag{3.19}$$

$$\sum_{n \geq 0} \sum_{1 \leqq i_1 < \cdots < i_{2n} \leqq N-1} \gamma_{i_{2n}} \gamma_{i_{2n-1}} \cdots \gamma_{i_1} \times \exp\left(-2ikl\left\{\sum_1^{i_{2n}} a_r c_r - \sum_1^{i_{2n-1}} a_r c_r + \cdots - \sum_1^{i_1} a_r c_r\right\}\right) \tag{3.20}$$

We recall that

$$\alpha_{j-1} = a_{j-1}c_j + a_j c_{j-1} \,.$$

Here $a_1$ and $c_1$ refer to $I_1 = ]-\infty, 0[$ so they are equal to $a_{-\infty}$ and $c_{-\infty}$. $a_j$ and $c_j$ tend to $a(x)$ and $c(x)$ respectively if $a$ and $c$ are continuous.

Now we also have:

$$\left(\frac{\alpha_1}{2} \cdots \frac{\alpha_{j-1}}{2}\right) \frac{1}{a_2 c_2 \cdots a_{j-1}c_{j-1}} = \left(\frac{a_1 c_2 + a_2 c_1}{2a_2 c_2}\right) \cdots \left(\frac{a_{j-1}c_j + a_j c_{j-1}}{2a_j c_j}\right) a_j c_j \,.$$

But

$$a_k = a_{k+1} - (a_{k+1} - a_k) \,, \qquad c_k = c_{k+1} - (c_{k+1} - c_k)$$

and the following finite product

$$\prod_{k=1}^{j-1} \left(\frac{a_k c_{k+1} + a_{k+1} c_k}{2a_{k+1} c_{k+1}}\right) = \prod_{k=1}^{j-1} \left(1 - \frac{a_{k+1} - a_k}{2a_{k+1}} - \frac{c_{k+1} - c_k}{2c_{k+1}}\right)$$

converges to

$$\exp\left(-\int_0^x \left(\frac{da(x)}{2a(x)} + \frac{dc(x)}{2c(x)}\right)\right) = \left(\frac{a_{-\infty} c_{-\infty}}{a(x)c(x)}\right)^{1/2}$$

by the definition of the Riemann-Stieltjes integral with respect to a bounded variation measure on the real line. In consequence the factor (3.18) converges to $(c_{-\infty} a(x)/a_{-\infty} c(x))^{1/2}$. We also see that the factor (3.19) converges to $\exp\left(ik \int_0^x a(\xi)c(\xi)d\xi - ikxa(x)c(x)\right)$, because $(j-2)L/N < x < (j-1)L/N$ and $l = L/N$. Concerning the sum (3.20), we note that

$$\gamma_j = \frac{\beta_j}{\alpha_j} = \frac{a_j c_{j+1} - a_{j+1} c_j}{a_j c_{j+1} + a_{j+1} c_j} = \frac{c_{j+1}/a_{j+1} - c_j/a_j}{c_{j+1}/a_{j+1} + c_j/a_j} \,.$$

In particular, we immediately see that each summand in (3.20) converges to

$$\int_0^x \frac{dV(x_{2n})}{2V(x_{2n})} \exp\left(-2ik \int_0^{x_{2n}} a(\xi)c(\xi)d\xi\right) \int_0^{x_{2n}} \frac{dV(x_{2n-1})}{2V} \tag{3.21}$$

$$\times \exp\Big(2ik\int_0^{x_{2n-1}} a(\xi)c(\xi)d\xi\Big)\cdots\int_0^{x_2}\frac{dV(x_1)}{2V}\exp\Big(2ik\int_0^{x_1} a(\xi)c(\xi)d\xi\Big)$$

again by the definition of the Riemann-Stieltjes integral where we have denoted $V = c/a$ which is by our hypothesis a continuous function such that

$$K \equiv \int_{-\infty}^{\infty} |d(\log c/a)| < \infty \, .$$

Let us denote $W(x) = \int_0^x |d(\log c/a)|$ which is an increasing function tending to $K$ if $x$ tends to $L$ or $\infty$. Since $|\gamma_j|$ is dominated by $|\log(c_{j+1}/a_{j+1}) - \log(c_j/a_j)|$, we have always an estimate from above of each summand of (3.20) by

$$\begin{aligned}\sum_{1\leqq i_1<\cdots<i_{2n}\leqq N-1} |\gamma_{i_{2n}}|\cdots|\gamma_{i_1}| &\leqq \int_0^x dW(x_{2n})\int_0^{x_{2n}} dW(x_{2n-1})\cdots\int_0^{x_2} dW(x_1)\\ &= W(x)^{2n}/(2n)! \leqq K^{2n}/(2n)!\, .\end{aligned}$$

By the Lebesgue dominated convergence theorem for series, the sum (3.20) tends as $N\to\infty$ to the infinite sum in $n$ of the term (3.21). Thus

$$\begin{aligned}A_j(\lambda) \to \Big(\frac{a(x)c_{-\infty}}{a_{-\infty}c(x)}\Big)^{1/2} &\exp(-ikxa(x)c(x))\exp\Big(-ik\int_0^x a(\xi)c(\xi)d\xi\Big)\\ &\times\sum_{n\geqq 0}\int_0^x \frac{dV(x_{2n})}{2V}\exp\Big(-2ik\int_0^{x_{2n}} a(\xi)c(\xi)d\xi\Big)\int_0^{x_{2n}}\frac{dV(x_{2n-1})}{2V}\\ &\times\exp\Big(2ik\int_0^{x_{2n-1}} a(\xi)c(\xi)d\xi\Big)\cdots\int_0^{x_2}\frac{dV(x_1)}{2V}\\ &\times\exp\Big(2ik\int_0^{x_1} a(\xi)c(\xi)d\xi\Big)\, .\end{aligned}\tag{3.22}$$

We denote this limit by $A(x, \lambda)$. In the same way

$$\begin{aligned}B_j(\lambda) \to \Big(\frac{a(x)c_{-\infty}}{a_{-\infty}c(x)}\Big)^{1/2} &\exp(ikxa(x)c(x))\exp\Big(-ik\int_0^x a(\xi)c(\xi)d\xi\Big)\\ &\times\sum_{n\geqq 0}\int_0^x \frac{dV(x_{2n+1})}{2V}\exp\Big(2ik\int_0^{x_{2n+1}} a(\xi)c(\xi)d\xi\Big)\int_0^{x_{2n+1}}\frac{dV(x_{2n})}{2V}\\ &\times\exp\Big(-2ik\int_0^{x_{2n}} a(\xi)c(\xi)d\xi\Big)\cdots\int_0^{x_2}\frac{dV(x_1)}{2V}\\ &\times\exp\Big(2ik\int_0^{x_1} a(\xi)c(\xi)d\xi\Big)\, .\end{aligned}\tag{3.23}$$

We denote this limit by $B(x, \lambda)$.

The case where $x > L$, so that $x \in I_{N+2}$ is slightly special; we denote this case by $A_\infty(\lambda)$ and $B_\infty(\lambda)$. We have

$$\frac{1}{c_\infty a_{-\infty}}\left(\frac{a_{-\infty}c_{-\infty}}{a_\infty c_\infty}\right)^{1/2}\exp(-ikLa_\infty c_\infty)\exp\left(ik\int_0^L a(\xi)c(\xi)d\xi\right)\sum_{n\geqq 0}\int_0^L\frac{dV(x_{2n})}{2V}\cdots$$

so that

$$\begin{aligned}A_\infty(\lambda) &= \left(\frac{a_\infty c_{-\infty}}{a_{-\infty}c_\infty}\right)^{1/2}\exp(-ikLa_\infty c_\infty)\exp\left(ik\int_0^L a(\xi)c(\xi)d\xi\right)\\ &\quad\times\sum_{n\geqq 0}\int_0^L\frac{dV(x_{2n})}{2V}\exp\left(-2ik\int_0^{x_{2n}}c(\xi)a(\xi)d\xi\right)\cdots\int_0^{x_2}\frac{dV(x_1)}{2V}\\ &\quad\times\exp\left(2ik\int_0^{x_1}a(\xi)c(\xi)d\xi\right)\\ B_\infty(\lambda) &= \left(\frac{a_\infty c_{-\infty}}{a_{-\infty}c_\infty}\right)^{1/2}\exp(ikLa_\infty c_\infty)\exp\left(-ik\int_0^L a(\xi)c(\xi)d\xi\right)\\ &\quad\times\sum_{n\geqq 0}\int_0^L\frac{dV(x_{2n+1})}{2V}\exp\left(2ik\int_0^{x_{2n+1}}a(\xi)c(\xi)d\xi\right)\cdots\int_0^{x_2}\frac{dV(x_1)}{2V}\\ &\quad\times\exp\left(2ik\int_0^{x_1}a(\xi)c(\xi)d\xi\right).\end{aligned}\tag{3.24}$$

Let us now take $x < y$ and $y > L$. We choose $j$ with $x \in I_j$; first if $x < L$, we have

$$\begin{aligned}G(x, y, \lambda) = -\frac{a_{N+2}c_{N+2}}{\lambda^{1/2}A_{N+2}(\lambda)}&\{A_j(\lambda)\exp(\lambda^{1/2}a_jc_jx) + B_j(\lambda)\exp(-\lambda^{1/2}a_jc_jx)\}\\ &\times\exp(-\lambda^{1/2}(a_{N+2}c_{N+2})y)\end{aligned}$$

and going to the limit $N\to\infty$, we obtain the Green function of the operator

$$\begin{aligned}G(x, y, \lambda) = -\frac{a_\infty c_\infty}{2\lambda^{1/2}A_\infty(\lambda)}&\{A(x, \lambda)\exp(\lambda^{1/2}a(x)c(x)x)\\ &+B(x, \lambda)\exp(-\lambda^{1/2}a(x)c(x)x)\}\exp(-\lambda^{1/2}a_\infty c_\infty y)\end{aligned}\tag{3.25}$$

were $A(x, \lambda)$ and $B(x, \lambda)$ are the limits given by (3.22) and (3.23).

If $L < x < y$, then

$$\begin{aligned}G(x, y, \lambda) = \frac{a_\infty c_\infty}{2\lambda^{1/2}}&\Big\{\exp(\lambda^{1/2}a_\infty c_\infty x)\\ &\times\frac{B(\infty, \lambda)}{A(\infty, \lambda)}\exp(-\lambda^{1/2}a_\infty c_\infty x)\Big\}\exp(-\lambda^{1/2}a_\infty c_\infty y)\end{aligned}\tag{3.26}$$

where $A(\infty, \lambda)$ and $B(\infty, \lambda)$ are given by (3.24).

*These are the Green function of*

$$\frac{1}{c^2(x)}\frac{d}{dx}\left(\frac{1}{a^2(x)}\frac{d}{dx}\right).$$

The heat kernel can be computed by the usual contour integral.

REMARK. We used the fact that the Green function $G_N(x, y, \lambda)$ for the operator $c_N^{-2}(x)\ (d/dx(a_N^{-2}(x)d/dx))$ converges to the Green function $G(x, y, \lambda)$ when $c_N$ and $a_N$ tend to $c$ and $a$ respectively. But this fact can be easily shown by routine argument of successive approximation.

**4. The continuum limit: the case of discontinuous coefficients.** We suppose now that $a^2(x)$ and $c^2(x)$ are functions constant for $x < 0$ and $x > L$ and that they are functions of bounded variation such that they may be discontinuous at a set which has only a finite number of accumulation points. We *define* $a/c$ at a point of discontinuity as the mean value of their left and right limits, so that the equalities

$$\frac{c}{a}(x_0) = \frac{1}{2}\left(\frac{c}{a}(x_0^+) + \frac{c}{a}(x_0^-)\right)$$

are valid for each point. We have now to be extremely careful to compute the limit of $A_j(\lambda)$ and $B_j(\lambda)$ when $j \to +\infty$ is such that $x \in I_j$. We suppose that $x$ is not a point of discontinuity: then, everything goes as in the previous section concerning (3.20) and (3.21). But the problem is the series in (3.14) at the points of discontinuity which are before $x$; if such a point $x_0$ appears in the interval $I_k$ $(k < j)$, we can refine the partition so that this point is the upper extremity of $I_k$, i.e., $(x_0) = \bar{I}_{k+1} \cap \bar{I}_k$; suppose now that $i_l = k$ in the series (3.14); then

$$i_1 < \cdots < i_{l-1} < k = i_l < i_{l+1} < \cdots$$

$$\gamma_{i_l} = \gamma_k = \frac{a_k c_{k+1} - c_k a_{k+1}}{a_k c_{k+1} + a_{k+1} c_k}$$

where $a_k$ is the limiting value on the left and $a_{k+1}$ the limiting value on the right. But this is exactly

$$\gamma_k = \frac{\dfrac{c_{k+1}}{a_{k+1}} - \dfrac{c_k}{a_k}}{\dfrac{c_{k+1}}{a_{k+1}} + \dfrac{c_k}{a_k}} = \frac{\left(\dfrac{c}{a}\right)(x_0^+) - \left(\dfrac{c}{a}\right)(x_0^-)}{2\left(\dfrac{c}{a}\right)(x_0)}\,.$$

But $c/a$ having a discontinuity at $x_0$, this is exactly the integral in $[x_0^-, x_0^+]$ of $(2(c/a)(x_0))^{-1}d(c/a)$ and we obtain formally the same expression as in (3.22)

$$\sum_{n\geq 0} \int_0^x \frac{dV}{2V}(x_{2n}) \exp\left(-2ik\int_0^{x_{2n}} a(\xi)c(\xi)d\xi\right) \int_{[0,x_{2n}[} \frac{dV}{2V}(x_{2n-1})$$
$$\times \exp\left(2ik\int_0^{x_{2n-1}} a(\xi)c(\xi)d\xi\right) \cdots \int_{[0,x_2[} \frac{dV}{2V}(x_1)\exp\left(2ik\int_0^{x_1} a(\xi)c(\xi)d\xi\right)\,.$$

But the intermediate integrals are taken on the semi-open set $[0, x_l[$ (because if $k = i_l$ for the same $l$ and corresponds to a discontinuity, then for $l' < l$, the $i_{l'}$ are different from $i_l$).

The only remaining case is the case where the upper bound $x$ of the integral is itself a point of discontinuity. We can assume that the partition in intervals is such that $x \in \bar{I}_j \cap \bar{I}_{j+1}$.

As we know that $x \to G(x, y, \lambda)$ is continuous, we can compute the value for $x' < x$ and let $x' \to x^-$, for example.

The final thing is to obtain the limit in (3.18) or (3.19) in the presence of points of discontinuity. Let us first suppose that $x$ itself is not a point of discontinuity and that $x$ is in $I_j$. First of all if there is only a finite number of discontinuities $x_1, \cdots, x_r$ before $x$ then by an easy modification of the argument of Section 3

$$\prod_{k=1}^{j-1} \frac{a_k c_{k+1} + a_{k+1} c_k}{2a_{k+1} c_{k+1}} = \left(\frac{a_{-\infty} c_{-\infty}}{a(x_1^-)c(x_1^-)}\right)^{1/2} \frac{a(x_1^-)c(x_1^+) + a(x_1^+)c(x_1^-)}{2a(x_1^+)c(x_1^-)} \left(\frac{a(x_1^+)c(x_1^+)}{a(x_2^-)c(x_2^-)}\right)^{1/2}$$
$$\times \cdots \times \left(\frac{a(x_r^+)c(x_r^+)}{a(x)c(x)}\right)^{1/2} = \left(\frac{a_{-\infty} c_{-\infty}}{a(x)c(x)}\right)^{1/2} \prod_{x_k < x} \left(\frac{c}{a}\right)(x_k) \left(\frac{a(x_k^-)a(x_k^+)}{c(x_k^-)c(x_k^+)}\right)^{1/2} .$$

If there is an infinite number of such points which accumulate to $x$, the only thing to check is that the infinite product

$$\prod_{x_k < x} \left(\frac{c}{a}\right)(x_k) \left(\frac{a(x_k^-)a(x_k^+)}{c(x_k^-)c(x_k^+)}\right)^{1/2}$$

is convergent.

Put $\xi_k = (c/a)(x_k^-)$, $\eta_k = (c/a)(x_k^+)$, so that by our definitions, $(c/a)(x_k) = 1/2\,(\xi_k + \eta_k)$. Put also $\delta_k = \eta_k - \xi_k$ (the jump at the discontinuity). Then the logarithm of the general term of the product is $\log((\xi_k + \eta_k)/2) - 1/2 \log(\xi_k \eta_k) = \log(1 + \delta_k/2\xi_k) - 1/2 \log(1 + \delta_k/\xi_k) = 0(\delta_k^2 \xi_k^{-2})$. On the other hand $d \log(c/a)$ is a bounded variation measure which implies that $\sum |\log \eta_k - \log \xi_k| = \sum |\log(1 + \delta_k/\xi_k)|$ is finite, so that $\sum |\delta_k/\xi_k|^2 < \infty$. Hence the infinite product converges. If $x$ is itself a point of discontinuity, we obtain if $x \in \bar{I}_j \cap \bar{I}_{j+1}$

$$\left(\frac{a_{-\infty} c_{-\infty}}{a(x^-)c(x^-)}\right)^{1/2} \prod_{x_k < x} \left(\frac{c}{a}\right)(x_k) \left(\frac{a(x_k^-)a(x_k^+)}{c(x_k^-)c(x_k^+)}\right)^{1/2}$$

and in (3.19), we obtain

$$\left(\frac{a(x^-)c_{-\infty}}{a_{-\infty} c(x^-)}\right)^{1/2} \prod_{x_k < x} \left(\frac{c}{a}\right)(x_k) \left(\frac{a(x_k^-)a(x_k^+)}{c(x_k^-)c(x_k^+)}\right)^{1/2}$$

where $x_k$ are the discontinuity points (we assume here that 0 is not a point of discontinuity for simplicity).

Finally (3.20) will not be changed and (3.21) will give

$$\exp(-ika(x^-)c(x^-)x) \quad (\text{recall } x \leqq L) .$$

Now in the summation in (3.14), we compute the limiting value for $x' < x$, $x' \in I_j$ and so the $i_{2n}$ (or $i_{2n+1}$) is $\leqq j-1$, and so at the limit when $N \to \infty$, we obtain

$$\sum_{n\geqq 0} \int_{[0,x[} \frac{dV}{2V}(x_{2n}) \exp\left(-2ik \int_0^{x_{2n}} a(\xi)c(\xi)d\xi \int_{[0,x_{2n}[} \frac{dV}{2V}(x_{2n-1}) \right.$$
$$\left. \times \exp\left(2ik \int_0^{x_{2n-1}} a(\xi)c(\xi)d\xi\right) \cdots \int_{[0,x_2[} \frac{dV}{2V}(x_1) \exp\left(2ik \int_0^{x_1} a(\xi)c(\xi)d\xi\right) \right. .$$

All these can be summarized in the following theorem.

THEOREM. *Let $a(x)$, $c(x)$ be functions of bounded variation such that $d(c/a)/(c/a)$ is a bounded measure. We suppose that $a$ and $c$ are constants in $]-\infty, 0[$ and $]L, \infty[$ and also that the set of discontinuous points of $a$ and $c$ has only a finite number of accumulation points.*

*Then the Green function of the operator*

$$L = \frac{1}{c^2(x)} \frac{d}{dx}\left(\frac{1}{a^2(x)}\left(\frac{d}{dx}\right)\right)$$

*is given by $G(x, y, \lambda)$ for $y \geqq L$, $x < y$ by the formulas*

$$G(x, y, \lambda) = -\frac{a_\infty c_\infty}{2\lambda^{1/2} A_\infty(\lambda)} \{A(x, \lambda)\exp(\lambda^{1/2}a(x^-)c(x^-)x)$$
$$+ B(x, \lambda)\exp(-\lambda^{1/2}a(x^-)c(x^-)x)\}\exp(-\lambda^{1/2}a_\infty c_\infty y) \qquad \textit{for} \quad x \leqq L$$

*and*

$$G(x,y,\lambda) = -\frac{a_\infty c_\infty}{2\sqrt{\lambda}}\left\{\exp(\sqrt{\lambda}\, a_\infty c_\infty(x-y)) + \frac{B_\infty(\lambda)}{A_\infty(\lambda)}\exp(-\sqrt{\lambda}\, a_\infty c_\infty(x+y))\right\}$$
$$\textit{for} \quad L \leqq x \leqq y$$

*with the following definitions*

$$A(x, \lambda) = \left(\frac{a(x^-)c_{-\infty}}{a_{-\infty}c(x^-)}\right)^{1/2} \exp(-\sqrt{\lambda}\, x a(x^-)c(x^-)) \exp\left(\sqrt{\lambda} \int_0^x a(\xi)c(\xi)d\xi\right)$$
$$\times\left[\sum_{n\geqq 0} \int_{[0,x[} \frac{dV}{2V}(x_{2n}) \exp\left(-2\sqrt{\lambda} \int_0^{x_{2n}} a(\xi)c(\xi)d\xi\right) \int_{[0,x_{2n}[} \frac{dV}{2V}(x_{2n-1})\right.$$
$$\times \exp\left(2\sqrt{\lambda} \int_0^{x_{2n-1}} a(\xi)c(\xi)d\xi\right) \cdots \int_{[0,x_2[} \frac{dV}{2V}(x_1)$$
$$\left.\times \exp\left(2\sqrt{\lambda} \int_0^{x_1} a(\xi)c(\xi)d\xi\right)\right] \prod_{x_k<x} \left(\frac{c}{a}\right)(x_k)\left(\frac{a(x_k^-)a(x_k^+)}{c(x_k^-)c(x_k^+)}\right)^{1/2}$$

$$B(x, \lambda) = \left(\frac{a(x^-)c_{-\infty}}{a_{-\infty}c(x^-)}\right)^{1/2} \exp(\sqrt{\lambda}\, x a(x^-)c(x^-)) \exp\left(-\sqrt{\lambda} \int_0^x a(\xi)c(\xi)d\xi\right)$$

$$\times\left[\sum_{n\geq 0} \int_{[0,x[} \frac{dV}{2V}(x_{2n+1}) \exp\left(2\sqrt{\lambda} \int_0^{x_{2n+1}} a(\xi)c(\xi)d\xi\right) \int_{[0,x_{2n+1}[} \frac{dV}{2V}(x_{2n})\right.$$

$$\times \exp\left(-2\sqrt{\lambda} \int_0^{x_{2n}} a(\xi)c(\xi)d\xi\right) \cdots \int_{[0,x_2[} \frac{dV}{2V}(x_1)$$

$$\left.\times \exp\left(2\sqrt{\lambda} \int_0^{x_1} a(\xi)c(\xi)d\xi\right)\right] \prod_{x_k<x} \left(\frac{c}{a}\right)(x_k) \left(\frac{a(x_k^-)a(x_k^+)}{c(x_k^-)c(x_k^+)}\right)^{1/2}$$

$$A_\infty(\lambda) = \left(\frac{a_\infty c_{-\infty}}{a_{-\infty}c_\infty}\right)^{1/2} \exp(-\sqrt{\lambda}\, L a_\infty c_\infty) \exp\left(\sqrt{\lambda} \int_0^L a(\xi)c(\xi)d\xi\right)$$

$$\times\left[\sum_{n\geq 0} \int_{[0,L]} \frac{dV}{2V}(x_{2n}) \exp\left(-2\sqrt{\lambda} \int_0^{x_{2n}} a(\xi)c(\xi)d\xi\right) \int_{[0,x_{2n}[} \frac{dV}{2V}(x_{2n-1})\right.$$

$$\times \exp\left(2\sqrt{\lambda} \int_0^{x_{2n-1}} a(\xi)c(\xi)d\xi\right) \cdots \int_{[0,x_2[} \frac{dV}{2V}(x_1)$$

$$\left.\times \exp\left(2\sqrt{\lambda} \int_0^{x_1} a(\xi)c(\xi)d\xi\right)\right] \prod_{x_k<L} \left(\frac{c}{a}\right)(x_k) \left(\frac{a(x_k^-)a(x_k^+)}{c(x_k^-)c(x_k^+)}\right)^{1/2}$$

$$B_\infty(\lambda) = \left(\frac{a_\infty c_{-\infty}}{a_{-\infty}c_\infty}\right)^{1/2} \exp(\sqrt{\lambda}\, L a_\infty c_\infty) \exp\left(-\sqrt{\lambda} \int_0^L a(\xi)c(\xi)d\xi\right)$$

$$\times \prod_{x_k<x} \left(\frac{c}{a}\right)(x_k) \left(\frac{a(x_k^-)a(x_k^+)}{c(x_k^-)c(x_k^+)}\right)^{1/2} \sum_{n\geq 0} \int_{[0,L]} \frac{dV}{2V}(x_{2n+1})$$

$$\times \exp\left(2\lambda^{1/2} \int_0^{x_{2n+1}} a(\xi)c(\xi)d\xi\right) \cdots \int_{[0,x_2[} \frac{dV}{2V}(x_1)$$

$$\times \exp\left(2\lambda^{1/2} \int_0^{x_1} a(\xi)c(\xi)d\xi\right)$$

*where* $V(x) = 2^{-1}((c/a)(x^+) + (c/a)(x^-))$.

REMARK 1. Clearly the case of piecewise constant coefficients is a particular case of these formulas where $dV$ is a pure jump measure (a sum of Dirac masses); but we needed first to examine this case to deduce the general case.

REMARK 2. This theorem can also be applied to the case where $V$ increases only on a set which is of Lebesgue measure 0, without being piecewise constant (i.e., $V$ is continuous).

**5. Comments about the form of the Green function.** (i) The preceding theorem gives a series converging to the Green function; this series is convergent provided that $d(\log c/a)$ is a measure of bounded variations and we have proved that it converges very rapidly because it is controlled by the series of sinh or cosh. Moreover, this series is a resummation of the trivial perturbation series which does not converge in general. The quantity which controls the convergence is only $d(\log c/a)$.

(ii) The problem of transmission of heat or waves through one-dimensional medium was posed to us by several physicists. In particular, physicists are interested in propagation of waves in random media (which means that $a(x)$ and $c(x)$ are random functions). There are two main problems: the first one is to find the total transmission or reflexion coefficients by the medium; or, equivalently, to find $G(x, y, \lambda)$ for $x$ and $y$ separated by the medium. The other problem is the inverse scattering problem: namely to obtain information about the medium by measuring the total transmission or reflexion coefficients, or by knowing $G(x, y, \lambda)$; explicit expressions for the Green function are interesting because they give partial answers to these questions.

(iii) In higher dimensions, it is hopeless to find such explicit expressions in general. On the other hand, using projection technique and comparison theory, we can hope to obtain estimates for the Green function by one-dimensional Green function (see Malliavin [7] and Debiard-Gaveau-Mazet [1] for example).

**CHAPTER IV. An example of singular perturbation: limit of operators with irregular coefficients.** In this chapter, we give a new kind of example of the singular perturbation theory and we examine the limit behaviour of a sequence of operators with irregular coefficients. The limit behaviour is rather complicated and depends strongly on the kind of limit that we take.

**1. An example of a sequence of operators and their heat kernels.** We shall take the following formal operators

$$L = \left(\mathbb{I}_{[x<0]} + \frac{1}{c_2^2}\mathbb{I}_{[0<x<l]} + \mathbb{I}_{[x>l]}\right)\frac{d}{dx}\left(\left(\mathbb{I}_{[x<0]} + \frac{1}{a_2^2}\mathbb{I}_{[0<x<l]} + \mathbb{I}_{[x>l]}\right)\frac{d}{dx}\right). \tag{4.1}$$

and we shall suppose that the boundary layer $0 < x < l$ tends to 0 and that $a_2$ and/or $c_2$ tend to $+\infty$. We define $\mu$ and $\nu$ by

$$\mu = \frac{a_2 - c_2}{a_2 + c_2}, \qquad \nu = l a_2 c_2 . \tag{4.2}$$

Recall from Chapter II, n°5 (formulas (2.24) and (2.25)) that then $c_1 = c_3 = a_1 = a_3 = 1$, we have for $x < y$

$$p_t^{(1,1)}(x, y) = g(t, x - y) + \mu h(t, x + y, -\mu^2, -2\nu)$$

$$- \mu h(t, x + y - 2\nu, -\mu^2, -2\nu)$$

$$p_t^{(1,3)}(x, y) = (1 - \mu^2) h(t, x - y + l - \nu, -\mu^2, -2\nu) .$$

Recalling the definition (2.17) of the function $h$, we can rewrite this more explicitly as

$$(4.3) \qquad p_t^{(1,1)}(x, y) = g(t, x - y) + \frac{\mu}{2\pi} \int_{-\infty}^{\infty} e^{-k^2 t} e^{ik(x+y)} \frac{(1 - e^{-2ik\nu})}{1 - \mu^2 e^{-2ik\nu}} dk$$

$$(4.4) \qquad p_t^{(1,3)}(x, y) = (1 - \mu^2) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-k^2 t} \frac{e^{ik(x-y+l-\nu)}}{1 - \mu^2 e^{-2\nu ki}} dk .$$

*Formally* we see that $L$ tends to the operator $d^2/dx^2$. In fact, we shall see at the end of this chapter that this conclusion is entirely misleading and that we can have a great variety of cases.

2. **The case where $\mu$ tends to 1.** (a) The case where $\nu$ tends to a limit $0 < \nu_0 < \infty$.

We examine $p_t^{(1,3)}(x, y)$ given by (4.4); because $\nu_0$ is finite $> 0$ and $\mu^2 \to 1$, this kernel is the integral of a function which tends to 0 pointwise; the only problem is for $k$ near $\pi n/\nu_0$ for $n \in \boldsymbol{Z}$. But on a small neighborhood of such a $k$, we have

$$e^{-k^2 t} \left| \frac{1 - \mu^2}{1 - \mu^2 e^{-2\nu ki}} \right| \sim \left| \frac{e^{-k^2 t}}{1 + 2\nu i \mu^2 (k - (\pi n/\nu_0))/(1 - \mu^2)} \right|$$

and this is bounded by $Ce^{-k^2 t}$; so by the Lebesgue theorem $p_t^{(1,3)}(x, y) \to 0$. On the other hand, if we examine the second term of $p_t^{(1,1)}(x, y)$ we see that

$$\left| \frac{1 - e^{-2ik\nu}}{1 - \mu^2 e^{-2ik\nu}} \right| \leqq C \qquad \text{where} \quad \mu \to 1, \ \nu \to \nu_0$$

and so

$$p_t^{(1,1)}(x, y) \to g_t(x - y) + g_t(x + y) .$$

In that case $L$ tends to $d^2/dx^2$ with the pure reflexion condition at 0.

(b) The case where $\nu \to \infty$.

We expand in series the denominator in the integral (4.4)

$$p_t^{(1,3)} = \frac{1 - \mu^2}{2\pi} \sum_{m \geqq 0} \int_{-\infty}^{\infty} e^{-k^2 t} \mu^{2m} e^{-2m\nu ki} e^{ik(x-y+l-\nu)}$$

$$= \frac{1 - \mu^2}{\pi} \sum_{m \geqq 0} \mu^{2m} g(t, x - y + l - \nu - 2m\nu) .$$

It is clear that this tends to 0 if $\nu \to \infty$ and $\mu \to 1$. On the other hand, in the integral in (4.3) we have

$$\frac{1-e^{-2ik\nu}}{1-\mu^2 e^{-2ik\nu}}=\left[1-\frac{(\mu^2-1)e^{-2ik\nu}}{1-e^{-2ik\nu}}\right]^{-1}\to 1$$

and so

$$p_t^{(1,1)}(x, y)\to g_t(x-y)+g_t(x+y)$$

and we have the same conclusion as in (a).

(c) The case where $\nu\to 0$.

Let us examine $p_t^{(1,3)}$; then

$$\frac{1-\mu^2}{1-\mu^2 e^{-2\nu ki}}=\left[1-\mu^2\frac{(e^{-2\nu ki}-1)}{1-\mu^2}\right]^{-1}$$

and the denominator is equivalent to $1+\nu ki/(1-\mu)$. So if $\nu/(1-\mu)\to 0$, then $p_t^{(1,3)}\to g_t(x-y)$; if $\nu/(1-\mu)\to\infty$, then $p_t^{(1,3)}\to 0$; if $\nu/(1-\mu)\to\lambda_0$, then

$$p_t^{(1,3)}(x, y)\to\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-k^2 t}\frac{e^{ik(x-y)}}{1+\lambda_0 ik}dk\,. \tag{4.5}$$

We examine the integral term in $p_t^{(1,1)}$ (cf. (4.3))

$$\frac{1-e^{-2ik\nu}}{-1\mu^2 e^{-2ik\nu}}\sim\frac{2ik\nu}{1-((1-\mu)-1)^2(1-2ik\nu)}\sim\frac{ik\nu}{+\nu ki+(1-\mu)}$$

so if $\nu/(1-\mu)\to 0$, then $p_t^{(1,1)}\to g_t(x-y)$; if $\nu/(1-\mu)\to\infty$, then $p_t^{(1,1)}\to g_t(x-y)+g_t(x+y)$; if $\nu/(1-\mu)\to\lambda_0$, then

$$p_t^{(1,1)}\to g_t(x-y)+\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-k^2 t}e^{ik(x+y)}\frac{dk}{1-i/\lambda_0 k}\,. \tag{4.5$'$}$$

**3. The case where $\mu\to\mu_0$ with $-1<\mu_0<1$.** (a) The case where $\nu$ tends to a limit $0<\nu_0<\infty$. Then

$$p_t^{(1,3)}(x, y)\to(1-\mu_0^2)\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-k^2 t}e^{ik(x-y-\nu_0)}/(1-\mu_0^2 e^{-2\nu_0 ki})dk \tag{4.6}$$

and

$$p_t^{(1,1)}(x, y)\to g_t(x-y)+\frac{\mu_0}{2\pi}\int_{-\infty}^{\infty}e^{-k^2 t}e^{ik(x+y)}(1-e^{-2ik\nu_0})/(1-\mu_0^2 e^{-2ik\nu_0})dk \tag{4.7}$$

(b) If $\nu\to 0$, then $p_t^{(1,3)}(x, y)\to g_t(x-y)$ and $p_t^{(1,1)}(x, y)\to g_t(x-y)$.

(c) If $\nu\to\infty$, then we again write

$$\frac{1}{1-\mu^2 e^{-2\nu ki}}=\sum_{j\geq 0}\mu^{2j}e^{-2\nu kij}\,.$$

Then in $p_t^{(1,3)}$ we obtain $\sum_{j=0}^{\infty} \mu^{2j} g(t, x - y + l - (2j+1)\nu)$ which tends to 0 if $\nu \to \infty$, so $p_t^{(1,3)} \to 0$.

In the same manner, we expand the denominator in the integral of the second member of (4.3) and we obtain

$$(4.8) \qquad p_t^{(1,1)}(x, y) \to g(t, x - y) + \mu_0 g(t, x + y) .$$

4. **The case where** $\mu \to -1$. It is similar to the case $\mu \to 1$.

(a) If $\nu$ tends to a limit $0 < \nu_0 < \infty$, then $p_t^{(1,3)}(x, y)$ tends to 0 and $p_t^{(1,1)}$ tends to $g(t, x - y) - g(t, x + y)$.

(b) If $\nu$ tends to $\infty$, then $p_t^{(1,3)}(x, y)$ tends to 0 and $p_t^{(1,1)}(x, y)$ tends to $g(t, x - y) - g(t, x + y)$.

(c) If $\nu$ tends to 0, then

$$\frac{1 - \mu^2}{1 - \mu^2 e^{-2\nu k i}} \sim \left(1 + \frac{\nu k i}{1 + \mu}\right)^{-1} .$$

If $\nu/(1 + \mu) \to 0$, then $p_t^{(1,3)}(x, y) \to g(t, x - y)$ and $p_t^{(1,1)}(x, y) \to g(t, x - y)$;

If $\nu/(1 + \mu) \to \infty$, then $p_t^{(1,3)}(x, y) \to 0$ and $p_t^{(1,1)}(x, y) \to g(t, x - y) - g(t, x + y)$;

(4.9) If $\nu/(1 + \mu) \to \lambda_0$, then

$$p_t^{(1,3)}(x, y) \to \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-k^2 t} \frac{e^{ik(x-y)}}{1 + \lambda_0 ik} dk$$

$$p_t^{(1,1)}(x, y) \to g_t(x - y) - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-k^2 t} \frac{e^{ik(x+y)}}{1 - i/(\lambda_0 k)} dk .$$

5. **Conclusion.** Let us take the family of operators $L$ defined by (4.1) and suppose that $l$ tends to 0 and $a_2$ and/or $c_2$ tend to $\infty$. Define $\mu$, $\nu$ by (4.2).

Then the heat kernel $q_t(x, y)$ $(x < y)$ tends to the following situation:

(A) Suppose $\mu \to 1$.

(a) If $\nu \to \nu_0$, $0 < \nu_0 \leqq \infty$, then to a heat kernel with pure reflexion at 0.

(b) If $\nu \to 0$ and

(1°) if $\nu/(1 - \mu) \to 0$, then to a free heat kernel on $\boldsymbol{R}$;

(2°) if $\nu/(1 - \mu) \to \infty$, then to a heat kernel with pure reflexion at 0;

(3°) if $\nu/(1 - \mu) \to \lambda_0$, then to the limits (4.5) and (4.5)′.

(B) Suppose $\mu \to \mu_0$ and $-1 < \mu_0 < +1$.

(a) If $\nu \to \nu_0$, $0 < \nu_0 < +\infty$, then to the limits (4.6) and (4.7).

(b) If $\nu \to 0$, then to the free heat kernel.

(c) If $\nu \to \infty$, then to the heat kernel with partial absorption at 0 and partial reflexion, the formula being (4.8).

(C) Suppose $\mu \to -1$.

(a) If $\nu \to \nu_0$, $0 < \nu_0 \leqq +\infty$, then to the heat kernel with absorption at 0.

(b) If $\nu \to 0$ and

(1°) if $\nu/(1+\mu) \to 0$, then to the free heat kernel;

(2°) if $\nu/(1+\mu) \to \infty$, then to the heat kernel with absorption at 0;

(3°) if $\nu/(1+\mu) \to \lambda_0$ $(0 < \lambda_0 < +\infty)$, then to the limit (4.9).

In particular, we see that, the approximating operators $L^{(\varepsilon)}$ can be conservative, but the limit diffusion may not be conservative (when $\varepsilon \to 0$), for example in cases (B), (c); (C), (a); (C), (b), 2°; which seems surprising.

## CHAPTER V. Diffusion operators with spherical symmetry in $\boldsymbol{R}^3$.

**1. Transfer matrix for a self-adjoint operator with piecewise constant coefficients.** In this chapter, we shall only consider a self-adjoint operator in $\boldsymbol{R}^3$ having a spherical symmetry around 0. If $x$ is a vector, $r = |x|$ is its length. We begin with the case of piecewise constant coefficients; formally the operator can be written as

$$L = \operatorname{div}\left(\left(\sum_{j=1}^{N} \frac{1}{a_j^2} \mathbb{I}_{\{(j-1)l<|x|<jl\}} + \frac{1}{a_{N+1}^2} \mathbb{I}_{\{|x|>Nl\}}\right)\nabla\right), \tag{5.1}$$

where $a_j$ are constant (and we can always assume that the spheres where $a_j$ changes its value has radius $(j-1)l$).

A generalized eigenfunction $u(x, k)$ satisfies

$$\frac{1}{a_j^2}\Delta u_j = -k^2 u_j \quad \text{on } (j-1)l < |x| < jl \text{ or } |x| > Nl \text{ if } j = N+1 \tag{5.2}$$

$$\begin{aligned} u_j|_{S(0,jl)} &= u_{j+1}|_{S(0,jl)} \\ \frac{1}{a_j^2}\frac{\partial u_j}{\partial r}\bigg|_{S(0,jl)} &= \frac{1}{a_{j+1}^2}\frac{\partial u_{j+1}}{\partial r}\bigg|_{S(0,jl)} \end{aligned} \tag{5.3}$$

where $S(0, jl)$ is the sphere of centre 0 and radius $jl$ and $u_j = u|_{(j-1)l<|x|<jl}$.

We consider only the case of radial functions $u_j(r, k)$. Define $u_j(r, k) = v_j(r, k)/r$. Then on $(j-1)l < r < jl$, we have $d^2v_j/dr^2 = -k^2 v_j$ so that

$$v_j(r, k) = A_j(k)\exp(ika_j r) + B_j(k)\exp(-ika_j r)\,. \tag{5.4}$$

The second condition (5.3) becomes

$$\frac{1}{a_j^2}\left(\frac{\partial v_j}{\partial r} - \frac{1}{R_j}v_j\right)\Big|_{r=jl} = \frac{1}{a_{j+1}^2}\left(\frac{\partial v_{j+1}}{\partial r} - \frac{1}{R_j}v_{j+1}\right)\Big|_{r=jl}$$

so that if we take into account the continuity condition, then

$$\begin{aligned} &A_{j+1}\exp(ika_{j+1}jl) - B_{j+1}\exp(-ika_{j+1}jl) \\ &\quad = \frac{a_{j+1}}{a_j}(A_j\exp(ika_jjl) - B_j\exp(-ika_jjl)) \\ &\qquad + \frac{a_{j+1}}{ikjl}\left(\frac{1}{a_{j+1}^2} - \frac{1}{a_j^2}\right)(A_j\exp(ika_jjl) + B_j\exp(-ika_jjl)) \,. \end{aligned}$$

The continuity condition is just

$$A_{j+1}\exp(ika_{j+1}jl) + B_{j+1}\exp(-ika_{j+1}jl) = A_j\exp(ika_jjl) + B_j\exp(-ika_jjl)$$

so that

$$\begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix} = T_j\begin{pmatrix} A_j \\ B_j \end{pmatrix} \tag{5.5}$$

with $T_j$ being the following transfer matrix

$$T_j = \frac{1}{2a_j}\begin{pmatrix} t1 & t2 \\ t3 & t4 \end{pmatrix}$$

where

$$\begin{aligned} t1 &= \exp(ik(a_j - a_{j+1})jl)(a_j + a_{j+1})\left(1 + \frac{a_j - a_{j+1}}{ikjla_ja_{j+1}}\right) \\ t2 &= \exp(-ik(a_j + a_{j+1})jl)(a_j - a_{j+1})\left(1 + \frac{a_j + a_{j+1}}{ikjla_ja_{j+1}}\right) \\ t3 &= \exp(ik(a_j + a_{j+1})jl)(a_j - a_{j+1})\left(1 - \frac{a_j + a_{j+1}}{ikjla_ja_{j+1}}\right) \\ t4 &= \exp(-ik(a_j - a_{j+1})jl)(a_j + a_{j+1})\left(1 - \frac{a_j - a_{j+1}}{ikjla_ja_{j+1}}\right) \end{aligned}$$

and $\det T_j = a_{j+1}/a_j$. We define $\alpha_j = a_j + a_{j+1}$, $\beta_j = a_j - a_{j+1}$ and

$$R_j = 2a_jT_j \,. \tag{5.6}$$

Then

$$R_j = \begin{pmatrix} r1 & r2 \\ r3 & r4 \end{pmatrix} \tag{5.7}$$

where

$$r1 = \exp(ik\beta_jjl)\alpha_j\left(1 + \frac{1}{ijkl}\,\frac{\beta_j}{a_ja_{j+1}}\right)$$

$$r2 = \exp(-ik\alpha_j jl)\beta_j\Big(1 + \frac{1}{ikjl}\,\frac{\alpha_j}{a_j a_{j+1}}\Big)$$

$$r3 = \exp(ik\alpha_j jl)\beta_j\Big(1 - \frac{1}{ikjl}\,\frac{\alpha_j}{a_j a_{j+1}}\Big)$$

$$r4 = \exp(-ik\beta_j jl)\alpha_j\Big(1 - \frac{1}{ikjl}\,\frac{\beta_j}{a_j a_{j+1}}\Big)\,.$$

We have to compute the product

$$R_N R_{N-1} \cdots R_1 \equiv \begin{pmatrix} A_{N+1,1} & B_{N+1,1} \\ B^*_{N+1,1} & A^*_{N+1,1} \end{pmatrix} \tag{5.8}$$

so that

$$\begin{pmatrix} A_{N+1,1} \\ B^*_{N+1,1} \end{pmatrix} = R_N \begin{pmatrix} A_{N,1} \\ B^*_{N,1} \end{pmatrix} \tag{5.9}$$

$$A_{N+1,1} = \exp(-iklNa_{N+1})\Big\{\exp(iklNa_N)\alpha_N\Big(1 + \frac{1}{iklN}\,\frac{\beta_N}{a_N a_{N+1}}\Big)A_{N,1} + \exp(-iklNa_N)\beta_N\Big(1 + \frac{1}{iklN}\,\frac{\alpha_N}{a_N a_{N+1}}\Big)B^*_{N,1}\Big\}$$

$$B^*_{N+1,1} = \exp(iklNa_{N+1})\Big\{\exp(iklNa_N)\beta_N\Big(1 - \frac{1}{iklN}\,\frac{\alpha_N}{a_N a_{N+1}}\Big)A_{N,1} + \exp(-iklNa_N)\alpha_N\Big(1 - \frac{1}{iklN}\,\frac{\beta_N}{a_N a_{N+1}}\Big)B^*_{N,1}\Big\}\,.$$

We define as in Chapter III

$$A_{N+1,1} = \alpha_1 \cdots \alpha_N C_{N+1}\,, \quad B^*_{N+1,1} = \alpha_1 \cdots \alpha_N D_{N+1}\,, \quad \gamma_N = \beta_N/\alpha_N\,, \tag{5.10}$$

and then

$$\begin{aligned} E_{N+1} &= \exp(-ikl(a_1 + \cdots + a_N))C_{N+1}\,, \\ F_{N+1} &= \exp(ikl(a_1 + \cdots + a_N))D_{N+1}\,. \end{aligned} \tag{5.11}$$

We obtain

$$\begin{aligned} E_{N+1} &= \exp(-iklNa_{N+1})\Big\{\exp(ikl(N-1)a_N)\Big(1 + \frac{1}{iklN}\,\frac{\beta_N}{a_N a_{N+1}}\Big)E_N \\ &\quad + \exp(-ikla_N(N+1))\exp(-2ikl(a_1 + \cdots + a_{N-1}))\gamma_N \\ &\quad \times\Big(1 + \frac{1}{iklN}\,\frac{\alpha_N}{a_N a_{N+1}}\Big)F_N\Big\} \\ F_{N+1} &= \exp(iklNa_{N+1})\Big\{\exp(ikla_N(N+1))\exp(2ikl(a_1 + \cdots + a_{N-1})) \\ &\quad \times\gamma_N\Big(1 - \frac{1}{iklN}\,\frac{\alpha_N}{a_N a_{N+1}}\Big)E_N \end{aligned} \tag{5.12}$$

$$+ \exp(-ikl(N-1)a_N)\Big(1 - \frac{1}{iklN}\frac{\beta_N}{a_N a_{N+1}}\Big)F_N\Big\} .$$

The formulas for solving (5.12) are of the same type as those found in Chapter III; namely we obtain

$$\begin{aligned}(5.13)\qquad E_N = \exp(-ikl(N-1)a_N)\Big[&\prod_{j=1}^{N-1}\Big(1+\frac{1}{iklj}\frac{\beta_j}{a_j a_{j+1}}\Big)+\sum_n \sum_{1\leq j_1<j_2<\cdots<j_{2n}\leq N-1}\\ &\prod_{r<j_1}\Big(1+\frac{1}{iklr}\frac{\beta_r}{a_r a_{r+1}}\Big)\gamma_{j_1}\Big(1-\frac{1}{iklj_1}\frac{\alpha_{j_1}}{a_{j_1}a_{j_1+1}}\Big)\\ &\times\exp\Big(2ikl\sum_{r=1}^{j_1}a_r\Big)\prod_{j_1<r<j_2}\Big(1-\frac{1}{iklr}\frac{\beta_r}{a_r a_{r+1}}\Big)\gamma_{j_2}\Big(1+\frac{1}{iklj_2}\frac{\alpha_{j_2}}{a_{j_2}a_{j_2+1}}\Big)\\ &\times\exp\Big(-2ikl\sum_{r=1}^{j_2}a_r\Big)\prod_{j_2<r<j_3}\Big(1+\frac{1}{iklr}\frac{\beta_r}{a_r a_{r+1}}\Big)\cdots\\ &\times\gamma_{j_{2n}}\Big(1+\frac{1}{ilkj_{2n}}\frac{\alpha_{j_{2n}}}{a_{j_{2n}}a_{j_{2n}+1}}\Big)\\ &\times\exp\Big(-2ikl\sum_{r=1}^{j_{2n}}a_r\Big)\prod_{j_{2n}<r\leq N-1}\Big(1+\frac{1}{iklr}\frac{\beta_r}{a_r a_{r+1}}\Big)\Big] .\end{aligned}$$

$$\begin{aligned}(5.14)\qquad F_N = \exp(ikl(N-1)a_N)&\sum_n\sum_{1\leq j_1<j_2\cdots<j_{2n+1}\leq N-1}\prod_{r<j_1}\Big(1+\frac{1}{iklr}\frac{\beta_r}{a_r a_{r+1}}\Big)\gamma_{j_1}\\ &\times\Big(1-\frac{1}{iklj_1}\frac{\alpha_{j_1}}{a_{j_1}a_{j_1+1}}\Big)\\ &\times\exp\Big(2ikl\sum_{r=1}^{j_1}a_r\Big)\prod_{j_1<r<j_2}\Big(1-\frac{1}{iklr}\frac{\beta_r}{a_r a_{r+1}}\Big)\gamma_{j_2}\Big(1+\frac{1}{iklj_2}\frac{\alpha_{j_2}}{a_{j_2}a_{j_2+1}}\Big)\\ &\times\exp\Big(-2ikl\sum_{r=1}^{j_2}a_r\Big)\cdots\gamma_{j_{2n+1}}\Big(1-\frac{1}{iklj_{2n+1}}\frac{\alpha_{j_{2n+1}}}{a_{j_{2n+1}}a_{j_{2n+1}+1}}\Big)\\ &\times\exp\Big(2ikl\sum_{r=1}^{j_{2n+1}}a_r\Big)\prod_{j_{2n+1}<r\leq N-1}\Big(1-\frac{1}{iklr}\frac{\beta_r}{a_r a_{r+1}}\Big) .\end{aligned}$$

To check that this is the correct solution, we have to substitute $E_N$ and $F_N$ in (5.12) by those given in the preceding formulas. We then see that we obtain the same formulas as (5.14) but for $E_{N+1}$ and $F_{N+1}$.

We then have from (5.10) and (5.11)

$$(5.15)\qquad \begin{aligned}A_{N+1,1} &= \alpha_1\cdots\alpha_N\exp(ikl(a_1+\cdots+a_N))E_{N+1}\\ B^*_{N+1,1} &= \alpha_1\cdots\alpha_N\exp(-ikl(a_1+\cdots+a_N))F_{N+1}\end{aligned}$$

and so

$$T_N\cdots T_1 = \frac{1}{2^N}\frac{1}{a_1\cdots a_N}R_N\cdots R_1$$

$$T_N \cdots T_1 = \frac{1}{2^N} \frac{\alpha_1 \cdots \alpha_N}{a_1 \cdots a_N} \begin{pmatrix} s1 & s2 \\ s3 & s4 \end{pmatrix} \tag{5.16}$$

where

$$s1 = \exp(ikl(a_1 + \cdots + a_N))E_{N+1}$$
$$s2 = \exp(ikl(a_1 + \cdots + a_N))F^*_{N+1}$$
$$s3 = \exp(-ikl(a_1 + \cdots + a_N))F_{N+1}$$
$$s4 = \exp(-ikl(a_1 + \cdots + a_N))E^*_{N+1} .$$

**2. Spectral resolution for a self-adjoint operator with piecewise constant coefficients.** We must now compute a spectral resolution of identity for $L$. Because we are on a half line $\boldsymbol{R}^+$, each eigenvalue $-k^2$ for the $v$ function is non-degenerate and there is only one $v(k, r)$: we must find $v$ such that

$$\delta(r - r') = \int_0^\infty v(k, r)v^*(k, r')dk . \tag{5.17}$$

We can also suppose that $v$ is a real function, so that

$$A_j^* = B_j . \tag{5.18}$$

Let us write (5.17) for $r, r' > Nl$; then $r - r'$ can take any positive or negative value and we must have

$$\begin{aligned} \delta(r - r') &= \int_0^\infty dk\{A_{N+1}\exp(ika_{N+1}r) + A^*_{N+1}\exp(-ika_{N+1}r)\} \\ &\quad \times\{A^*_{N+1}\exp(-ika_{N+1}r') + A_{N+1}\exp(ika_{N+1}r')\} \\ &= 2\int_0^\infty dk|A_{N+1}|^2\cos ka_{N+1}(r-r') + \int_0^\infty dk(A^2_{N+1} + A^{*2}_{N+1})\cos ka_{N+1}(r+r') \\ &\quad + i\int_0^\infty dk(A^2_{N+1} - A^{*2}_{N+1})\sin ka_{N+1}(r + r') . \end{aligned}$$

This gives

$$|A_{N+1}|^2 = \frac{a_{N+1}}{4\pi} . \tag{5.19}$$

Moreover, we have $v(k, 0)=0$ because $u(k, r) = v(k, r)/r$ has to be regular at $r = 0$, so that

$$A_1^* = -A_1 . \tag{5.20}$$

Now if we want to find a kernel $K(0, r')$ of some function $F(L)$ between 0 (the center of symmetry) and $r'$, we take

$$\begin{aligned} K(0, r') &= \lim_{r\to 0}\int_0^\infty F(-k^2)\frac{v(k, r)}{r}\,\frac{v^*(k, r')}{r'}dk \\ &= \frac{a_1}{r'}\int_0^\infty F(-k^2)ik(A_1 - A_1^*)(A_j\exp(ika_jr') + A_j^*\exp(-ika_jr'))dk \end{aligned} \tag{5.21}$$

$$= \frac{2a_1}{r'} \int_0^\infty F(-k^2) ikA_1(A_j \exp(ika_j r') + A_j^* \exp(-ika_j r'))dk \ ,$$

if $(j-1)l < r' \leqq jl$. But by (5.5) and (5.8)

$$\begin{pmatrix} A_{N+1} \\ A_{N+1}^* \end{pmatrix} = T_N T_{N-1} \cdots T_1 \begin{pmatrix} A_1 \\ -A_1 \end{pmatrix}$$

and by (5.16)

$$A_{N+1} = \frac{\alpha_1 \cdots \alpha_N}{2^N a_1 \cdots a_N} \exp(ikl(a_1 + \cdots + a_N))(E_{N+1} - F_{N+1}^*)A_1 \ .$$

Taking the modulus we have by (5.19)

$$|A_1| = \left(\frac{a_{N+1}}{4\pi}\right)^{1/2} \frac{2^N a_1 \cdots a_N}{\alpha_1 \cdots \alpha_N} \frac{1}{|E_{N+1} - F_{N+1}^*|} \qquad \text{and} \quad \arg A_1 = \frac{\pi}{2} \tag{5.22}$$

because $A_1 = -A_1^*$

$$A_{N+1} = \left(\frac{a_{N+1}}{4\pi}\right)^{1/2} \exp(ikl(a_1 + \cdots + a_N)) \frac{E_{N+1} - F_{N+1}^*}{|E_{N+1} - F_{N+1}^*|} e^{i\pi/2} \tag{5.23}$$

and more generally

$$\begin{pmatrix} A_j \\ A_j^* \end{pmatrix} = T_{j-1} \cdots T_1 \begin{pmatrix} A_1 \\ -A_1 \end{pmatrix} ,$$

$$A_j = \frac{\alpha_1 \cdots \alpha_{j-1}}{2^{j-1} a_1 \cdots a_{j-1}} \exp(ikl(a_1 + \cdots + a_{j-1}))(E_j - F_j^*)A_1$$

so that

$$\begin{aligned} A_1 A_j = & -\frac{a_j \cdots a_N 2^{N-j+1}}{\alpha_j \cdots \alpha_N} \frac{a_{N+1}}{4\pi} \frac{2^N a_1 \cdots a_N}{\alpha_1 \cdots \alpha_N} \frac{E_j - F_j^*}{|E_{N+1} - F_{N+1}^*|^2} \\ & \times \exp(ikl(a_1 + \cdots + a_{j-1})) \end{aligned} \tag{5.24}$$

$$A_1 A_{N+1} = -\frac{a_{N+1}}{4\pi} \frac{2^N a_1 \cdots a_N}{\alpha_1 \cdots \alpha_N} \frac{E_{N+1} - F_{N+1}^*}{|E_{N+1} - F_{N+1}^*|^2} \exp(ikl(a_1 + \cdots + a_N)) \ . \tag{5.25}$$

So, putting together formulas (5.21), (5.24) or (5.25) and the values of $E_j$ and $F_j$ given by (5.13) and (5.14), we have an explicit representation of the spectral measures of $L$ and of the functional calculus for $L$.

**3. Spectral resolution for a general self-adjoint operator (continuous coefficients).** We shall now assume that $L$ is of the form

$$L = \operatorname{div}\left(\frac{1}{a^2(x)} \nabla\right) , \tag{5.26}$$

where $a^2(x)$ is, first of all, a continuous function that we suppose to be constant for $|x| > L$. As in Chapter III, we divide the ball of radius $L$ in small corona

$$I_j = \{(j-1)l < |x| < jl\} \quad \text{where} \quad l = L/N$$

and call $I_{N+1} = \{|x| > L\}$. We take an approximation of $a^2(x)$ by piecewise constant functions in each $I_j$ in an obvious manner. Fix an $x$ with $|x| < L$ and choose $j$ such that $x \in I_j$ so that $j \leqq N$. We first look at the behaviour of

$$\frac{\alpha_1 \cdots \alpha_{j-1}}{2^{j-1} a_1 \cdots a_{j-1}} = \frac{(a_1 + a_2)(a_2 + a_3) \cdots (a_{j-1} + a_j)}{2^{j-1}(a_1 \cdots a_{j-1})} = \prod_{k=2}^{j} \left(1 + \frac{(a_k - a_{k-1})}{2a_{k-1}}\right)$$

which leads to

$$\exp\left(\int_0^x \frac{da}{2a}\right) = (a(x)/a(0))^{1/2} . \tag{5.27}$$

Again we have

$$\exp(ikl(a_1 + \cdots + a_{j-1})) \to \exp\left(ik \int_0^x a(\xi) d\xi\right), \quad a_{N+1} = a_\infty . \tag{5.28}$$

Now $E_j$ and $F_j$ are given by (5.13) and (5.14) in which

$$e^{-ik(j-1)a_j} \to e^{-ikxa(x)} \quad \text{because} \quad jl \simeq x \tag{5.29}$$

$$\begin{aligned} &\prod_{r=1}^{j-1} \left(1 + \frac{1}{iklr} \frac{\beta_r}{a_r a_{r+1}}\right) \\ &= \prod_{r=1}^{j-1} \left(1 + \frac{1}{iklr} \frac{a_r - a_{r+1}}{a_r a_{r+1}}\right) \to \exp\left(-\frac{1}{ik} \int_0^x \frac{da(\xi)}{\xi a^2(\xi)}\right) . \end{aligned} \tag{5.30}$$

Then, the structure of (5.13) is rather elementary. Define the kernel for $y < x$ by

$$\begin{aligned} \psi(x, dy) = &-\exp\left(-\frac{2}{ik} \int_y^x \frac{da(\xi)}{\xi a^2(\xi)}\right) \\ &\times \exp\left(-2ik \int_0^y a(\xi) d\xi\right)\left(1 + \frac{2}{ikya(y)}\right) \frac{da(y)}{a(y)} . \end{aligned} \tag{5.31}$$

Then we obtain

$$\begin{aligned} E_j \to e^{-ikxa(x)} \Bigg( &\exp\left(-\frac{1}{ik} \int_0^x \frac{da(\xi)}{\xi a^2(\xi)}\right) + \sum_{n \geqq 1} \int_0^x \psi(x, dx_{2n}) \int_0^{x_{2n}} \psi^*(x_{2n}, dx_{2n-1}) \\ &\times \cdots \int_0^{x_2} \psi^*(x_2, dx_1) \exp\left(-\frac{1}{ik} \int_0^{x_1} \frac{da(\xi)}{\xi a^2(\xi)}\right)\Bigg) \end{aligned} \tag{5.32}$$

and the corresponding formula for $F_j$

$$(5.33) \qquad F_j \to e^{ikxa(x)} \sum_{n\geq 0} \int_0^x \psi^*(x, dx_{2n+1}) \int_0^{x_{2n+1}} \psi(x_{2n+1}, dx_{2n}) \times \cdots \int_0^{x_2} \psi^*(x_2, dx_1) \exp\left(-\frac{1}{ik} \int_0^{x_1} \frac{da(\xi)}{\xi a^2(\xi)}\right).$$

Then, if $|x| > L$,

$$(5.34) \qquad E_{N+1} \to e^{-ikLa_\infty}\left(\exp\left(-\frac{1}{ik}\int_0^L \frac{da(\xi)}{\xi a^2(\xi)}\right) + \sum_{n\geq 1} \int_0^L \psi(L, dx_{2n}) \int_0^{x_{2n}} \psi^*(x_{2n}, dx_{2n-1}) \times \cdots \int_0^{x_2} \psi^*(x_2, dx_1) \exp\left(-\frac{1}{ik}\int_0^{x_1} \frac{da(\xi)}{\xi a^2(\xi)}\right)\right)$$

and

$$(5.35) \qquad F_{N+1} \to e^{ikLa_\infty} \sum_{n\geq 0} \int_0^L \psi^*(L, dx_{2n+1}) \int_0^{x_{2n+1}} \psi(x_{2n+1}, dx_{2n}) \times \cdots \int_0^{x_2} \psi^*(x_2, dx_1) \exp\left(-\frac{1}{ik}\int_0^{x_1} \frac{da(\xi)}{\xi a^2(\xi)}\right).$$

Let us now look at $E_j - F_j^*$ appearing in $A_1 A_j$ in (5.24).

$$(5.36) \qquad E_j - F_j^* = e^{-ikxa(x)}\left(\exp\left(-\frac{1}{ik}\int_0^x \frac{da(\xi)}{\xi a^2(\xi)}\right) + \sum_{p\geq 1} (-1)^p \int_0^x \psi(x, dx_p) \times \int_0^{x_p} \psi^*(x_p, dx_{p-1}) \cdots \int_0^{x_2} C^{p+1}\left(\psi(x_2, dx_1) \times \exp\left(\frac{1}{ik}\int_0^{x_1} \frac{da(\xi)}{\xi a^2(\xi)}\right)\right)\right)$$

where $C$ denotes the complex conjugation and $C^k$ its $k$-th power. As usual, this series will converge if

$$(5.37) \qquad \int_0^L \frac{|da(\xi)|}{a(\xi)} < \infty, \quad \int_0^L \frac{|da(\xi)|}{\xi a^2(\xi)} < \infty.$$

**4. The general case when $a$ has discontinuities.** We redefine $a$ by the formula

$$(5.38) \qquad a(x) = \frac{1}{2}(a(x^+) + a(x^-)).$$

As in Chapter III, we assume that $a$ has finite right and left limits at each point and that $a$ is a constant $a$ for $|x| > L$. We can obtain the same kind of formulas as in Chapter III, n°4.

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(*) UNIVERSITÉ P. ET M. CURIE
MATHÉMATIQUES UA 761
TOUR 45-46, 5 ÈME ÉTAGE 4
PLACE JUSSIEU 75230 PARIS
PARIS CEDEX 05-
FRANCE

(**) DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
TÔHOKU UNIVERSITY
KAWAUCHI, SENDAI 980
JAPAN

(***) TOKYO METROPOLITAN COLLEGE
OF AERONAUTICAL ENGINEERING
8-53-1 MINAMI-SENJU, ARAKAWA-KU
TOKYO, 116
JAPAN