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# RICCI CURVATURES OF CONTACT RIEMANNIAN MANIFOLDS

SHUKICHI TANNO

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**1. Introduction.** Let $(M, \eta, g) = (M, \phi, \xi, \eta, g)$ be a contact Riemannian manifold of dimension $2n+1$. If $\xi$ is a Killing vector field, then it is called a $K$-contact Riemannian manifold. Further, if the covariant derivative $\nabla\phi$ of $\phi$ satisfies some relation, then it is called a Sasakian manifold. The model spaces of contact metric structure are complete and simply connected Sasakian manifolds of constant $\phi$-sectional curvature $H$. These Sasakian manifolds admit the maximal dimensional automorphism groups (Tanno [6]). The Riemannian curvature tensor $R$ of a Sasakian manifold of constant $\phi$-sectional curvature is determined (Ogiue [3]). However, we know almost nothing about geometry on contact Riemannian manifolds of constant $\phi$-sectional curvature. One good result is due to Olszak [4], who showed an inequality on $H$ and the scalar curvature $S$ of a contact Riemannian manifold of constant $\phi$-sectional curvature $H$. Generalizing this inequality, we obtain the following.

THEOREM 3.1. *Let $(M, \eta, g)$ be a contact Riemannian manifold of constant $\phi$-sectional curvature $H$. Then the Ricci curvatures satisfy*

$$\mathrm{Ric}(X, X) + \mathrm{Ric}(\phi X, \phi X) \leqq 3n - 1 + (n+1)H$$

*for each unit vector $X \in T_xM$, $x \in M$, such that $\eta(X) = 0$. Equality holds for any $x \in M$ and for any unit vector $X \in T_xM$ such that $\eta(X) = 0$, if and only if $(M, \eta, g)$ is Sasakian.*

Generalizing the theorem of Blair [1], Olszak [4] proved that any contact Riemannian manifold of constant curvature $k$ and of dimension $2n+1 \geqq 5$ is a Sasakian manifold of constant curvature $k = 1$. We generalize this by replacing the constancy of sectional curvature by the conditions on the Ricci tensor and the $k$-nullity distribution. Namely, we obtain the following.

THEOREM 5.2. *Let $(M, \eta, g)$ be an Einstein contact Riemannian manifold of dimension $2n+1 \geqq 5$. If $\xi$ belongs to the $k$-nullity distribution, then $k = 1$ and $(M, \eta, g)$ is Sasakian.*

**2. Preliminaries.** Let $(M, \eta, g)$ be a contact Riemannian manifold of dimension $2n+1$. Following Blair [1], we define $h=(h^i_j)$ by $h=(1/2)L_\xi\phi$, where $L_\xi$ denotes the Lie derivation by $\xi$. Then the structure tensors of $(M, \eta, g)$ satisfy the following relations:

$$\eta_r\xi^r = 1\,, \quad \phi^i_r\xi^r = 0\,, \quad \eta_r\phi^r_j = 0\,,$$
$$\phi^i_r\phi^r_j = -\delta^i_j + \xi^i\eta_j\,,$$
$$g_{rs}\phi^r_j\phi^s_k = g_{jk} - \eta_j\eta_k\,, \quad g_{jr}\xi^r = \eta_j\,,$$
$$\nabla_i\eta_j - \nabla_j\eta_i = 2\phi_{ij} = 2g_{ir}\phi^r_j\,,$$
$$\nabla_r\phi^r_j = -2n\eta_j\,, \quad \xi^r\nabla_r\phi^i_j = 0\,, \tag{2.1}$$
$$\nabla_i\eta_j = \phi_{ij} - \phi_{ir}h^r_j\,, \tag{2.2}$$
$$h_{ij} = h_{ji} = g_{jr}h^r_i\,,$$
$$\phi^i_r h^r_j = -h^i_r\phi^r_j\,, \quad h_{ij}\xi^j = 0\,.$$

$h=0$ is equivalent to the condition that $(M, \eta, g)$ is a $K$-contact Riemannian manifold. We prepare some relations which hold on a contact Riemannian manifold. By (2.2) we obtain

$$\nabla_r\eta_i\nabla^r\eta_j = h_{ir}h^r_j - 2h_{ij} + g_{ij} - \eta_i\eta_j\,. \tag{2.3}$$

The next two relations are obtained by Blair [1], [2].

$$R_{irjs}\xi^r\xi^s + R_{arbs}\xi^r\xi^s\phi^a_i\phi^b_j = -2h_{ir}h^r_j + 2g_{ij} - 2\eta_i\eta_j\,, \tag{2.4}$$
$$\mathrm{Ric}(\xi, \xi) = 2n - \|h\|^2\,, \tag{2.5}$$

where $\|T\|^2 = g^{ir}g^{js}T_{ij}T_{rs}$ for $T=(T_{ij})$.

LEMMA 2.1. *The Ricci tensor satisfies the following.*

$$R_{jr}\xi^r = \nabla_r\nabla_j\xi^r = \nabla^r\nabla_r\eta_j + 4n\eta_j\,, \tag{2.6}$$
$$\begin{aligned}\phi^s_j\nabla^r\nabla_r\phi_{ks} + \phi^s_k\nabla^r\nabla_r\phi_{js} &= 2\nabla_r\phi_{sj}\nabla^r\phi^s_k + R_{jr}\xi^r\eta_k + R_{kr}\xi^r\eta_j \\ &\quad + 2h_{jr}h^r_k - 4h_{jk} + 2g_{ij} - 2(4n+1)\eta_j\eta_k\,.\end{aligned} \tag{2.7}$$

PROOF. Contracting $R^i_{rkl}\xi^r = \nabla_k\nabla_l\xi^i - \nabla_l\nabla_k\xi^i$ with respect to $i$ and $k$, we obtain the first equality of (2.6). To verify the second equality we rewrite $\nabla^r\nabla_r\eta_j$ as $\nabla^r\nabla_r\eta_j = \nabla^r(2\phi_{rj}) + \nabla^r\nabla_j\eta_r$. Then, applying (2.1), we get (2.6). Next, operating $\nabla^r\nabla_r$ to $\phi^s_j\phi_{ks} = -g_{jk} + \eta_j\eta_k$, we obtain

$$\phi^s_j\nabla^r\nabla_r\phi_{ks} + \phi^s_k\nabla^r\nabla_r\phi_{js} - 2\nabla_r\phi_{sj}\nabla^r\phi^s_k = \nabla^r\nabla_r\eta_j\eta_k + \eta_j\nabla^r\nabla_r\eta_k + 2\nabla_r\eta_j\nabla^r\eta_k\,.$$

Applying (2.3) and (2.6) to the last equation, we get (2.7). q.e.d.

We define $P=(P_{rsi})$ on a contact Riemannian manifold by

$$P_{rsi} = \nabla_r\phi_{si} - \eta_s g_{ri} + \eta_i g_{rs}\,. \tag{2.8}$$

LEMMA 2.2. *$P_{rsi}P^{rs}{}_j$ is given by*

(2.9) $$P_{rsi}P^{rs}{}_j = \nabla_r\phi_{si}\nabla^r\phi^s_j - 2h_{ij} - g_{ij} - (2n-1)\eta_i\eta_j\ .$$

PROOF. First we get

$$P_{rsi}P^{rs}{}_j = \nabla_r\phi_{si}\nabla^r\phi^s_j - \eta_s\nabla_i\phi^s_j - \eta_s\nabla_j\phi^s_i + g_{ij} - (2n+1)\eta_i\eta_j\ .$$

Since $\eta_s\nabla_i\phi^s_j = -\nabla_i\eta_s\phi^s_j$, applying (2.2) to the last equation, we obtain (2.9). q.e.d.

We define $R^*_{ij}$ by the same way as in the Kählerian case:

$$2R^*_{ij} = -R_{irkl}\phi^r_j\phi^{kl}\ .$$

By the Bianchi identity $R^*_{ij}$ is written also as

$$R^*_{ij} = -R_{ikrl}\phi^r_j\phi^{kl}\ .$$

We define $S^*$ by $S^* = R^*_{ij}g^{ij}$.

LEMMA 2.3. *$R^*_{ij}$ satisfies the following.*

(2.10) $$R^*_{ij} + R^*_{ji} = R_{ij} + R_{rs}\phi^r_i\phi^s_j - 2(2n-1)g_{ij} + 2(n-1)\eta_i\eta_j + P_{rsi}P^{rs}{}_j + h_{ir}h^r_j\ .$$

PROOF. By the Ricci identity for $\phi$, we obtain

$$\nabla_l\nabla_k\phi^i_j - \nabla_k\nabla_l\phi^i_j = -R^i_{skl}\phi^s_j + R^s_{jkl}\phi^i_s\ .$$

Contracting the last equation with respect to $i$ and $k$, we get

(2.11) $$-2n\nabla_l\eta_j - \nabla_i\nabla_l\phi^i_j = -R_{sl}\phi^s_j + R_{sjrl}\phi^{rs}\ .$$

Transvecting (2.11) by $-\phi^l_k$, we obtain

(2.12) $$2n\nabla_l\eta_j\phi^l_k + \phi^l_k\nabla_i\nabla_l\phi^i_j = R_{sl}\phi^s_j\phi^l_k - R^*_{jk}\ .$$

Transvecting (2.11) by $\phi^j_k$, we obtain

(2.13) $$-2n\nabla_l\eta_j\phi^j_k - \phi^j_k\nabla_i\nabla_l\phi^i_j = R_{kl} - R_{ls}\xi^s\eta_k - R^*_{lk}\ .$$

Change $l$ to $j$ in (2.13). Then the result and (2.12) imply

$$4n\phi_{rj}\phi^r_k + \phi^r_k\nabla^i(\nabla_r\phi_{ij} - \nabla_j\phi_{ir}) = R_{jk} + R_{rs}\phi^r_j\phi^s_k - R_{js}\xi^s\eta_k - 2R^*_{jk}\ .$$

Since $\nabla_r\phi_{ij} + \nabla_i\phi_{jr} + \nabla_j\phi_{ri} = 0$, the above is written as

$$4n(g_{jk} - \eta_j\eta_k) - \phi^r_k\nabla^i\nabla_i\phi_{jr} = R_{jk} + R_{rs}\phi^r_j\phi^s_k - R_{js}\xi^s\eta_k - 2R^*_{jk}\ .$$

Taking the symmetric part of the last equation and using (2.7) and (2.9), we obtain (2.10). q.e.d.

We define $P(X)$ for a vector field (or tangent vector) $X$ by $P(X) = (P_{rsi}X^i)$. Then we get $\|P(X)\|^2 = (P_{rsi}P^{rs}{}_jX^iX^j)$. By (2.9) it is easy to verify

(2.14) $$\|P(\xi)\| = \|h\|\ .$$

Therefore, $(M, \eta, g)$ is a $K$-contact Riemannian manifold, if and only if $P(\xi)=0$. A contact Riemannian manifold $(M, \eta, g)$ satisfying $P=0$ is called Sasakian.

By Lemma 2.3 we obtain the following.

PROPOSITION 2.4. *A contact Riemannian manifold $(M, \eta, g)$ is Sasakian, if and only if*

$$R^*_{ij} + R^*_{ji} = R_{ij} + R_{rs}\phi^r_i\phi^s_j - 2(2n-1)g_{ij} + 2(n-1)\eta_i\eta_j \,.$$

REMARK. (2.5) and (2.10) give the Olszak's inequality;

$$S^* - S + 4n^2 = (1/2)(\|\nabla\phi\|^2 - 4n) + \|h\|^2 \geqq 0 \,, \tag{2.15}$$

where $\|P\|^2 = \|\nabla\phi\|^2 - 4n$ (cf.[4]). $S^* - S + 4n^2 = 0$ is a necessary and sufficient condition for $(M, \eta, g)$ to be Sasakian.

## 3. Constant $\phi$-sectional curvature.

By $D$ we denote the contact distribution of a contact Riemannian manifold $(M, \eta, g)$ defined $\eta = 0$. $(M, \eta, g)$ is said to be of constant $\phi$-sectional curvature if at any point $x \in M$ the sectional curvature $K(X, \phi X)$ is independent of the choice of non-zero $X \in D_x$. In this case, the $\phi$-sectional curvature $H$ is a function on $M$.

THEOREM 3.1. *Let $(M, \eta, g)$ be a $(2n+1)$-dimensional contact Riemannian manifold of constant $\phi$-sectional curvature $H$. Then the Ricci curvatures satisfy the following inequality*

$$\mathrm{Ric}(X, X) + \mathrm{Ric}(\phi X, \phi X) \leqq 3n - 1 + (n+1)H \tag{3.1}$$

*for each unit $X \in D_x$, $x \in M$. Equality holds for any point $x \in M$ and for any unit $X \in D_x$, if and only if $(M, \eta, g)$ is Sasakian.*

PROOF. We define $A$ and $B$ by

$$\begin{aligned} A_{ijkl} &= R_{arbs}\phi^r_c\phi^s_d(\delta^a_i - \xi^a\eta_i)(\delta^c_j - \xi^c\eta_j)(\delta^b_k - \xi^b\eta_k)(\delta^d_l - \xi^d\eta_l) \\ &= R_{irks}\phi^r_j\phi^s_l + R_{arbs}\xi^a\xi^b\phi^r_j\phi^s_l\eta_i\eta_k - R_{arks}\xi^a\phi^r_j\phi^s_l\eta_i - R_{irbs}\xi^b\phi^r_j\phi^s_l\eta_k \,, \\ B_{ijkl} &= H(g_{ik} - \eta_i\eta_k)(g_{jl} - \eta_j\eta_l) \,. \end{aligned}$$

Then $K(X, \phi X) = H$ for any non-zero $X \in D_x$ is equivalent to

$$(A_{ijkl} - B_{ijkl})Y^iY^jY^kY^l = 0 \tag{3.2}$$

for any $Y \in T_xM$. Put $Q = A - B$. Then (3.2) is equivalent to

$$\begin{aligned} Q_{ijkl} + Q_{ijlk} + Q_{ikjl} + Q_{iklj} + Q_{ilkj} + Q_{iljk} + Q_{jikl} + Q_{jilk} \\ + Q_{kijl} + Q_{kilj} + Q_{likj} + Q_{lijk} = 0 \,. \end{aligned}$$

Transvecting the last equation by $g^{jl}$, we obtain

$$R_{ik} + R_{rs}\phi_i^r\phi_k^s - R_{aibk}\xi^a\xi^b - R_{arbs}\xi^a\xi^b\phi_i^r\phi_k^s + 3R_{ik}^* + 3R_{ki}^* - R_{ir}\xi^r\eta_k - R_{kr}\xi^r\eta_i$$
$$- 3R_{ri}^*\xi^r\eta_k - 3R_{rk}^*\xi^r\eta_i + R_{rs}\xi^r\xi^s\eta_i\eta_k - 4(n+1)H(g_{ik} - \eta_i\eta_k) = 0 .$$

Let $X \in D_x$ such that $\|X\| = 1$. Transvecting the last equation by $X^iX^k$ and applying (2.4) and (2.10), we obtain

$$4\,\mathrm{Ric}(X, X) + 4\,\mathrm{Ric}(\phi X, \phi X) = 12n - 4 + 4(n+1)H - 3\|P(X)\|^2 - 5\|hX\|^2 . \tag{3.3}$$

Therefore we obtain (3.1). Equality of (3.1) for any $X \in D$ implies $P(X) = 0$ and $hX = 0$ for any $X \in D$. Since $h\xi = 0$, $hX = 0$ for any $X \in D$ implies $h = 0$. Thus, we obtain $P(\xi) = 0$ by (2.14). Therefore, $P(X) = 0$ for any $X \in D$ implies $P = 0$, and $(M, \eta, g)$ is Sasakian.

REMARK. Let $\{e_\alpha, \phi e_\alpha, \xi; 1 \leqq \alpha \leqq n\}$ be an adapted frame of $T_xM$ of a contact Riemannian manifold of constant $\phi$-sectional curvature $H$. Since $\|h\phi X\| = \|\phi hX\| = \|hX\|$, (3.3) gives $\|P(\phi X)\| = \|P(X)\|$. Thus, we obtain $\|P\|^2 = 2\sum_\alpha \|P(e_\alpha)\|^2 + \|h\|^2$ and $\|h\|^2 = 2\sum_\alpha \|he_\alpha\|^2$. Then, by (2.5) and (3.3), the scalar curvature $S$ is given by

$$\begin{aligned} S &= \mathrm{Ric}(\xi, \xi) + \textstyle\sum_\alpha \mathrm{Ric}(e_\alpha, e_\alpha) + \sum_\alpha \mathrm{Ric}(\phi e_\alpha, \phi e_\alpha) \\ &= 3n^2 + n + n(n+1)H - \|h\|^2 - (3/4)\textstyle\sum_\alpha \|P(e_\alpha)\|^2 - (5/4)\sum_\alpha \|he_\alpha\|^2 \\ &= 3n^2 + n + n(n+1)H - (3/8)\|P\|^2 - (5/4)\|h\|^2 \leqq 3n^2 + n + n(n+1)H . \end{aligned}$$

The last inequality is due to Olszak [4].

REMARK. Let $(M, \eta, g)$ be a $K$-contact Riemannian manifold of constant $\phi$-sectional curvature $H$. If $H$ is constant on $M$, then $H$ can be deformed by a $D$-homothetic deformation of the structure tensors. For example, if $H > -3$, then choosing a constant $\theta = (H+3)/4$, we get a $K$-contact Riemannian manifold

$$(M, \phi, (1/\theta)\xi, \theta\eta, \theta g + (\theta^2 - \theta)\eta \otimes \eta)$$

of constant $\phi$-sectional curvature 1 (cf. (2.14) of Tanno [5]).

REMARK. It seems to be an open problem if there exist contact Riemannian manifolds of constant $\phi$-sectional curvature, which are not Sasakian.

## 4. Conformally flat contact Riemannian manifolds.

Let $(M, \eta, g)$ be a conformally flat contact Riemannian manifold. Then the Riemannian curvature tensor $R$ is expressed as

$$\begin{aligned} R^i_{jkl} = (1/(2n-1))(\delta^i_k R_{jl} - \delta^i_l R_{jk} + R^i_k g_{jl} - R^i_l g_{jk}) \\ - (S/2n(2n-1))(\delta^i_k g_{jl} - \delta^i_l g_{jk}) . \end{aligned}$$

Hence, $R^*_{ij}$ is given by

$$R^*_{ij} = (1/(2n-1))(R_{ij} + R_{rs}\phi^r_i\phi^s_j - R_{ir}\xi^r\eta_j) - (S/2n(2n-1))(g_{ij} - \eta_i\eta_j) .$$

Let $X \in D$ such that $\|X\| = 1$. Then

$$R^*_{ij}X^iX^j = (1/(2n-1))(\mathrm{Ric}(X, X) + \mathrm{Ric}(\phi X, \phi X)) - S/2n(2n-1) .$$

On the other hand, (2.10) gives

$$2R^*_{ij}X^iX^j = \mathrm{Ric}(X, X) + \mathrm{Ric}(\phi X, \phi X) - 2n(2n-1) + \|P(X)\|^2 + \|hX\|^2 .$$

Combining the last two equations we obtain

$$\begin{aligned}(4.1)\qquad &(2n-3)(\mathrm{Ric}(X, X) + \mathrm{Ric}(\phi X, \phi X)) \\ &\quad = 2(2n-1)^2 - S/n - (2n-1)(\|P(X)\|^2 + \|hX\|^2) .\end{aligned}$$

Therefore we obtain the following.

THEOREM 4.1. *Let $(M, \eta, g)$ be a conformally flat contact Riemannian manifold of dimension $2n+1 \geqq 5$. Then, for any unit $X \in D$,*

$$(4.2)\qquad \mathrm{Ric}(X, X) + \mathrm{Ric}(\phi X, \phi X) \leqq 4n + [2n(2n+1) - S]/n(2n-3)$$

*holds. Equality holds for any unit $X \in D$, if and only if $(M, \eta, g)$ is Sasakian.*

REMARK. Let $\{e_\alpha, \phi e_\alpha, \xi\}$ be an adapted frame of $T_xM$ of a conformally flat contact Riemannian manifold. Then, using (2.5) and (4.1), we can show that the scalar curvature $S$ is given by

$$S = 2n(2n+1) - ((2n-1)/4(n-1))\|P\|^2 - ((2n-3)/2(n-1))\|h\|^2 \leqq 2n(2n+1) .$$

This is the inequality due to Olszak [4].

**5. $k$-nullity distribution.** Let $k$ be a real number. By $N(k)\colon x \to N_x(k)$ we denote the $k$-nullity distribution of a Riemannian manifold $(M, g)$:

$$N_x(k) = \{Z \in T_xM;\ R(X,Y)Z = k(g(Y, Z)X - g(X, Z)Y),\ X, Y \in T_xM\} .$$

Considering the second theorem of Blair [2] as $k = 0$ case, we prove the following.

PROPOSITION 5.1. *Let $(M, \eta, g)$ be a contact Riemannian manifold. If $\xi$ belongs to the $k$-nullity distribution, then $k \leqq 1$. If $k < 1$, then $(M, \eta, g)$ admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$, defined by the eigenspaces of $h$, where $\lambda = \sqrt{1-k}$.*

PROOF. By $\xi \in N(k)$ we can verify $\mathrm{Ric}(\xi, \xi) = 2nk$. Then, (2.5) implies $k \leqq 1$. Now we suppose $k < 1$. Olszak ([4], p. 250, p. 251) proved that $\xi \in N(k)$ with $k < 1$ implies $h^2 = (k-1)\phi^2$ and

$$\nabla_r \phi_s^i = (g_{rs} + h_{rs})\xi^i - \eta_s(\delta_r^i + h_r^i) . \tag{5.1}$$

Since $h\xi = 0$ and $h$ is symmetric, $h^2 = (k-1)\phi^2$ implies that the restriction $h|D$ of $h$ to the contact distribution $D$ has eigenvalues $\lambda = \sqrt{1-k}$ and $-\lambda$. By $D(\lambda)$ and $D(-\lambda)$ we denote the distributions defined by the eigenspaces of $h$ corresponding to $\lambda$ and $-\lambda$, respectively. By $D(0)$ we denote the distribution defined by $\xi$. Then these three distributions are mutually orthogonal. Let $X \in D(\lambda)$. Then $hX = \lambda X$ and $\phi h = -h\phi$ imply $h(\phi X) = -\lambda(\phi X)$, and hence $\phi X \in D(-\lambda)$. This means that the dimension of $D(\lambda)$ and $D(-\lambda)$ are equal to $n$. We prove that $D(\lambda)$ ($D(-\lambda)$, resp.) is integrable. Let $X, Y \in D(\lambda)$ ($D(-\lambda)$, resp.). Then,

$$\nabla_X \xi = -\phi X - \phi h X = -(1 \pm \lambda)\phi X$$

and $\nabla_Y \xi = -(1 \pm \lambda)\phi Y$. Therefore, $g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X) = 0$ holds. Thus, $d\eta(X, Y) = 0$ and $\eta([X, Y]) = 0$ follow. $X, Y \in D$ and $\xi \in N(k)$ imply $R(X, Y)\xi = 0$. On the other hand,

$$\begin{aligned} 0 &= \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi \\ &= -(1 \pm \lambda)\nabla_X(\phi Y) + (1 \pm \lambda)\nabla_Y(\phi X) + \phi[X, Y] + \phi h[X, Y] \\ &= -(1 \pm \lambda)\{(\nabla_X \phi)Y - (\nabla_Y \phi)X\} \mp \lambda\phi[X, Y] + \phi h[X, Y] . \end{aligned}$$

By (5.1) the first term of the last line vanishes. And so we obtain $\phi h[X, Y] = \pm\lambda\phi[X, Y]$, which together with $\eta([X, Y]) = 0$ implies $[X, Y] \in D(\lambda)$ ($D(-\lambda)$, resp.). q.e.d.

REMARK. (i) In Proposition 5.1, if $k = 0$, then $D(0) + D(-\lambda)$ is also integrable ([2]).

(ii) In a Sasakian manifold, $\xi \in N(1)$ holds.

THEOREM 5.2. *Let $(M, \eta, g)$ be an Einstein contact Riemannian manifold of dimension $2n + 1 \geqq 5$. If $\xi$ belongs to the $k$-nullity distribution, then $k = 1$ and $(M, \eta, g)$ is Sasakian.*

PROOF. By $\xi \in N(k)$ we obtain $\|\nabla\phi\|^2 = 4n(2-k)$ (cf. [4], p. 251) and $\|h\|^2 = 2n(1-k)$. We obtain also $\mathrm{Ric}(\xi, \xi) = 2nk$. Since $(M, g)$ is an Einstein manifold, we get $R_{ij} = 2nkg_{ij}$, and hence $S = 2n(2n+1)k$. Operating $\nabla^j$ to $\xi^i R_{ijkl} = k(\eta_k g_{jl} - \eta_l g_{jk})$, we get

$$\phi^{ji} R_{ijkl} + \xi^i \nabla^j R_{ijkl} = 2k\phi_{lk} . \tag{5.2}$$

By the second Bianchi identity and $R_{ij} = 2nkg_{ij}$, we see that $\nabla^j R_{ijkl}$ vanishes. Hence, transvecting (5.2) by $\phi^{kl}$, we get $S^* = 2nk$. Substituting these values into (2.15), we obtain $4n^2(1-k) = 4n(1-k)$. Since $n \geqq 2$, we get $k = 1$. Therefore, we get $h = 0$ and $\|\nabla\phi\|^2 = 4n$, and $(M, \eta, g)$ is Sasakian. q.e.d.

REMARK. Theorem 5.2 is a generalization of Olszak's theorem [4] that any contact Riemannian manifold of constant curvature $k$ and of dimension $2n+1 \geqq 5$ is a Sasakian manifold of constant curvature $k=1$.

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DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
MEGURO-KU, TOKYO 152
JAPAN