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PIECEWISE LINEAR HOMEOMORPHISMS OF A CIRCLE AND FOLIATIONS

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0. Introduction. $\mathrm{Homeo}_+(S^1)$ denotes the group of all orientation preserving homeomorphisms of the circle S^1. Its universal covering group $\widetilde{\mathrm{Homeo}}_+(S^1)$ is identified with the group of all orientation preserving homeomorphisms of $\boldsymbol{R}$ which commute with the translation by 1. Let $\widetilde{\mathrm{PL}}_+(S^1)$ be the subgroup of $\widetilde{\mathrm{Homeo}}_+(S^1)$ each element of which satisfies the following:

(1) $\tilde{f}$ is a piecewise linear homeomorphism of $\boldsymbol{R}$.

(2) The set of all non-differentiable points of $\tilde{f}$ has no accumulation point in $\boldsymbol{R}$.

We put $\mathrm{PL}_+(S^1)=p(\widetilde{\mathrm{PL}}_+(S^1))$, where the map $p\colon \widetilde{\mathrm{Homeo}}_+(S^1)\to \mathrm{Homeo}_+(S^1)$ is the universal covering projection. For any $a\in\boldsymbol{R}$, the translation $T_a\colon \boldsymbol{R}\to\boldsymbol{R}$ by a belongs to $\widetilde{\mathrm{PL}}_+(S^1)$. Hence the rotation $R_a=p(T_a)$ of S^1 ($a\in\boldsymbol{R}$) belongs to $\mathrm{PL}_+(S^1)$. Namely $SO(2)\subset \mathrm{PL}_+(S^1)$.

For any element $\tilde{f}\in\widetilde{\mathrm{Homeo}}_+(S^1)$, the following invariants were introduced in [E-H-N].

$$\overline{m}(\tilde{f})=\max_{x\in\boldsymbol{R}}(\tilde{f}(x)-x)\,,$$
$$\underline{m}(\tilde{f})=\min_{x\in\boldsymbol{R}}(\tilde{f}(x)-x)\,.$$

We note that $\overline{m}(T_a)=\underline{m}(T_a)=a$ for any $a\in\boldsymbol{R}$. In [E-H-N], the following theorem was proved:

THEOREM (Eisenbud-Hirsch-Neumann). *Let $\tilde{f}$ be an element of $\widetilde{\mathrm{Homeo}}_+(S^1)$. $\tilde{f}$ can be written as a product of k (≥ 1) commutators of elements of $\widetilde{\mathrm{Homeo}}_+(S^1)$ if and only if $\underline{m}(\tilde{f})<2k-1$ and $\overline{m}(\tilde{f})>1-2k$.*

In this paper, we consider the PL-version of this theorem. First we show that the theorem with $\widetilde{\mathrm{Homeo}}_+(S^1)$ simply replaced by $\widetilde{\mathrm{PL}}_+(S^1)$ does not hold. Indeed, we have the following theorem by using a property of the leaf holonomy of a transversely PL-foliation (see §1).

THEOREM 1. *There exists an element $\tilde{f}\in\widetilde{\mathrm{PL}}_+(S^1)$ such that*

(1) $\underline{m}(\tilde{f})<1$ *and* $\overline{m}(\tilde{f})>-1$,

(2) $\tilde{f}$ *is not a commutator in* $\widetilde{\mathrm{PL}}_+(S^1)$.

On the other hand, we can prove the following theorem by using the method in [Min].

THEOREM 2. *Let T_a be a translation of $\boldsymbol{R}$. T_a can be written as a product of $k \geq 1$ commutators of elements of $\widetilde{\mathrm{PL}}_+(S^1)$ if and only if $|a| < 2k-1$.*

We note that the condition $|a| < 2k-1$ is equivalent to the condition "$\underline{m}(T_a) < 2k-1$ and $\overline{m}(T_a) > 1-2k$". Therefore Theorem 2 says that for every translation T_a ($a \in \boldsymbol{R}$), a theorem of Eisenbud-Hirsch-Neumann type holds in $\widetilde{\mathrm{PL}}_+(S^1)$. Applying Theorem 2 to translations by integers, we get the following PL-version of a theorem due to Milnor [Mil] and Wood [Wo]:

THEOREM 3. *Let Σ be an oriented closed surface of genus ≥ 1 and E a circle bundle over Σ with the structural group* $\mathrm{Homeo}_+(S^1)$. *Then the following two conditions are equivalent:*

(1) $|\mathrm{eu}(E)| \leq |\chi(\Sigma)|$, *where $\chi(\Sigma)$ is the Euler characteristic of Σ,*
(2) *E is induced by a representation $\phi\colon \pi_1(\Sigma) \to \mathrm{PL}_+(S^1)$.*

1. Leaf holonomy of codimension-one transversely PL-foliation. Let $\mathscr{F}$ be a codimension-one, transversely PL-foliation on an m-dimensional closed manifold M. That is, there exists a finite family $\{(U_i, \varphi_i)\}_{i=1,\cdots,n}$ which satisfies the following four conditions:

(1) $\bigcup_{i=1}^n U_i = M$.
(2) $\varphi_i\colon (U_i, \mathscr{F} \mid U_i) \to (D^{m-1} \times (a_i, b_i), \{D^{m-1} \times \{y\}\}_{y \in (a_i, b_i)})$ for every $1 \leq i \leq n$, is a foliation-preserving homeomorphism. Here D^{m-1} denotes the compact unit disk of $\boldsymbol{R}^{m-1}$.
(3) If $U_i \cap U_j \neq \varnothing$ $(1 \leq i \leq j \leq n)$, then there exists a simple foliation chart (U, φ) such that $U \supset U_i \cup U_j$. Here a foliation chart (U, φ) is simple if it satisfies the condition (2).
(4) For every coordinate transformation $\varphi_i \circ \varphi_j^{-1} = (f_{ij}, \gamma_{ij})$, there exists an element $g \in \mathrm{PL}(\boldsymbol{R})$ such that $\gamma_{ij} = g$ on the domain of γ_{ij}. Here $\mathrm{PL}(\boldsymbol{R})$ denotes the group of piecewise linear homeomorphism of $\boldsymbol{R}$.

EXAMPLE. Let N be a topological manifold and $\phi\colon \pi_1(N) \to \mathrm{PL}(S^1)$ a homomorphism. $\pi_1(N)$ acts on the universal covering space $\tilde{N}$ and on S^1 through ϕ then on $\tilde{N} \times S^1$. This last action preserves the foliation $\mathscr{F} = \{\tilde{N} \times t\}_{t \in S^1}$ of $\tilde{N} \times S^1$. Then the quotient manifold $N \times_\phi S^1 = \pi_1(N) \backslash (\tilde{N} \times S^1)$ has the foliation $\mathscr{F}_\phi$ induced by $\mathscr{F}$, which is a codimension-one, transversely PL-foliation. $(N \times_\phi S^1, \mathscr{F}_\phi)$ is called a suspension foliation of ϕ.

Let M, $\mathscr{F}$, L and $\{(U_i, \varphi_i)\}_{i=1,\cdots,n}$ be as above and $L \in \mathscr{F}$ a leaf. For every loop in L, the associated holonomy can be written as a composite of γ_{ij}'s. The following proposition plays an important role in the proof of Theorem 1.

PROPOSITION 1.1. *Let M, $\mathscr{F}$, L and $\{(U_i, \varphi_i)\}_{i=1,\cdots,n}$ be as above. Then there exists a compact set K in L which satisfies the following condition: For every loop $\sigma\colon [0, 1] \to L - K$ and every representation $\gamma_\sigma = \gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \cdots \gamma_{i_k i_0}$ of the holonomy associated to the loop σ,*

$\varphi_{i_0}^{\mathrm{tr}}(\sigma(0))$ is a differentiable point of γ_σ. Here $\varphi_i^{\mathrm{tr}}=\pi_i\circ\varphi_i$ and $\pi_i\colon D^{m-1}\times(a_i, b_i)\to(a_i, b_i)$ is the natural projection.

PROOF. For every γ_{ij}, its graph has at most finitely many non-differentiable points, which we denote by

$$(x_1^{ij}, y_1^{ij}), \cdots, (x_{l_{ij}}^{ij}, y_{l_{ij}}^{ij}) .$$

We define a compact set K by

$$K=\Bigg(\bigcup_{\substack{1\le i,j\le n\\ 1\le l\le l_{ij}}}\varphi_i^{-1}(D^{m-1}\times\{y_l^{ij}\})\Bigg)\cup\Bigg(\bigcup_{\substack{1\le i,j\le n\\ 1\le l\le l_{ij}}}\varphi_j^{-1}(D^{m-1}\times\{x_l^{ij}\})\Bigg) .$$

Then $K\cap L$ is the required compact set.

A leaf of a codimension-one foliation is said to be *without one-sided holonomy* if for every loop in the leaf, the associated holonomy germ is either trival or nontrivial on both sides.

PROPOSITION 1.2. *Let M, $\mathscr{F}$, L and $\{(U_i, \varphi_i)\}_{i=1,\cdots,n}$ be as above. Suppose that L is homeomorphic to $N\times\boldsymbol{R}$ for some topological manifold N. For every loop $\sigma\colon[0, 1]\to L$ and every representation $\gamma_\sigma=\gamma_{i_0i_1}\circ\gamma_{i_1i_2}\circ\cdots\gamma_{i_ki_0}$ of the holonomy associated to the loop σ, $\varphi(\sigma(0))$ is a differentiable point of γ_σ. Especially the leaf L is without one-sided holonomy.*

PROOF. Let K be as in Proposition 1.1. Since $L\cong N\times\boldsymbol{R}$, every loop in L is free homotopic to a loop in $L-K$. For any two free homotopic loops in L, the associated holonomies are germinally PL-conjugate to each other. Moreover a differentiability of a PL-map of $\boldsymbol{R}$ at the fixed points is invariant under PL-conjugations. Then Proposition 1.1 completes the proof.

PROOF OF THEOREM 1. Define $\tilde{f}_1, \tilde{g}_1\in\widetilde{\mathrm{PL}}_+(S^1)$ as in Figure 1. Here $-3/8<b<-1/4$, $c=9/16$, the left derivative of $\tilde{f}_1$ at c is not equal to 1 and the right derivative of $\tilde{f}_1$ at c is equal to 1.

By construction, $[\tilde{f}_1, \tilde{g}_1](3/4)=b$. Then we have $T_2\circ[\tilde{f}_1, \tilde{g}_1](3/4)=2+b$. Since $0<T_2\circ[\tilde{f}_1, \tilde{g}_1](3/4)-3/4<1$, we have

$$\underline{m}(T_2\circ[\tilde{f}_1, \tilde{g}_1])<1 \quad\text{and}\quad \overline{m}(T_2\circ[\tilde{f}_1, \tilde{g}_1])>-1 .$$

To prove the theorem, it is enough to show that $T_2\circ[\tilde{f}_1, \tilde{g}_1]$ is not a commutator of $\widetilde{\mathrm{PL}}_+(S^1)$.

Suppose it is a commutator. Then there exist $\tilde{f}_2, \tilde{g}_2\in\widetilde{\mathrm{PL}}_+(S^1)$ such that

$$T_2\circ[\tilde{f}_1, \tilde{g}_1]=[\tilde{g}_2, \tilde{f}_2] .$$

Let Σ be a closed surface of genus 2. The fundamental group $\pi_1(\Sigma)$ is presented as

$$\pi_1(\Sigma)=\langle\alpha_1, \beta_1, \alpha_2, \beta_2\,|\,[\alpha_1, \beta_1]\#[\alpha_2, \beta_2]=1\rangle .$$

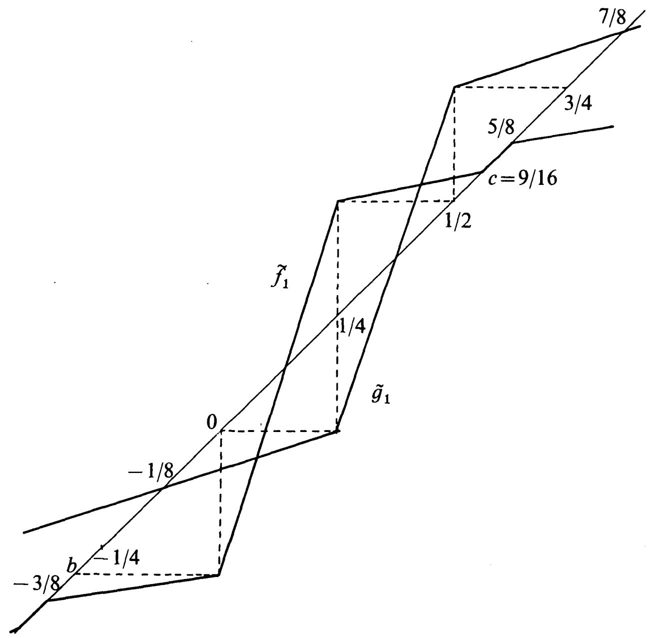

FIGURE 1

Then we can define a homomorphism $\phi: \pi_1(\Sigma) \to \mathrm{PL}(S^1)$ by relations $\phi(\alpha_i)=p(\tilde{f}_i)$, $\phi(\beta_i)=p(\tilde{g}_i)$ $(i=1,2)$. Indeed, the fact $[\tilde{f}_1, \tilde{g}_1][\tilde{f}_2, \tilde{g}_2]=T_{-2}$ guarantees that the map ϕ is a well-defined homomorphism. For the suspension foliation $(E_\phi, \mathscr{F}_\phi)$ of ϕ, E_ϕ is a foliated circle bundle over Σ. The Euler number $\mathrm{eu}(E_\phi)$ of E_ϕ is equal to 2 by the algorithm of Milnor ([Mil, Lemma 2], [Wo, Lemma 2.1]), that is, $\mathrm{eu}(E_\phi)=\chi(\Sigma)$. Since $\mathrm{eu}(E_\phi)=2\neq 0$, $\mathscr{F}_\phi$ has no compact leaf. Then every leaf of $\mathscr{F}_\phi$ is homeomorphic to $\boldsymbol{R}^2$ or $S^1 \times \boldsymbol{R}$ ([Gh, Thm. 3]). We identify a typical fiber of E_ϕ with S^1. By the construction of ϕ, the leaf $L_{p(c)}$ through $p(c)$ is with one-sided holonomy. On the other hand, the leaf $L_{p(c)}$ is without one-sided holonomy by Proposition 1.2, a contradiction. This completes the proof of Theorem 1.

2. Proof of Theorem 2.

PROPOSITION 2.1 ([Min]). *Let Γ be a group and f, g elements of Γ. For every integer $k\geq 1$, $[f, g]^{2k-1}$ can be written as a product of k commutators of Γ.*

PROOF OF THEOREM 2. The "if" part was proved in [E-H-N]. We here prove the "only if" part.

For any real number $x\geq 1$, we define $F_x, G_x: [0, (x+1)^2]\to[0, (x+1)^2]$ by

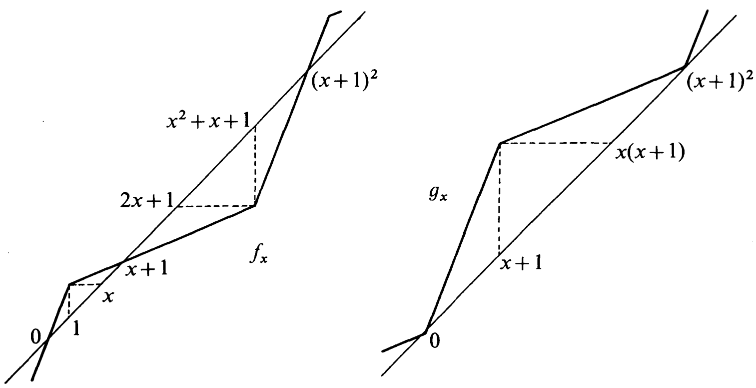

FIGURE 2

$$F_x(y)=\begin{cases} xy & \text{if } y\in[0,1], \\ (y+x^2-1)/x & \text{if } y\in[1,(x+1)^2-1], \\ xy-(x-1)(x+1)^2 & \text{if } y\in[(x+1)^2-1,(x+1)^2]. \end{cases}$$

$$G_x(y)=\begin{cases} xy & \text{if } y\in[0,x+1], \\ ((y-(x+1)^2)/x)+(x+1)^2 & \text{if } y\in[x+1,(x+1)^2]. \end{cases}$$

Using F_x, G_x, we have homeomorphsms $f_x, g_x: \boldsymbol{R}\to\boldsymbol{R}$ which satisfy the following two conditions (see Figure 2):

(1) $f_x|_{[0,(x+1)^2]}=F_x$, $g_x|_{[0,(x+1)^2]}=G_x$,

(2) f_x, g_x commute with $T_{(x+1)^2}$.

By construction, f_x and g_x satisfy the following relation:

$$T_{1-x}\circ g_x\circ f_x\circ g_x^{-1}\circ T_{1-x}^{-1}=T_{(1-x)^2}\circ f_x .$$

By a straightforward calculation, we have

$$[T_{1-x}\circ g_x, f_x]=T_{(1-x)^2} .$$

Taking a conjugation of f_x, g_x, T_{1-x} by the multiplication map $M_{1/(x+1)^2}: \boldsymbol{R}\to\boldsymbol{R}$, $M_{1/(x+1)^2}(y)=y/(x+1)^2$, we have that $T_{\{(1-x)/(1+x)\}^2}$ $(x\geq 1)$ is a commutator of $\widetilde{\mathrm{PL}}_+(S^1)$. This implies that for every real number $b\in R$ $(|b|<1)$, the translation T_b is a commutator of $\widetilde{\mathrm{PL}}_+(S^1)$. If a real number a satisfies $|a|<2k-1$ for some integer $k\geq 1$, then $|a/(2k-1)|<1$. Therefore, the translation $T_a=(T_{a/(2k-1)})^{2k-1}$ can be written as a product of k commutators of $\widetilde{\mathrm{PL}}_+(S^1)$ by Proposition 2.1. This completes the proof of Theorem 2.

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