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NORMAL TWO-DIMENSIONAL HYPERSURFACE TRIPLE POINTS AND THE HORIKAWA TYPE RESOLUTION

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Abstract. Here we solve Durfee's conjecture for 2-dimensional hypersurface singularities of multiplicity 3. We show that the Milnor number is greater than or equal to six times the geometric genus plus two in this case. The equality holds if and only if it is a simple elliptic singularity. For the proof, we consider an analog for triple coverings of Horikawa's canonical resolution for double coverings. We express these invariants in terms of our resolution process and the covering base surface.

Introduction. This paper consists of two parts. In the former (§§1 and 2), we consider an analog for triple coverings of Horikawa's canonical resolution for double coverings [H1, §2]. In the latter (§§3 and 4), we apply our method to normal 2-dimensional hypersurface singularities of multiplicity 3. Especially, we solve Durfee's conjecture [D, p. 97] in this case.

Horikawa introduced a method of resolving singularities of normal surfaces of double section type in the total space of a line bundle over a surface. This method is sometimes useful for global or local study of surfaces (cf. [H1], [H2], [H3], [P], [T1], etc.). We generalize this method to a normal surface S of triple section type in a certain form. The cyclic version of our method is already in [AK, §3].

In §1, we construct some reduction process of singularities on S as follows: First we "transpose" an isolated singularity of multiplicity 3 on S to a codimension-one singularity supported on a line by an easy base change. Next we produce the elementary transformation of the ambient vector bundle along this line, and obtain another surface whose singularities are improved. By applying this process successively finitely many times, we obtain a surface S_r whose singularities are "standardized" in some sense (Theorem 1.9). Namely, S_r has singularities of three types, i.e., relative nodes, relative cusps (these are codimension-one singularities supported on projective lines which are the relativization of curve singularities of ordinary double points and simple cusps, respectively) and some isolated singularities of multiplicity 2.

In §2, we study topological properties of the exceptional sets and triple covering structure which the resolution S^* of S_r naturally possesses. We have formulas for $\chi(\mathcal{O}_{S^*})-\chi(\mathcal{O}_S)$ and $\omega_{S^*}^2-\omega_S^2$ (Proposition 2.4).

In §3, for a germ (V, p) of a normal 2-dimensional hypersurface singularity of

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multiplicity 3, we express the geometric genus $p_g(V,p)$ and the Milnor number $\mu(V,p)$ in terms of our canonical reduction process and the covering base surface (Proposition 3.5). Three examples (Examples 3.6–3.8) are also presented.

§4 is devoted to the "geographical" problem concerning (p_g, μ) which was posed by Durfee [D], who studied the signature of the intersection form of the 2-homology for the Milnor fiber of a 2-dimensional hypersurface singularity. He also conjectured the inequality $\mu \geq 6p_g$, and proved that this inequality implies the negativity of the signature. For this problem, Xu and Yau [XY1] proved $\mu \geq 12p_g - 4$ for weakly elliptic hypersurface singularities, and recently they [XY2] also proved $\mu \geq 6p_g + \nu - 1$ for weighted homogeneous hypersurface singularities of multiplicity ν. Tomari [T2] proved $\mu \geq 8p_g + 1$ for hypersurface singularities of multiplicity 2.

With respect to some special classes of hypersurface singularities, Fukuhara-Matumoto-Sakamoto [FMS] and Neumann-Wahl [NW] proved an equality for the signature σ and the Casson invariant of the link of the singularities, which induce the negativity of σ in this case. Note that Wahl [W, p. 240] showed that some non-complete-intersection singularities have positive signature.

Saito [S2] generalized Durfee's conjecture from the viewpoint of his theory of exponents.

Our result is the following:

THEOREM. *Let (V,p) be a normal 2-dimensional hypersurface singularity of multiplicity 3. Then we have*

$$\mu(V,p) \geq 6p_g(V,p) + 2 .$$

Especially the signature of the Milnor fiber of (V,p) is negative.

Moreover, the equality $\mu(V,p) = 6p_g(V,p) + 2$ holds if and only if (V,p) is a simple elliptic singularity of type $\tilde{E}_6$ in the sense of Saito [S1].

Our proof proceeds as follows: We first express the number $\mu - 6p_g - 2$ as the sum of some numbers d_i $(1 \leq d_i \leq r)$ which are determined by each step of our canonical reduction process of the singularity, and then estimate these numbers d_i.

Note that we have $(p_g, \mu) = (1, 8)$ for $\tilde{E}_6$. So a sharper inequality is likely to exist. Concerning this point, see Remark 4.13.

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1. Canonical reduction by triplet blow-ups. The aim in this section is to improve the singularities of normal surfaces of triple section type by the method analogous to [H1, §2]. The cyclic version of our method is already in [AK, §3].

1.1. Let W be a nonsingular complex analytic surface and L a line bundle on W. We denote by $\pi: \bar{L}=\boldsymbol{P}(\mathcal{O}_W \oplus \mathcal{O}_W(L)) \to W$ the $\boldsymbol{P}^1$-bundle associated with L. We set $T=\mathcal{O}_{\bar{L}}(1)$. Let S be an irreducible reduced divisor on $\bar{L}$ which is linearly equivalent to $3T$. We call (S, W, L) *a triple section surface*.

Let T_∞ be the ∞-section of π. T_∞ is linearly equivalent to $T-\pi^*L$. Let $Y_0 \in |T|$ (the complete linear system of T) and $Y_1 \in |T_\infty|$ be the canonical members associated with the projections to each direct summand of $\mathcal{O}_W \oplus \mathcal{O}_W(L)$. The pair (Y_0, Y_1) induces a system of homogeneous fiber coordinate of π. Then by the isomorphism

$$H^0(\bar{L}, 3T) \simeq H^0(W, \mathrm{Symm}^3(\mathcal{O}_W \oplus \mathcal{O}_W(L)),$$

the equation of S is given by

$$\sum_{i=0}^{3} \phi_{iL} Y_0^{3-i} Y_1^i = 0,$$

where $\phi_{iL} \in H^0(W, \mathcal{O}_W(iL))$ for $0 \le i \le 3$. Now by putting $Z_0 = Y_0 + (1/3)\phi_L Y_1 \in |T|$, $Z_1 = Y_1$, $\psi_{2L} = \phi_{2L} - (1/3)\phi_L^2 \in H^0(W, \mathcal{O}(2L))$ and $\psi_{3L} = \phi_{3L} - (1/3)\phi_{2L}\phi_L + (2/27)\phi_L^3 \in H^0(W, \mathcal{O}(3L))$, the equation of S is given by

$$Z_0^3 + \psi_{2L} Z_0 Z_1^2 + \psi_{3L} Z_1^3 = 0.$$

The divisors $G=(\psi_{2L})$ and $H=(\psi_{3L})$ on W are called the *assistant curves* of S with respect to the fiber coordinate (Z_0, Z_1). We also call the divisor $G+H$ the *assistant divisor* of S. By the irreducibility of S, H is not identically zero, although G may be identically zero. We denote by T_0 the member of $|T|$ which is associated with (Z_0).

We also put $\Delta := 4\psi_{2L}^3 + 27\psi_{3L}^2$, which is the discriminant locus for S. The divisor associated with Δ is linearly equivalent to $6L$.

Since $\hat{\pi} = \pi|_S : S \to W$ is a finite triple covering, the fiber consists of one, two or three distinct points. We easily have the following:

LEMMA-DEFINITION 1.2. *Let P be a point on W. The following two conditions are equivalent, and we call such P a target point on W*:

(1) *If $G \not\equiv 0$, then both G and H pass through P. If $G \equiv 0$, then H passes through P.*

(2) *The fiber $(\hat{\pi})^{-1}(P)$ consists of one point $\bar{P}$ which is the point of intersection of $\pi^{-1}(P)$ and T_0 on $\bar{L}$.*

1.3. Let $\bar{Q}$ be a singular point on S. Set $Q=\pi(\bar{Q})$. Let $\mathfrak{m}_Q$ be the maximal ideal at Q. Then by Miranda [Mir, Lemma 5.1], one of the following conditions is satisfied:

(1) $\psi_{2L} \in \mathfrak{m}_Q$ and $\psi_{3L} \in \mathfrak{m}_Q^2$,

(2) $\psi_{2L} \notin \mathfrak{m}_Q$, $\psi_{3L} \notin \mathfrak{m}_Q$ and $\Delta \in \mathfrak{m}_Q^2$.

If the case (1) occurs, then Q is a target point. We call such $\bar{Q}$ a *target singularity*. Assume that the case (2) occurs. Then $(\hat{\pi})^{-1}(Q)$ consists of two points $\bar{Q}$ and $\bar{Q}'$ such that $\bar{Q}$ is a singularity of multiplicity 2 and $\bar{Q}'$ is a nonsingular point. We call such $\bar{Q}$ an *inner double point.*

The aim of this section is to construct a certain "reduction process" for target singularities.

Let P be a target point on W. If $G\not\equiv 0$, then we put $m=\mathrm{mult}_P(G)$ (the multiplicity of the curve G at P). If $G\equiv 0$, then we let $m=+\infty$. We also put $n=\mathrm{mult}_P(H)$. We set

$$l_1=\min([m/2],[n/3])\,,$$

where $[m/2]$ is the greatest integer not exceeding $m/2$. We call l_1 the *twisting order.*

Now let $\tau_1: W_1\to W$ be the blow-up at P, and set $E_1=\tau^{-1}(P)$. We put

$$L_1=\tau_1^*L\otimes\mathcal{O}_{W_1}(-l_1E_1)\,.$$

Set $\pi_1: \bar{L}_1=P(\mathcal{O}_{W_1}\oplus\mathcal{O}_{W_1}(L_1))\to W_1$. We denote by G' (resp. H') the proper transform of G (resp. H) by τ_1. Set

$$G_1=G'+(m-2l)E_1\,,\quad H_1=H'+(n-3l)E_1\,.$$

Then G_1 (resp. H_1) is linearly equivalent to $2L_1$ (resp. $3L_1$).

1.4. From now, we construct naturally a triple section surface S_1 on $\bar{L}_1$ whose assistant curves are G_1 and H_1, and a birational morphism $\bar{\tau}_1: S_1\to S$ such that the following diagram commutes:

$$\begin{array}{ccc}\bar{L}\supset S & \xleftarrow{\bar{\tau}_1} & S_1\subset\bar{L}_1\\ \downarrow\hat{\pi} & & \downarrow\hat{\pi}_1\\ W & \xleftarrow{\tau_1} & W_1\end{array}$$

where $\hat{\pi}_1$ is the restriction of π_1 to S_1.

First, let $\tau^{(0)}: L^{(0)}\to\bar{L}$ be the blow-up along $\pi^{-1}(P)$. Then $L^{(0)}$ is isomorphic to $\boldsymbol{P}(\mathcal{O}_{W_1}\oplus\mathcal{O}_{W_1}(\tau_1^*L))$ with the diagram

$$\begin{array}{ccc}\bar{L} & \xleftarrow{\tau^{(0)}} & L^{(0)}\\ \downarrow\pi & & \downarrow\pi^{(0)}\\ W & \xleftarrow{\tau_1} & W_1\end{array}$$

cartesian, where $\pi^{(0)}$ is the bundle projection. Let $S^{(0)}$, $T_0^{(0)}$ and $T_\infty^{(0)}$ be the proper transforms of $S^{(0)}$, T_0 and T_∞, respectively. Then $S^{(0)}$ is the triple section surface on $L^{(0)}$ such that τ_1^*G and τ_1^*H are assistant curves of $S^{(0)}$ with respect to the fiber coordinate associated with $(T_0^{(0)}, T_\infty^{(0)})$.

If $l_1=0$, then by putting $S_1=S^{(0)}$, we obtain the desired diagram.

Assume that $l_1>0$. Let Γ be the line on $L^{(0)}$ which is the intersection of $T_0^{(0)}$ with $\mathscr{E}^{(0)}=(\pi^{(0)})^{-1}(E_1)$. We produce the elementary transformation along Γ as follows: Let $\tilde{\tau}^{(0)}: \tilde{L}^{(0)}\to L^{(0)}$ be the blow-up along Γ, and let $\tilde{S}^{(0)}$ and $\tilde{\mathscr{E}}^{(0)}$ be the proper transform of $S^{(0)}$ and $\mathscr{E}^{(0)}$, respectively. The $\boldsymbol{P}^1$-bundle $\tilde{\mathscr{E}}^{(0)}$ is smoothly contractible to a line (see [FN]). Let $\tau^{(00)}: \tilde{L}^{(0)}\to L^{(0)}$ be the contraction. Then $L^{(1)}$ is isomorphic to

$P(\mathcal{O}_{W_1} \oplus \mathcal{O}_{W_1}(\tau_1^* L - E_1))$.

Set $S^{(1)} = \tau^{(00)}(\tilde{S}^{(0)})$. It is easy to see that $\tilde{S}^{(0)}$ does not intersect $\tilde{\mathscr{E}}^{(0)}$. Therefore the biraional map $\tilde{\tau}^{(0)} \circ (\tau^{(00)})^{-1} : L^{(1)} \to L^{(0)}$ induces a birational morphism $\tilde{\tau}^{(1)} : S^{(1)} \to S^{(0)}$.

The local description of $S^{(1)}$ is as follows: Let (x, t) be a local coordinate on W_1 such that E_1, G' and H' are given by $t=0$, $g'(x, t)=0$ and $h'(x, t)=0$. Let ξ be the inhomogeneous fiber coordinate of $\pi^{(0)}$ such that $T_0^{(0)}$ is given by $\xi=0$. With respect to the local coordinate (ξ, t, x) on $L^{(0)}$, the surface $S^{(0)}$ is given by

$$\xi^3 + g'(x, t)t^m \xi + h'(x, t)t^n = 0 .$$

Then, with respect to the local coordinate (ξ', x, t) with $\xi' = \xi/t$ on $L^{(1)}$, the surface $S^{(1)}$ is given by

$$\xi'^3 + g'(x, t)t^{m-2}\xi' + h'(x, t)t^{n-3} = 0 .$$

We set $G^{(1)} = G' + (m-2)E_1$ and $H^{(1)} = H' + (n-3)E_1$, which are linearly equivalent to $2(\tau_1^* L - E_1)$ and $3(\tau_1^* L - E_1)$, respectively. Note that the images by $\tau^{(00)}$ of the proper transforms of $T_0^{(0)}$ and $T_\infty^{(0)}$ by $\tilde{\tau}^{(0)}$ defines a member of $|\mathcal{O}_{L^{(1)}}(1)|$ and an ∞-section on $L^{(1)}$, respectively, so that they determine a system of homogeneous fiber coordinates of $L^{(1)} \to W_1$. Then $S^{(1)}$ is the triple section surface on $L^{(1)}$ whose assistant curves are $G^{(1)}$ and $H^{(1)}$ with respect to such fiber coordinates.

We apply this "elementary transformation" process successively for l_1 times. Then the P^1-bundle $L^{(l_1)}$ obtained is isomorphic to $\bar{L}_1$, and the surface $S_1 := S^{(l_1)}$ obtained is the triple section surface whose assistant curves are G_1 and H_1. Thus by putting $\bar{\tau}_1 = \tilde{\tau}^{(0)} \circ \cdots \circ \tilde{\tau}^{(l_1)}$, we obtain the desired diagram.

Note that the local equation of S_1 with respect to the coordinate (ξ_1, x, t) with $\xi_1 = \xi/t^{l_1}$ on $\bar{L}_1$ is written as

$$\xi_1^3 + g'(x, t)t^{m-2l_1}\xi_1 + h'(x, t)t^{n-3l_1} = 0 .$$

The surface S_1 is isomorphic to the strict transform of $S^{(0)}$ by the blow-up of $L^{(0)}$ along the ideal $(\mathscr{I}_{T_0^{(0)}}, (\mathscr{I}_{\mathscr{E}^{(0)}})^{l_1})$, where $\mathscr{I}_{T_0^{(0)}}$ and $\mathscr{I}_{\mathscr{E}^{(0)}}$ are the defining ideals of $T_0^{(0)}$ and $\mathscr{E}^{(0)}$ on $L^{(0)}$, respectively.

We denote this process by

$$\hat{\tau}_1 = (\bar{\tau}_1, \tau_1) : (S_1, W_1, L_1) \to (S, W, L)$$

and call it the *triplet blow-up* at P. We remark that, even if S is normal, S_1 is not necessarily normal.

REMARK 1.5. In [AK, §3], our process was vague for lack of the use of elementary transformations of ambient projective bundles. Especially, the Lines 1–2 on p. 236 should be corrected as follows:

"$\cdots$ tensoring $\mathcal{O}_{V_1}(-[m_1/3]E_1)$ induces a rational map $\tilde{\mu}_1 : X_1 \to X$. Since S is not contained in the set of points of indeterminacy of $\tilde{\mu}_1$, we get a birational morphism $\mu_1 = \tilde{\mu}|_{S_1} : S_1 \to S$."

1.6. For later use, we define the following: Let C be any irreducible reduced curve on W. If $G\not\equiv 0$, then we denote by a_C the multiplicity of C in the divisor G. If $G\equiv 0$, then we put $a_C=+\infty$. We also denote by b_C the multiplicity of C in the divisor H. We set

$$\boldsymbol{Z}_{(G,H)}(C):=(a_C,\, b_C)$$

and call it the $\boldsymbol{Z}^2$*-weighting with respect to* (G, H). (We often write this as $\boldsymbol{Z}^2(C)$ if there is no confusion.)

If C is not a component of the divisor $G+H$, then $(a_C, b_C)=(0, 0)$. If S is normal, then we have either $a_C=0$ or $0\le b_C\le 1$ for any C.

Next let C_1 be an irreducible reduced curve on W_1. We consider the $\boldsymbol{Z}^2$-weighting of C_1 with respect to (G_1, H_1). If C_1 is the proper transform by τ_1 of a curve C on W, then $\boldsymbol{Z}^2_{(G_1,H_1)}(C_1)$ coincides with $\boldsymbol{Z}^2_{(G,H)}(C)$. If C_1 is the exceptional curve E_1, then we have $\boldsymbol{Z}^2_{(G_1,H_1)}(C_1)=(m-2l_1, n-3l_1)$. Therefore if S is normal, then we have either $0\le a_{C_1}\le 1$ or $0\le b_{C_1}\le 2$ for any C_1.

So we classify the curve C_1 into the following six types according to its $\boldsymbol{Z}^2$-weighting:

(1) type **C** if the $\boldsymbol{Z}^2$-weighting is $(\alpha, 2)$ for $\alpha\ge 2$,
(2) type **N** if the $\boldsymbol{Z}^2$-weighting is $(1, \beta)$ for $\beta\ge 2$,
(3) type **I** if the $\boldsymbol{Z}^2$-weighting is $(\gamma, 1)$ for $\gamma\ge 1$,
(4) type **G** if the $\boldsymbol{Z}^2$-weighting is $(\delta, 0)$ for $\delta\ge 1$,
(5) type **H** if the $\boldsymbol{Z}^2$-weighting is $(0, \varepsilon)$ for $\varepsilon\ge 1$,
(6) type **O** if the $\boldsymbol{Z}^2$-weighting is $(0, 0)$.

We sometimes say "C_1 is of type $\mathbf{C}_\alpha$" instead of "$\boldsymbol{Z}^2$-weighting of C_1 is $(\alpha, 2)$", and so on.

From now on, we are always assuming that a triple section surface is normal, or is obtained by a succession of triplet blow-ups of a normal one.

DEFINITION 1.7. Let (S_i, W_i, L_i) be a triple section surface with its assistant curves (G_i, H_i). Let $P\in W_i$ be a target point. Let $\{C_1, \dots, C_s\}$ (for $s\ge 1$) be the set of irreducible reduced components of the assistant divisor G_i+H_i which pass through P. We say P is *good* if one of the following conditions is satisfied:

(1) One of $C_1, \dots, C_s$ is of type **C**, and the others are all of type **G**.
(2) One of $C_1, \dots, C_s$ is of type **N**, and the others are all of type **H**.
(3) One of $C_1, \dots, C_s$, say C_1, is of type **I** or $\mathbf{H}_1$, and the others are all of type **G**. Moreover, C_1 is nonsingular at P.

A *bad* target point is a target point which is not good.

LEMMA 1.8. *Let P be a target point on W_i as above. Then $\bar{P}$ is nonsingular on S_i if and only if the condition* (3) *in Definition* 1.7 *is satisfied.*

PROOF. Assume that P is a target point on W_i such that $\bar{P}$ is nonsingular. Let C_j be any one of $C_1, \dots, C_s$. Then C_j is neither of type **C** nor of type **N**, for otherwise S_i is singular along $(\pi_i)^{-1}(C_j)$. Since C_j is a component of the assistant divisor, C_j is not of type **O**. So C_j is one of types **G**, **H** and **I**. Moreover, one of $C_1, \dots, C_s$, say C_1, is of type **I** or of type **H**, for otherwise P is not a target point.

Assume that C_1 is of type **I**. If one of $C_2, \dots, C_s$ is of type **I** or of type **H**, then $\bar{P}$ is singular. So any one of $C_2, \dots, C_s$ is of type **G**. Furthermore, C_1 is nonsingular at P, for otherwise $\bar{P}$ is singular on S_i.

Assume that C_1 is of type **H** and any one of $C_2, \dots, C_s$ is of type **H** or of type **G**. Then one of $C_2, \dots, C_s$, say C_2, is of type **G**, for otherwise P is not a target point. Moreover, if one of $C_3, \dots, C_s$ is of type **H**, then $\bar{P}$ is singular. So any one of $C_2, \dots, C_s$ is of type **G**. Furthermore, if $\boldsymbol{Z}^2(C_1)=(0, \varepsilon)$ for $\varepsilon \geq 2$ or C_1 is singular at P, then $\bar{P}$ is singular on S_i. Therefore the condition (3) in Definition 1.7 is satisfied.

The converse is clear. q.e.d.

We note that the number of bad target points on a normal triple section surface (S, W, L) is finite, because it coincides with the number of isolated target singularities on S.

Let $\hat{\tau}_{i+1}: (S_{i+1}, W_{i+1}, L_{i+1}) \to (S_i, W_i, L_i)$ be the triplet blow-up at a bad target point $P \in W_i$. Then the number of bad target points on $E_{i+1}=\tau_{i+1}^{-1}(P)$ is finite. In fact, if S_{i+1} has only isolated singularities along $\pi_{i+1}^{-1}(E_{i+1})$, the assertion is clear. Assume that S_{i+1} is singular along $\pi_{i+1}^{-1}(E_{i+1})$. Then E_{i+1} is of type **C** or **N**. Since the number of points of intersection of E_{i+1} with the proper transform of the assistant curve G_i+H_i by τ_{i+1} is finite, the assertion is also clear.

Now we produce a process to improve singularities of a normal triple section surface (S, W, L) by a succession of triplet blow-ups. We first apply triplet blow-ups at all the bad target points on W. Then the triple section surface $(S_{i_1}, W_{i_1}, L_{i_1})$ obtained has finite bad target points. Next we apply triplet blow-ups at all the bad target points on W_{i_1}, and obtain $(S_{i_2}, W_{i_2}, L_{i_2})$ whose bad target points are finite. We continue this process successively in the same way. Our next claim is the termination of this process.

THEOREM 1.9. *Let (S, W, L) be a triple section surface such that S is normal. Let*

$$(S, W, L) \xleftarrow{\hat{\tau}_1} (S_1, W_1, L_1) \xleftarrow{\hat{\tau}_2} \cdots \xleftarrow{\hat{\tau}_r} (S_r, W_r, L_r) \xleftarrow{\hat{\tau}_{r+1}} \cdots$$

be the reduction process by successive triplet blow-ups at bad target points introduced above. Then this process terminates in finite steps. Namely, there exists r such that (S_r, W_r, L_r) has no bad target points.

PROOF. Step 1. We may assume that, for a sufficiently large number r, the reduced scheme $(G_r+H_r)_{\text{red}}$ of the assistant divisor of (S_r, W_r, L_r) has simple normal crossing at any bad target point on W_r.

Indeed, let P be a bad target point on W_r at which $(G_r+H_r)_{\text{red}}$ does not have simple normal crossing. Let $\hat{\tau}_{r+1}: (S_{r+1}, W_{r+1}, L_{r+1}) \to (S_r, W_r, L_r)$ be the triplet blow-up at P. If there exist bad target points on $E_{r+1}=\tau_{r+1}^{-1}(P)$ at which $(G_{r+1}+H_{r+1})_{\text{red}}$ does not have simple normal crossing, we apply triplet blow-ups at these points and obtain another triple section surface $(S_{r+i}, W_{r+i}, L_{r+i})$ for some i. We continue this process successively in the same way. Then, since the reduced scheme of the total transform of $G_{r+j}+H_{r+j}$ by τ_{r+j+1} coincides with $(G_{r+j+1}+H_{r+j+1})_{\text{red}}$ or $(G_{r+j+1}+H_{r+j+1})_{\text{red}}+E_{r+j+1}$ for any j, there exists s such that $(G_{r+s}+H_{r+s})_{\text{red}}$ of $(S_{r+s}, W_{r+s}, L_{r+s})$ has simple normal crossing at any bad target points on W_{r+s}. So replacing r by $r+s$, the assertion follows.

Step 2. In the situation of Step 1, let P be a bad target point on W_r, and let C and D be the irreducible components of $(G_r+H_r)_{\text{red}}$ which pass through P. We show that, by replacing r by a sufficiently large number if necessary, the $\boldsymbol{Z}^2$-weighting (a_C, b_C) (resp. (a_D, b_D)) of C (resp. D) with respect to (G_r, H_r) satisfy one of the following two conditions:

(A) $0\leq a_C\leq 1, 0\leq a_D\leq 1$ and $(a_C, a_D)\neq(0, 0)$,

(B) $0\leq b_C\leq 2, 0\leq b_D\leq 2$ and $(b_C, b_D)\neq(0, 0)$.

Indeed, suppose that $a_C\geq 2$, $0\leq b_C\leq 2$, $0\leq a_D\leq 1$ and $b_D\geq 3$. Then define

$$\operatorname{diag}(C, D):=a_C+b_D\,,$$

which is not less than 5.

Now let $P(C\cap D)$ be the point of intersection of C and D. Let $\hat{\tau}_{r+1}\colon (S_{r+1}, W_{r+1}, L_{r+1})\to(S_r, W_r, L_r)$ be the triplet blow-up at $P(C\cap D)$, and let E_{r+1} be the exceptional curve for τ_{r+1}. We denote by C' and D' the proper transform by τ_{r+1} of C and D, respectively. We remark that the twisting order l_{r+1} is greater than 0.

Since $C+D$ has normal crossing at P, the reduced scheme of the total transform $C'+E_{r+1}+D'$ by τ_{r+1} has normal crossing along E_{r+1}, and further we have $\boldsymbol{Z}^2_{(G_{r+1},H_{r+1})}(C')=(a_C, b_C)$, $\boldsymbol{Z}^2_{(G_{r+1},H_{r+1})}(E_{r+1})=(a_C+a_D-2l_{r+1}, b_C+b_D-3l_{r+1})$, $\boldsymbol{Z}^2_{(G_{r+1},H_{r+1})}(D')=(a_D, b_D)$. For simplicity, we say in the above situation that the $\boldsymbol{Z}^2$-weighted graph changes as

$$(a_C, b_C)\text{—}(a_D, b_D) \Longleftarrow (a_C, b_C)\text{—}(a_C+a_D-2l_{r+1}, b_C+b_D-3l_{r+1})\text{—}(a_D, b_D)\,.$$

If l_{r+1} coincides with $[(a_C+a_D)/2]$, then we have $0\leq a_C+a_D-2l_{r+1}\leq 1$. For the first pair (C', E_{r+1}), we have

$$\operatorname{diag}(C', E_{r+1})=\operatorname{diag}(C, D)-(3l_{r+1}-b_C)<\operatorname{diag}(C, D)\,,$$

and the second pair (D', E_{r+1}) satisfies the condition (A). If the point $P(C'\cap E_{r+1})$ is a bad target point, then we apply the next triplet blow-up at this point. Otherwise, we stop the precess.

On the other hand, if l_{r+1} coincides with $[(b_C+b_D)/3]$, then we have

$$\operatorname{diag}(E_{r+1}, D')<\operatorname{diag}(C, D)\,,$$

and the pair (C', E_{r+1}) satisfies the condition (B). If $P(D'\cap E_{r+1})$ is a bad target point, we produce the next triplet blow-up at this point.

Then after a finite succession of this process, the "diag" becomes less than 5. We remark that, if $a_C=a_D=0$ or $b_C=b_D=0$, then $P(C\cap D)$ is not a target point. Hence by replacing r by a sufficiently large number, the condition in Step 2 is satisfied.

Sept 3. We show that the above conditions (A) and (B) are reduced to the following conditions (A′) and (B′) by replacing r by a sufficiently large number.

(A′) "$a_C=0$ and $a_D=1$" or "$a_D=0$ and $a_C=1$",

(B′) "$b_C=0$ and $1\le b_D\le 2$" or "$b_D=0$ and $1\le b_C\le 2$".

We first assume the condition (A) in Step 2. Assume that $a_C=a_D=1$. We produce the triplet blow-up τ_{r+1} at $P(C\cap D)$ as in Step 2. We have $l_{r+1}\le 1$. If $l_{r+1}=1$, then the pairs (C', E_{r+1}) and (D', E_{r+1}) satisfy the condition (A′). Assume that $l_{r+1}=0$. Then we may assume that $b_C=b_D=1$. In this case, by applying triplet blow-ups three times in total, our condition is satisfied, i.e., the $\boldsymbol{Z}^2$-weighted graphs change as follows:

$$(1,1)\text{—}(1,1)\Longleftarrow(1,1)\text{—}(2,2)\text{—}(1,1)\Longleftarrow(1,1)\text{—}(1,0)\text{—}(2,2)\text{—}(1,0)\text{—}(1,1)\,.$$

We next assume the condition (B) in Step 1. We may assume $a_C+a_D\ge 2$. (Otherwise the condition (A′) is satisfied.) We need to consider the following cases:

(1) $b_C=b_D=1$, (2) $b_C=1$, $b_D=2$, (3) $b_C=b_D=2$, $a_C+a_D=2$,

(4) $b_C=b_D=2$, $a_C+a_D\ge 3$.

In each case, our condition is satisfied after a finite succession of triplet blow-ups. For example, assume that the case (4) occurs. Then the $\boldsymbol{Z}^2$-weighted graph $(a_C,2)\text{—}(a_D,2)$ changes to

$$(a_C,2)\text{—}(2a_C+a_D-4,0)\text{—}(a_C+a_D-2,1)\text{—}(a_C+2a_D-4,0)\text{—}(a_D,2)$$

after three triplet blow-ups. We omit the proof of the other cases.

Step 4. Under the conditions (A′) or (B′) in Step 3, the point $P(C\cap D)$ is a bad target point if the types of (C, D) is one of the following:

$$(\mathbf{G},\mathbf{H}_2),\ (\mathbf{G}_1,\mathbf{H}_\varepsilon)\quad\text{for}\quad \varepsilon\ge 2\,,\ (\mathbf{I}_1,\mathbf{H}),\ (\mathbf{N}_2,\mathbf{G})\,,$$

where "(C, D) is of type $(\mathbf{G},\mathbf{H}_2)$", for instance, means that one of C and D is of type $\mathbf{G}$ and the other is of type $\mathbf{H}_2$.

However we can reduce these points to good target points and non-target points by a succession of triplet blow-ups.

Indeed, we first assume that (C, D) is of type $(\mathbf{G}_\delta,\mathbf{H}_2)$. If $\delta\ge 4$, then the $\boldsymbol{Z}^2$-weighted graph changes to

$$\mathbf{G}_\delta\text{—}\mathbf{C}_\delta\text{—}\mathbf{G}_{2\delta-4}\text{—}\mathbf{I}_{\delta-2}\text{—}\mathbf{G}_{\delta-4}\text{—}\mathbf{H}_2$$

after a succession of triplet blow-ups four times. By applying this process inductively, we may assume that $1\le\delta\le 3$. In these cases, the $\boldsymbol{Z}^2$-weighted graph changes to the following:

(1) $\delta=1$, $\mathbf{G}_1\text{—}\mathbf{C}_2\text{—}\mathbf{G}_1\text{—}\mathbf{I}_1\text{—}\mathbf{O}\text{—}\mathbf{N}_2\text{—}\mathbf{H}_2$,

(2) $\delta=2$, $\mathbf{G}_2\text{—}\mathbf{C}_2\text{—}\mathbf{O}\text{—}\mathbf{H}_1\text{—}\mathbf{H}_2$,

(3) $\delta=3$, $\mathbf{G}_3\text{—}\mathbf{C}_3\text{—}\mathbf{G}_2\text{—}\mathbf{I}_1\text{—}\mathbf{O}\text{—}\mathbf{N}_2\text{—}\mathbf{H}_1\text{—}\mathbf{N}_3\text{—}\mathbf{H}_2$.

Any point of intersection of two components of the above graph is a good target point or is not a target point. Thus the case $(\mathbf{G}, \mathbf{H}_2)$ is settled.

Assume that (C, D) is of type $(\mathbf{G}_1, \mathbf{H}_\varepsilon)$ for $\varepsilon \geq 2$. If $\varepsilon \geq 4$, then after two triplet blow-ups, the Z^2-weighted graph changes to $\mathbf{G}_1$—$\mathbf{H}_{\varepsilon-3}$—$\mathbf{N}_\varepsilon$—$\mathbf{H}_\varepsilon$. By applying this process inductively, we may assume that $\varepsilon \leq 3$. If $\varepsilon = 3$, then after two triplet blow-ups, the Z^2-weighted graph changes to $\mathbf{G}_1$—$\mathbf{O}$—$\mathbf{N}_3$—$\mathbf{H}_3$. If $\varepsilon = 2$, then after five triplet blow-ups, the Z^2-weighted graph changes to $\mathbf{G}_1$—$\mathbf{C}_2$—$\mathbf{G}_1$—$\mathbf{I}_1$—$\mathbf{O}$—$\mathbf{N}_2$—$\mathbf{H}_2$.

Assume that (C, D) is of type $(\mathbf{I}_1, \mathbf{H}_\varepsilon)$. If $\varepsilon \geq 2$, then after two triplet blow-ups, the Z^2-weighted graph changes to $\mathbf{I}_1$—$\mathbf{H}_{\varepsilon-1}$—$\mathbf{N}_{\varepsilon+1}$—$\mathbf{H}_\varepsilon$. So we may assume that $\varepsilon = 1$. In this case, after two triplet blow-ups, the Z^2-weighted graph changes to $\mathbf{I}_1$—$\mathbf{O}$—$\mathbf{N}_2$—$\mathbf{H}_1$.

Assume that (C, D) is of type $(\mathbf{N}_2, \mathbf{G}_\delta)$. After four triplet blow-ups, the Z^2-weighted graph changes to $\mathbf{N}_2$—$\mathbf{G}_{\delta-1}$—$\mathbf{I}_\delta$—$\mathbf{G}_{2\delta-1}$—$\mathbf{C}_{\delta+1}$—$\mathbf{G}_\delta$. Therefore after triplet blow-ups 4δ times in total, our assertion is satisfied.

This completes the proof of Theorem 1.9.

The triple section surface (S_r, W_r, L_r) which enjoy the condition in Theorem 1.9 for the least number r is clearly unique, although one has many choice for the order of triplet blow-ups to obtain S_r. For such r, we set $\bar{\tau} = \bar{\tau}_r \circ \cdots \circ \bar{\tau}_1$, $\tau = \tau_r \circ \cdots \circ \tau_1$, $\pi_r \colon \bar{L}_r \to W_r$ and $\hat{\pi}_r = \pi_r|_{S_r}$. We obtain a commutative diagram:

$$\begin{array}{ccc} \bar{L} \supset S & \xleftarrow{\bar{\tau}} & S_r \subset \bar{L}_r \\ \downarrow \hat{\pi} & & \downarrow \hat{\pi}_r \\ W & \xleftarrow{\tau} & W_r \end{array}$$

We simply denote this diagram by $\hat{\tau} = (\bar{\tau}, \tau) \colon (S_r, W_r, L_r) \to (S, W, L)$, and call it the *canonical reduction*.

The surface S_r is not necessarily normal, but the multiplicities of the singularities on S_r are at most 2. We will discuss this point in detail in the next section.

The following lemma is proved by the same argument as in [AK, Proposition 3.6]:

LEMMA 1.10. *Let (S_r, W_r, L_r) be the canonical reduction of (S, W, L). Let l_i be the twisting order at the i-th step of this process. Then we have*

$$\chi(\mathcal{O}_{S_r}) - \chi(\mathcal{O}_S) = -\frac{1}{2} \sum_{i=1}^{r} l_i(5l_i - 3)\,,$$

$$\omega_{S_r}^2 - \omega_S^2 = -3 \sum_{i=1}^{r} (2l_i - 1)^2\,,$$

where ω_S is the dualizing sheaf of S.

2. Singularities on S_r. Let $\hat{\tau} = (\bar{\tau}, \tau) \colon (S_r, W_r, L_r) \to (S_0, W_0, L_0) = (S, W, L)$ be the canonical reduction of a triple section surface (S, W, L) such that S is normal. The aim of this section is to study the singularities on S_r and the topological properties of the exceptional set of the resolution of S_r.

2.1. Let $\boldsymbol{E}$ be the exceptional set on W_r with respect to τ, and let $\mathscr{E}$ be an irreducible component of $\boldsymbol{E}$. We denote by $\mathscr{E}^*$ the set-theoretic pull-back of $\mathscr{E}$ by $\hat{\pi}_r$.

(1) Assume that $\mathscr{E}$ is of type **C**. Let P be any point on $\mathscr{E}$. Then $(\hat{\pi}_r)^{-1}(P)$ consists of one point $\bar{P}=\pi_r^{-1}(P)\cap T_0$. We choose a local parameter (ξ, x, t) at $\bar{P}$ of $\bar{L}_r$ as follows: (x, t) gives a local parameter at P of W_r such that $\mathscr{E}$ is defined by $t=0$, and ξ is an inhomogeneous fiber coordinate of π_r such that T_0 is defined by $\xi=0$. Further, we can choose a local parameter so that the local equation of S_r at $\bar{P}$ is written as

$$\xi^3+t^\alpha f(x, t)\xi+t^2=0\,.$$

Let $\bar{\sigma}\colon M\to\bar{L}_r$ be the blow-up with the center $\mathscr{E}^*$. Let S' be the proper transform of S_r, and $\sigma\colon S'\to S_r$ the natural morphism. The surface S' is nonsingular along $\sigma^{-1}(\mathscr{E}^*)$. We call this singularity a *relative cusp*. We consider the triple covering $\pi'=\hat{\pi}_r\circ\sigma\colon S'\to W_r$. Let $\bar{\mathscr{E}}$ be the set-theoretic pull-back of $\mathscr{E}$ by π'. Since the scheme-theoretic pull-back $\pi'^*\mathscr{E}$ coincides with $3\bar{\mathscr{E}}$, it follows that

$$\mathscr{E}^2=\mathscr{E}\cdot\pi'_*\bar{\mathscr{E}}=\pi'^*\mathscr{E}\cdot\bar{\mathscr{E}}=3\bar{\mathscr{E}}^2\,.$$

Moreover by the same argument as in [AK, Proposition 3.6], we have

$$\chi(\mathcal{O}_{S'})-\chi(\mathcal{O}_S)=1-\frac{1}{3}\mathscr{E}^2\,,\qquad \omega_{S'}^2-\omega_S^2=8\,.$$

(2) Assume that $\mathscr{E}$ is of type **N**. In the same way as in (1), the local equation at any point on $\mathscr{E}^*$ is written as

$$\xi^3+t\xi+t^\beta f(x, t)=0$$

for $\beta\geq 2$. Let $\bar{\sigma}\colon M\to\bar{L}_r$ be the blow-up with center $\mathscr{E}^*$. Let S' be the proper transform of S by $\bar{\sigma}$, and $\sigma\colon S'\to S$ the natural morphism. The surface S' is nonsingular along $\sigma^{-1}(\mathscr{E}^*)$. We call this singularity a *relative node*. The curve $\sigma^{-1}(\mathscr{E}^*)$ consists of two disjoint nonsingular rational curves N' and N''. Set $\pi'=\pi_r\circ\sigma\colon S'\to W_r$. Around one of N' and N'', the morphism π' is locally a double covering. When N'' is the curve, we have $\pi'^*\mathscr{E}=N'+2N''$. Thus similarly as in 2.1, (1), we have

$$(N')^2=\mathscr{E}^2\,,\qquad (N'')^2=\frac{1}{2}\mathscr{E}^2\,.$$

Moreover, we have the following:

$$\chi(\mathcal{O}_{S'})-\chi(\mathcal{O}_{S_r})=1\,,\qquad \omega_{S'}^2-\omega_{S_r}^2=8+\frac{3}{2}\mathscr{E}^2\,.$$

Indeed, let $\mathscr{D}\subset M$ be the exceptional divisor for $\bar{\sigma}$. We have $S'\sim\bar{\sigma}^*S_r-2\mathscr{D}$. From the exact sequences

$$0 \to \mathcal{O}_{2\mathscr{D}}(-S') \to \mathcal{O}_{\bar{\sigma}^*S_r} \to \mathcal{O}_{S'} \to 0\ ,$$

$$0 \to \mathcal{O}_{\mathscr{D}}(-S'-\mathscr{D}) \to \mathcal{O}_{2\mathscr{D}}(-S') \to \mathcal{O}_{\mathscr{D}}(-S') \to 0\ ,$$

we have $\chi(\mathcal{O}_{S'})-\chi(\mathcal{O}_{\bar{\sigma}^*S_r})=-\chi(\mathcal{O}_{\mathscr{D}}(-S'-\mathscr{D}))-\chi(\mathcal{O}_{\mathscr{D}}(-S'))$. On the other hand, we have

$$\chi(\mathcal{O}_{\bar{\sigma}^*S_r})=\chi(\mathcal{O}_M)-\chi(\mathcal{O}_M(-\bar{\sigma}^*S_r))=\chi(\mathcal{O}_{\bar{L}_r})-\chi(\mathcal{O}_{\bar{L}_r}(-S_r))=\chi(\mathcal{O}_{S_r})\ .$$

Moreover, since $-S'-\mathscr{D}$ is linearly equivalent to $\mathscr{D}-\bar{\sigma}^*S_r$ and $\mathscr{D}$ is an exceptional divisor which can be identified with a $\boldsymbol{P}^1$-bundle over $\bar{\mathscr{E}}$, we have $\chi(\mathcal{O}_{\mathscr{D}}(-S'-\mathscr{D}))=0$. Note that we have $\mathcal{O}_{\mathscr{D}}(-S')\simeq\mathcal{O}_{\mathscr{D}}(-N'-N'')$. Therefore from the exact sequences

$$0 \to \mathcal{O}_{\mathscr{D}}(-N'-N'') \to \mathcal{O}_{\mathscr{D}} \to \mathcal{O}_{N'+N''} \to 0\ ,$$

$$0 \to \mathcal{O}_{N'}(-N'') \to \mathcal{O}_{N'+N''} \to \mathcal{O}_{N''} \to 0\ ,$$

we get $\chi(\mathcal{O}_{\mathscr{D}}(-S'))=-\chi(\mathcal{O}_{N'}(-N''))$. Since N' and N'' are disjoint, we have $\chi(\mathcal{O}_{N'})=\chi(\mathcal{O}_{N'}(-N''))=1$. Therefore we get the first assertion.

On the other hand, the sheaf $\mathcal{O}_{S'}(\sigma^*\omega_{S_r})$ is isomorphic to $\mathcal{O}_{S'}(\omega+N'+N'')$. Thus by the virtual genus formula, we get

$$\omega_{S_r}^2=(\sigma^*\omega_{S_r})^2=\omega_{S'}^2-8-(N')^2-(N'')^2\ ,$$

which proves the second assertion.

Lemma 2.2. *If the point* $\bar{Q}$ *is an inner double point* (*in the sense of* 1.3) *of* (S_r, W_r, L_r), *then* $\bar{Q}$ *is an isolated singularity.*

Proof. We may assume that $Q=\pi_r(\bar{Q})$ lies on an exceptional set $\mathscr{E}$ for τ of type **O**, for otherwise the assertion is clear, because S is normal. Then $\mathscr{E}$ is the proper transform by $\tau_{i+1}\circ\cdots\circ\tau_r$ of the exceptional curve $E_i=\tau_i^{-1}(P_{i-1})$ for some i $(0\le i\le r)$, where P_{i-1} is the center of the blow-up τ_i.

We have $m_i=2l_i$ and $n_i=3l_i$, where $m_i=\mathrm{mult}_{P_{i-1}}(G_{i-1})$ and $n_i=\mathrm{mult}_{P_{i-1}}(H_{i-1})$. Moreover G_i and H_i coincide with the proper transforms of G_{i-1} and H_{i-1} by τ_i, respectively. Thus the discriminant divisor Δ_i for π_i coincides with the proper transform by τ_i of the discriminant divisor Δ_{i-1} for π_{i-1}. Especially, the exceptional curve E_i is not a component of the divisor Δ_i. Therefore the generic point of $(\hat{\pi}_r)^{-1}(\mathscr{E})$ is nonsingular.

Note that none of the components of type **C** and type **N** intersect $\mathscr{E}$ at Q. Hence the assertion is clear. q.e.d.

The following lemma is now clear.

Lemma 2.3. *There is no singularity on* S_r *except relative cusps, relative nodes and isolated inner double points.*

To resolve singularities on S_r, we first blow-up all the locus of relative cusps and relative nodes. We then resolve all inner double points in such a way that the reduced scheme of the total transform of $\hat{\pi}_r^*E$ has simple normal crossings. Let $\bar{\rho}\colon S^*\to S$ be such a resolution. Note that S^* is not necessarily minimal even if S is minimal.

We denote by the symbol "Inn" the set of inner double points on S_r. Denote by $\{C_i \mid 1 \le i \le n(\mathbf{C})\}$ (resp. $\{N_i \mid 1 \le i \le n(\mathbf{N})\}$) the set of the reduced curves on W_r of type $\mathbf{C}$ (resp. type $\mathbf{N}$). From the previous argument and by Lemma 1.10, we have:

PROPOSITION 2.4.

$$\chi(\mathcal{O}_{S^*}) - \chi(\mathcal{O}_S) = -\frac{1}{2}\sum_{i=1}^{r} l_i(5l_i - 3) + n(\mathbf{C}) + n(\mathbf{N}) - \frac{1}{3}\sum_{i=1}^{n(\mathbf{C})} C_i^2 - \sum_{\bar{Q} \in \mathrm{Inn}} p_g(\bar{Q}),$$

$$\omega_{S^*}^2 - \omega_S^2 = -3\sum_{i=1}^{r}(2l_i - 1)^2 + 8n(\mathbf{C}) + 8n(\mathbf{N}) + \frac{3}{2}\sum_{i=1}^{n(\mathbf{N})} N_i^2 + \sum_{\bar{Q} \in \mathrm{Inn}} (Z_{\bar{Q}})^2,$$

where $p_g(\bar{Q})$ is the geometric genus of the singularity $\bar{Q}$ and $Z_{\bar{Q}}$ is a certain divisor on S^* supported on $(\bar{\rho})^{-1}(\bar{Q})$.

REMARK 2.5. With respect to isolated inner double points on a triple section surface, one can also construct a reduction process similar to §1. Since one chooses a member of $|T|$ which passes through this singular point, one should not use Tschirnhausen transformation as in §1. However this process is not needed for the main purpose of this paper.

2.6. For later use, we introduce some notation: Let (S_i, W_i, L_i) be the triple section surface appearing for the i-th step $(0 \le i \le r)$ of our reduction process. Let C be an irreducible reduced curve on W_i. For instance, a point P on C is said to be of type $P(C \cap \mathbf{C})$ if another component of type $\mathbf{C}$ intersects C at P. Denote by $n(C \cap \mathbf{C})$ the cardinality of the points of type $P(C \cap \mathbf{C})$ on C. We use similar notation for the other types $\mathbf{N}, \mathbf{I}, \ldots$, etc.

Assume that C is of type $\mathbf{O}$. Let $\{C_j \mid 1 \le j \le n(\mathbf{C})\}$, $\{N_k \mid 1 \le k \le n(\mathbf{N})\}$ and $\{I_s \mid 1 \le s \le n(\mathbf{I})\}$ be the set of irreducible reduced curves on W_i of types $\mathbf{C}$, $\mathbf{N}$ and $\mathbf{I}$, respectively. Then the discriminant divisor Δ_i decomposes as

$$\Delta_i = \sum_{j=1}^{n(\mathbf{C})} 4C_j + \sum_{k=1}^{n(\mathbf{N})} 3N_k + \sum_{s=1}^{n(\mathbf{I})} 2I_s + \Delta_i'$$

where Δ_i' is an effective divisor. Let Q be a point at which Δ_i' and C intersect each other. Then $(\pi_r)^{-1}(Q)$ consists of two points Q' and $\bar{Q}$ such that

(a) π_r is a local isomorphism around Q', and

(b) the other point $\bar{Q}$ is either a nonsingular point or an inner double point.

We say Q is of type $P(C \cap \Delta_i'')$ or $P(C \cap \mathrm{Inn})$ according as $\bar{Q}$ is a nonsingular point or an inner double point, respectively. We set $n(C \cap \Delta_i'')$ (resp. $n(C \cap \mathrm{Inn})$) to be the cardinality of the points of type $P(\mathscr{E} \cap \Delta_i'')$ (resp. $P(C \cap \mathrm{Inn})$).

2.7. Now we set $\rho = \pi_r \circ \bar{\rho}\colon S^* \to W_r$, which is a triple covering. Let $\bar{\mathscr{E}}$ be the set-theoretic pull-back of $\mathscr{E}$. We calculate the topological Euler number $e(\bar{\mathscr{E}})$ of $\bar{\mathscr{E}}$.

(1) If $\mathscr{E}$ is of type $\mathbf{I}$, then $\bar{\mathscr{E}}$ is a nonsingular rational curve, i.e., $e(\bar{\mathscr{E}}) = 2$.

(2) Assume that $\mathscr{E}$ is of type **G**. If P is a point of type **C** or **I**, then the fiber $\rho^{-1}(P)$ consists of one point. Otherwise it consists of three points. We apply the Hurwitz formula (in the irreducible case) and the Mayer-Vietoris exact sequence (in the reducible case) to $\bar{\mathscr{E}}$. Then we have

$$e(\bar{\mathscr{E}})=6-2n(\mathscr{E}\cap \mathbf{C})-2n(\mathscr{E}\cap \mathbf{I})\,.$$

(3) Assume that $\mathscr{E}$ is of type **H**. If P is a point of type $P(\mathscr{E}\cap \mathbf{N})$, then $\rho^{-1}(P)$ consists of two points. Otherwise it consists of three points. It follows that

$$e(\mathscr{E})=6-n(\mathscr{E}\cap N)\,.$$

(4) Assume that $\mathscr{E}$ is of type **O**. Then $\mathscr{E}$ decomposes as

$$\bar{\mathscr{E}}=\bar{\mathscr{E}}'+\sum E_{\bar{Q}}\,,$$

where $E_{\bar{Q}}$ is the exceptional set for the inner double point $\bar{Q}$ on $\mathscr{E}$.

The fiber of $\rho'=\rho|_{\bar{\mathscr{E}}'}\colon \bar{\mathscr{E}}'\to\mathscr{E}$ over P consists of one, two, one and two points according as P is a point of type $P(\mathscr{E}\cap \mathbf{C})$, $P(\mathscr{E}\cap \mathbf{N})$, $P(\mathscr{E}\cap \mathbf{I})$ and $P(\mathscr{E}\cap \Delta_r'')$, respectively. If Q is a point of type $P(\mathscr{E}\cap \mathrm{Inn})$, then let ν be the number of points in the fiber $\rho^{-1}(Q)$, which is equal to 2 or 3. We have

$$e(\bar{\mathscr{E}}')=6-2n(\mathscr{E}\cap \mathbf{C})-n(\mathscr{E}\cap \mathbf{N})-2n(\mathscr{E}\cap \mathbf{I})-n(\mathscr{E}\cap \Delta_r'')-\sum_Q(\nu(Q)-1)\,,$$

where Q runs through the points of type $P(\mathscr{E}\cap \mathrm{Inn})$. From the Mayer-Vietoris exact sequence, it follows that

$$\begin{aligned}e(\bar{\mathscr{E}})&=e(\bar{\mathscr{E}}')+\sum_Q e(E_{\bar{Q}})-\sum_Q(3-\nu(Q))\\&=6-2n(\mathscr{E}\cap \mathbf{C})-n(\mathscr{E}\cap \mathbf{N})-2n(\mathscr{E}\cap \mathbf{I})-n(\mathscr{E}\cap \Delta_r'')+\sum_Q(e(E_{\bar{Q}})-2)\,.\end{aligned}$$

3. Local invariants and examples. In this section, we mainly calculate the Milnor number of normal 2-dimensional hypersurface singularities of multiplicity 3 by our method, and give some examples.

Let (V,p) be a germ of a hypersurface 2-dimensional analytic space V with an isolated singularity p. Assume that the multiplicity of (V,p) is 3. Let $\mathscr{F}=0$ be a defining equation of (V,p) at the origin of the complex affine 3-space. Then we have the following:

LEMMA 3.1. *There exists a triple section surface* (S, W, L) *and a target singularity* $\bar{P}_0$ *on* S *such that the local analytic equation of* S *at* $\bar{P}_0$ *coincides with* $\mathscr{F}$ (*in the completion of the local ring at* $\bar{P}_0$ *with respect to its maximal ideal*).

PROOF. By the Weierstrass preparation theorem and the Tschirnhausen transformation, $\mathscr{F}$ is written as

$$\mathscr{F}=\mathscr{Z}^3+\mathscr{G}(\mathscr{X},\mathscr{Y})\mathscr{Z}+\mathscr{H}(\mathscr{X},\mathscr{Y}).$$

Moreover, we may assume that $\mathscr{G}(\mathscr{X},\mathscr{Y})$ and $\mathscr{H}(\mathscr{X},\mathscr{Y})$ are polynomials by the well-known argument. (See, e.g., [Mil, p. 89].)

Now we set $W=\boldsymbol{P}^2$, and fix a point P_0 on W. Let $\hat{g}(x, y)=0$ (resp. $\hat{h}(x, y)=0$) be the plane curve of degree d_1 (resp. degree d_2) such that the local equation at P_0 coincides with $\mathscr{G}(\mathscr{X},\mathscr{Y})=0$ (resp. $\mathscr{H}(\mathscr{X},\mathscr{Y})=0$). We choose nonnegative integers (d, s_1, s_2) with $2d=d_1+s_1$ and $3d=d_2+s_2$. Let $H_1, \dots, H_{s_1}, H_1', \dots, H_{s_2}'$ be general hyperplanes on W which do not pass through P_0, and set

$$G=(\hat{g})+H_1+\cdots+H_{s_1}, \qquad H=(\hat{h})+H_1'+\cdots+H_{s_2}'.$$

The divisors G and H are linearly equivalent to $\mathcal{O}_{\boldsymbol{P}^2}(2d)$ and $\mathcal{O}_{\boldsymbol{P}^2}(3d)$, respectively.

Let $L=\mathcal{O}_{\boldsymbol{P}^2}(d)$ and let $\pi: \bar{L}\to W$ be the associated $\boldsymbol{P}^1$-bundle. Let (S, W, L) be the triple section surface such that G and H are its assistant curves with respect to some homogeneous fiber coordinate (Z_0, Z_1) of π. Then the point $\bar{P}_0=\{(Z_0=0)\cap\pi^{-1}(P_0)\}$ satisfies the desired property. q.e.d.

3.2. Now we first apply the triplet blow-up $\hat{\tau}_1=(\bar{\tau}_1, \tau_1): (S_1, W_1, L_1)\to(S, W, L)$ at P_0 as in Lemma 3.1. Next we apply the triplet blow-up $\hat{\tau}_2: (S_2, W_2, L_2)\to(S_1, W_1, L_1)$ at one of the bad target points on $E_1=\tau_1^{-1}(P_0)$. In this way, we continue to apply triplet blow-ups only at the points which are infinitely near to P_0. Let $\hat{\tau}=(\bar{\tau}, \tau): (S_r, W_r, L_r)\to(S, W, L)$ be the composite of these triplet blow-ups so that there is no bad target point on W_r which is infinitely near to P_0, and we take r to be the least number which enjoys this property. We call (S_r, W_r, L_r) the *canonical reduction for* P_0.

Let $\boldsymbol{D}$ be the total transform by τ of the assistant divisor $G+H$ for S. We decompose $\boldsymbol{D}$ as

$$\boldsymbol{D}=\boldsymbol{E}+\boldsymbol{B},$$

where $\boldsymbol{E}$ is the exceptional set for τ and $\boldsymbol{B}$ is the proper transform of $G+H$ by τ.

Let $\bar{\rho}: S^*\to S$ be the resolution of singularities on the locus $\tau^{-1}(P_0)$ of W_r as in §2, i.e., the reduced scheme $\bar{\boldsymbol{E}}$ of the pull back $\rho^*\boldsymbol{E}$ for $\rho=\pi_r\circ\bar{\rho}: S^*\to W$ has simple normal crossings. $\bar{\boldsymbol{E}}$ is the exceptional set for $\bar{P}_0$ by our resolution.

Next we calculate the topological Euler number $e(\bar{\boldsymbol{E}})$ of $\bar{\boldsymbol{E}}$. Let us denote, for instance, by $n(\mathbf{G})$ (resp. $n(\mathbf{G}_{\boldsymbol{E}})$, resp. $n(\mathbf{G}_{\boldsymbol{B}})$) the cardinality of the set of irreducible reduced curves of type $\mathbf{G}$ in the components of $\boldsymbol{D}$ (resp. $\boldsymbol{E}$, resp. $\boldsymbol{B}$), and so on. We denote, for instance, by $n(\mathbf{G}_{\boldsymbol{E}}\cap\mathbf{I}_{\boldsymbol{B}})$ the cardinality of the set of points of intersection of curves of type $\mathbf{G}$ in $\boldsymbol{E}$ and curves of type $\mathbf{I}$ in $\boldsymbol{B}$, and so on. We also denote by $n(\mathbf{G}_{\boldsymbol{E}}\cap\mathbf{G}_{\boldsymbol{E}})$ the cardinality of the set of points of intersection of mutually distinct curves of type $\mathbf{G}$ in $\boldsymbol{E}$.

LEMMA 3.3.

$$e(\bar{\boldsymbol{E}}) = -n(\mathbf{C}) + n(\mathbf{N}) - n(\mathbf{I}_E) + 3n(\mathbf{G}_E) + 3n(\mathbf{H}_E) + 3n(\mathbf{O}_E) + 3 - 2n(\mathbf{G}_E \cap \mathbf{I}_B)$$
$$-2n(\mathbf{O}_E \cap \mathbf{I}_B) - n(\mathbf{O}_E \cap \Delta_r'') + \sum_{\bar{Q} \in \mathrm{Inn}} (e(E_{\bar{Q}}) - 2) .$$

PROOF. Let $\boldsymbol{E} = \sum_{i=1}^{r} \mathscr{E}_i$ be the decomposition into irreducible components, and let $\bar{\mathscr{E}}_i$ be the reduced pull-back of $\mathscr{E}_i$ by ρ for $1 \leq i \leq r$. Then from the argument in 2.7, if follows that

$$\sum_{i=1}^{r} e(\bar{\mathscr{E}}_i) = 2n(\mathbf{C}) + 4n(\mathbf{N}) + 2n(\mathbf{I}_E) + 6n(\mathbf{G}_E) - 2n(\mathbf{G}_E \cap \mathbf{C}) - 2n(\mathbf{G}_E \cap \mathbf{I}) + 6n(\mathbf{H}_E)$$
$$-n(\mathbf{H}_E \cap \mathbf{N}) + 6n(\mathbf{O}_E) - 2n(\mathbf{O}_E \cap \mathbf{C}) - n(\mathbf{O}_E \cap \mathbf{N}) - 2n(\mathbf{O}_E \cap \mathbf{I})$$
$$-n(\mathbf{O}_E \cap \Delta_r'') + \sum_{\bar{Q} \in \mathrm{Inn}} (e(E_{\bar{Q}}) - 2) .$$

Furthermore, by the Mayer-Vietoris exact sequence for the topological manifold $\bar{\boldsymbol{E}}$, we have

$$e(\bar{\boldsymbol{E}}) = \sum_{i=1}^{r} e(\bar{\mathscr{E}}_i) - n(\mathbf{G}_E \cap \mathbf{C}) - n(\mathbf{G}_E \cap \mathbf{I}_E) - 3n(\mathbf{G}_E \cap \mathbf{G}_E) - 2n(\mathbf{H}_E \cap \mathbf{N})$$
$$-3n(\mathbf{H}_E \cap \mathbf{H}_E) - n(\mathbf{O}_E \cap \mathbf{C}) - 2n(\mathbf{O}_E \cap \mathbf{N}) - n(\mathbf{O}_E \cap \mathbf{I}_E) - 3n(\mathbf{O}_E \cap \mathbf{G}_E)$$
$$-3n(\mathbf{O}_E \cap \mathbf{H}_E) - 3n(\mathbf{O}_E \cap \mathbf{O}_E) .$$

Note that the cardinality of the set of irreducible components in $\boldsymbol{E}$ coincides with the cardinality of the set of double points in $\boldsymbol{E}$ plus 1. Thus by an easy calculation, the desired formula follows.

3.4. Next we calculate the geometric genus $p_g(\bar{P}_0)$ and the Milnor number $\mu(\bar{P}_0)$ of $\bar{P}_0$. We note that $p_g(\bar{P}_0) = -\chi(\mathcal{O}_{S^*}) + \chi(\mathcal{O}_S)$. Moreover by [L], we have

$$\mu(\bar{P}_0) = 12p_g(\bar{P}_0) + \omega_{S^*}^2 - \omega_S^2 + e(\bar{\boldsymbol{E}}) - 1 .$$

We remark that this formula holds even if the resolution is not minimal. So by Proposition 2.4 and Lemma 3.3, we obtain the following:

PROPOSITION 3.5.

$$p_g(\bar{P}_0) = \frac{1}{2} \sum_{i=1}^{r} l_i(5l_i - 3) - n(\mathbf{C}) - n(\mathbf{N}) + \frac{1}{3} \sum_{i=1}^{n(\mathbf{C})} C_i^2 + \sum_{Q \in \mathrm{Inn}} p_g(\bar{Q}) ,$$

$$\mu(\bar{P}_0) = \sum_{i=1}^{r} 3(6l_i^2 - 2l_i - 1) - 5n(\mathbf{C}) - 3n(\mathbf{N}) - n(\mathbf{I}_E) + 3n(\mathbf{G}_E) + 3n(\mathbf{H}_E)$$
$$+3n(\mathbf{O}_E) + 4 \sum_{i=1}^{n(\mathbf{C})} C_i^2 + \frac{3}{2} \sum_{i=1}^{n(\mathbf{N})} N_i^2 - 2n(\mathbf{G}_E \cap \mathbf{I}_B) - 2n(\mathbf{O}_E \cap \mathbf{I}_B)$$
$$-n(\mathbf{O}_E \cap \Delta_r'') + \sum_{\bar{Q} \in \mathrm{Inn}} (\mu(\bar{Q}) - 1) + 2 .$$

We now give some examples for resolving singularities defined by $\xi^3+g(x, y)\xi+h(x, y)=0$ by our method and calculate p_g and μ:

EXAMPLE 3.6. $\xi^3+x^4\xi+y^2=0$.

Set $G=(x^4)$ and $H=(y^2)$. After four triplet blow-ups, the total transform of $G+H$ by $\tau_1\circ\cdots\circ\tau_4$ becomes

$$(\mathbf{H}_2)—\mathbf{O}—\mathbf{I}_2—\mathbf{G}_4—\mathbf{C}_4—(\mathbf{G}_4)\,,$$

where $(\mathbf{H}_2)$ and $(\mathbf{G}_4)$ are the proper transforms of G and H, respectively. The self-intersection numbers of the exceptional curves $\mathbf{O}$, $\mathbf{I}_2$, $\mathbf{G}_4$ and $\mathbf{C}_4$ are -1, -3, -1 and -3, respectively. Therefore the dual graph of the exceptional set on S^* is

$$[-3]—(-1)—(-3)—(-1)\,.$$

Since $n(\mathbf{O}\cap\Delta_4'')=4$, the curve $[-3]$ is elliptic. The other three curves are rational. Contracting (-1)-curves three times, we get an elliptic curve with self-intersection number -1 as the exceptional set of the minimal resolution of this singularity. This is called a simple elliptic singularity of type $\tilde{E}_8$ (cf. [S1]).

In this case, we have $p_g=1$ and $\mu=10$.

EXAMPLE 3.7. $\xi^3+xy\xi+x^5+y^3=0$.

Set $G=(xy)$ and $H=(x^5+y^3)$. After eight triplet blow-ups, the total transform of $G+H$ becomes

$$\begin{tikzcd}[column sep=small] & & & & & & (\mathbf{H}_1) \arrow[d, dash] & & & \\ (\mathbf{G}_1) \arrow[r, dash] & \mathbf{C}_2 \arrow[r, dash] & \mathbf{G}_1 \arrow[r, dash] & \mathbf{I}_1 \arrow[r, dash] & \mathbf{O} \arrow[r, dash] & \mathbf{N}_2 \arrow[r, dash] & \mathbf{H}_3 \arrow[r, dash] & \mathbf{N}_3 \arrow[r, dash] & \mathbf{O} \arrow[r, dash] & (\mathbf{G}_1)\,. \end{tikzcd}$$

The self-intersection numbers of the exceptional curves $\mathbf{C}_2,\dots,\mathbf{O}$ are -3, -1, -3, -1, -6, -1, -2 and -3, respectively. The dual graph of the exceptional set on S^* is

$$\begin{tikzcd}[column sep=small] & & & & (-6) \arrow[r, dash] & (-1) \arrow[r, dash] & (-2) \arrow[dr, dash] & \\ (-1) \arrow[r, dash] & (-3) \arrow[r, dash] & (-1) \arrow[r, dash] & (-3) \arrow[ur, dash] \arrow[dr, dash] & & & & (-9)\,. \\ & & & & (-3) \arrow[r, dash] & (-2) \arrow[r, dash] & (-1) \arrow[ur, dash] & \end{tikzcd}$$

All these curves are rational. Contracting (-1)-curves nine times, we get

$$\begin{tikzcd} (-2) \arrow[r, dash, bend left] \arrow[r, dash, bend right] & (-5) \end{tikzcd}$$

as the minimal resolution.

In this case, we have $p_g=1$ and $\mu=10$.

EXAMPLE 3.8. $\xi^3+(\prod_{i=1}^{2l}(x+\alpha_i y))\xi+\prod_{j=1}^{3l}(x+\beta_j y)=0$, where $\alpha_1,\dots,\beta_{3l}$ are mutually distinct complex numbers. After one triplet blow-up, our resolution process is completed. The exceptional set consists of only one curve of genus $g=3l-2$ with self-intersection number -3. In this case, we have $p_g=l(5l-3)/2$ and $\mu=2(3l-1)^2$.

4. Proof of the main theorem.

4.1. The situation is the same as in the previous section. We fix a number i with $1 \leq i \leq r$, and consider the i-th triplet blow-up at P_{i-1}. Assume that the component $\mathscr{E}_i$ is the proper transform of $E_i = (\tau_i)^{-1}(P_{i-1})$ by $\tau_i \circ \cdots \circ \tau_r$. Then we define the number θ_i as follows:

(1) If $\mathscr{E}_i$ is of type **C** or **I**, then $\theta_i = -1$.

(2) If $\mathscr{E}_i$ is of type **N**, then $\theta_i = 3/2$.

(3) If $\mathscr{E}_i$ is of type **G**, then $\theta_i = 3 - 2n(\mathscr{E}_i \cap \mathbf{I}_B)$.

(4) If $\mathscr{E}_i$ is of type **H**, then $\theta_i = 3$.

(5) Assume that $\mathscr{E}_i$ is of type **O**. Assume further that another component $\mathscr{E}_j$ in $\boldsymbol{E}$ of type **O** intersect E_i at Q, such that there exists an inner double point $\bar{Q}$ of S_r over Q. Moreover, assume that $\mathscr{E}_j$ is the proper transform of $E_j = (\tau_j)^{-1}(P_{j-1})$ by $\tau_j \circ \cdots \circ \tau_r$ with $i < j$. We call such $\bar{Q}$ a negligible inner double point on $\mathscr{E}_i$. Denote by "PIN$_i$" the set of "not" negligible inner double points on $\pi_r^{-1}(\mathscr{E}_i)$. Then we put

$$\theta_i = 3 - 2n(\mathscr{E}_i \cap \mathbf{I}_B) - n(\mathscr{E}_i \cap \Delta_r'') + \sum_{\bar{Q} \in \mathrm{PIN}_i} (\mu(\bar{Q}) - 6p_g(\bar{Q}) - 1) .$$

DEFINITION 4.2. We define the number d_i by

$$d_i := 3(l_i^2 + l_i - 1) - 2n(E_i \cap \mathbf{C}) - \frac{3}{2} n(E_i \cap \mathbf{N}) + \theta_i .$$

LEMMA 4.3. $\mu(\bar{P}_0) - 6p_g(\bar{P}_0) - 2 = \sum_{i=1}^{r} d_i$.

PROOF. It follows from Proposition 3.5 that

$$\begin{aligned}\mu(P_0) - 6p_g(P_0) - 2 &= 3\sum_{i=1}^{r} (l_i^2 + l_i - 1) + n(\mathbf{C}) + 3n(\mathbf{N}) - n(\mathbf{I}_E) + 3n(\mathbf{G}_E) + 3n(\mathbf{H}_E) \\ &\quad + 3n(\mathbf{O}_E) + 2\sum_{i=1}^{n(\mathbf{C})} C_i^2 + \frac{3}{2}\sum_{i=1}^{n(\mathbf{N})} N_i^2 - 2n(\mathbf{G}_E \cap \mathbf{I}_B) - 2n(\mathbf{O}_E \cap \mathbf{I}_B) - n(\mathbf{O}_E \cap \Delta_r'') \\ &\quad + \sum_{\bar{Q} \in \mathrm{PIN}_i} (\mu(\bar{Q}) - 6p_g(\bar{Q}) - 1) .\end{aligned}$$

Thus by considering the contribution of the blow-up τ_i to the decrease of the self-intersection numbers of curves of type **C** and **N**, the desired formula follows. q.e.d.

LEMMA 4.4.

(1) *If E_i is of type* **C** *or* **I**, *then we have* $d_i \geq 3l_i^2 + 3l_i - 8$.

(2) *If E_i is of type* **N**, *then we have* $d_i \geq 3l_i^2 + 3l_i - 11/2$.

(3) *If E_i is of type* **G** *or* **O**, *then we have* $d_i \geq 3l_i^2 - 3l_i$.

(4) *If E_i is of type* **H**, *then we have* $d_i \geq 3l_i^2 + 3l_i - 4$.

Especially, if $l_i \geq 2$, then we always have $d_i > 0$.

PROOF. Since the curves of types **C** and **N** are always exceptional, we have

$$n(E_i \cap \mathbf{C}) + n(E_i \cap \mathbf{N}) \leq 2 .$$

From this, we have the assertions (1), (2) and (4).

Assume that E_i is of type **G**. It follows that $n_i = 3l_i$. Moreover, we have

$$n_i \geq 2n(E_i \cap \mathbf{C}) + 2n(E_i \cap \mathbf{N}) + n(E_i \cap \mathbf{I}) ,$$

$$n(E_i \cap \mathbf{I}) \geq n(\mathscr{E}_i \cap \mathbf{I}_{\boldsymbol{B}}) .$$

From this, the former part of (3) follows.

Assume that E_i is of type **O**. We clearly have $n(\mathscr{E}_i \cap \Delta_r'') \leq n(E_i \cap \Delta_i')$. Hence from the argument in 2.6, it follows that

$$2n(\mathscr{E}_i \cap \mathbf{I}_{\boldsymbol{B}}) + n(\mathscr{E}_i \cap \Delta_r'') + 2n(E_i \cap \mathbf{C}) + \frac{3}{2} n(E_i \cap \mathbf{N}) \leq 2n(E_i \cap \mathbf{I}) + n(E_i \cap \Delta_i')$$

$$+ 4n(E_i \cap \mathbf{C}) + 3n(E_i \cap \mathbf{N}) \leq \deg_{P_{i-1}}(\Delta_{i-1}) = 6l_i .$$

Note that $\mu(\bar{Q}) - 6p_g(\bar{Q}) - 1 \geq 0$ for an inner double point $\bar{Q}$ by [T2]. Thus the latter part of (3) follows. q.e.d.

4.5. For studying the case $l_i \leq 1$, we need new definitions and notation:

Let $\boldsymbol{P} = \{P_0, \ldots, P_{r-1}\}$ be the set of the centers $P_{j-1} \in W_{j-1}$ for the blow-ups τ_j $(1 \leq j \leq r)$. For fixed j, let $\boldsymbol{P}_j = \{P_j, P_{j_1}, \ldots, P_{j_s}\}$ $(j = j_0 < j_1 < \cdots < j_s)$ be the subset of $\boldsymbol{P}$ consisting of the points which are infinitely near to P_j. We define

$$\hat{d}(P_j) := \sum_{P_{j_k} \in \boldsymbol{P}_j} d_{j_k+1} .$$

If $\bar{P}_j$ is an isolated singularity on S_j, then $\hat{d}(P_j)$ coincides with $\mu(\bar{P}_j) - 6p_g(\bar{P}_j) - 2$ by Lemma 4.3.

We say P_j *is positively combined* if there exists an integer $s' \leq s$ such that

$$\sum_{k=0}^{s'} d_{j_k+1} > 0 .$$

Note that if $d_{j+1} > 0$ or $\hat{d}(P_j) > 0$, then P_j is positively combined.

Let $\{C_j\}_{1 \leq j \leq t}$ be the set of local analytic branches at P_{i-1} of the assistant divisor $G_{i-1} + H_{i-1}$. Assume that the $\boldsymbol{Z}^2$-weighting of C_j with respect to (G_{i-1}, H_{i-1}) is (α_j, β_j). Then we simply say that *the branches at* P_{i-1} *are* $\{C_j(\alpha_j, \beta_j)\}_{1 \leq j \leq t}$.

We denote by $C_j', C_j'', \ldots, C_j^{(k)}$ the proper transform of C_j by $\tau_i, \tau_i \circ \tau_{i+1}, \ldots, \tau_i \circ \cdots \circ \tau_{i+k-1}$, respectively.

LEMMA 4.6. *For a normal 2-dimensional hypersurface singularity P of multiplicity 2, we have $\mu(P) \geq 6p_g(P) + 1$. Moreover, the equality $\mu(P) = 6p_g(P) + 1$ holds if and only if P is a rational double point of type A_1.*

PROOF. If $P_g(P) \geq 1$, the assertion follows from Tomari [T2]. If P is a rational double point, the assertion follows from an explicit calculation. q.e.d.

LEMMA 4.7. *Assume $l_i=0$. If $\bar{P}_{i-1}$ is an isolated singularity of S_{i-1}, then we have $\hat{d}(\bar{P}_{i-1})\geq -1$. Moreover, the equality $\hat{d}(\bar{P}_{i-1})=-1$ holds if and only if one of the following three conditions is satisfied:*

(1) *The branches at P_{i-1} consist of $C_1(1,0)$, $C_2(0,1)$ and $C_3(0,1)$ such that*
 (i) C_j $(1\leq j\leq 3)$ is nonsingular, and
 (ii) *if we denote by $\mathscr{T}_j$ the tangent line of C_j at P_{i-1}, then $\mathscr{T}_1$ is distinct from $\mathscr{T}_2$ and $\mathscr{T}_3$ ($\mathscr{T}_2$ may coincide with $\mathscr{T}_3$).*

(2) *The branches at P_{i-1} consist of $C_1(1,0)$ and $C_2(0,1)$ such that*
 (i) *C_1 is nonsingular and C_2 is a reduced irreducible curve of multiplicity* 2, *and*
 (ii) *the tangent lines $\mathscr{T}_1$ of C_1 and $\mathscr{T}_2$ of C_2 are distinct from each other.*

(3) *The branches at P_{i-1} consist of $C_1(1,0)$ and $C_2(0,2)$ which are nonsingular and meeting transversally.*

PROOF. Since $l_i=0$, the multiplicity of the singularity $\bar{P}_{i-1}$ is 2. Therefore the first assertion follows from Lemma 4.6. To prove the second assertion, it sufficies to show that $\bar{P}_{i-1}$ is an A_1-singularity if and only if one of the conditions (1)$\sim$(3) is satisfied. Let

$$f:=\xi^3+\left(\sum_{j\geq 1} g^{(j)}(x,y)\right)\xi+\sum_{j\geq 2} h^{(j)}(x,y)$$

be the local equation at $\bar{P}_{i-1}$, where $g^{(j)}$ and $h^{(j)}$ are homogeneous polynomials of degree j. Then one of the following conditions is satisfied:

(a) $h^{(2)}$ is a product of two linear functions distinct from each other such that
 (a1) $g^{(1)}$ coincides with one of them, or
 (a2) $g^{(1)}$ does not coincide with any of them, or
 (a3) $g^{(1)}$ is identically zero.

(b) $h^{(2)}$ is the square of a linear function such that
 (b1) $g^{(1)}$ coincides with it, or
 (b2) $g^{(1)}$ does not coincide with it, or
 (b3) $g^{(1)}$ is identically zero.

(c) $h^{(2)}$ is identically zero and $g^{(1)}$ is not identically zero.

In each case, the degree 2 part of f is written as

(a1) $x\xi+xy$, (a2) $x\xi+(x+cy)y$ for $c\neq 0$, (a3) xy,
(b1) $x\xi+x^2$, (b2) $y\xi+x^2$, (b3) x^2, (c) $x\xi$.

Note that the rank of the above quadratic forms (a1), ..., (c) are 2, 3, 2, 2, 3, 1 and 2, respectively. Thus f defines an A_1-singularity if and only if (a2) or (b2) is satisfied. From this, the second assertion follows. q.e.d.

LEMMA 4.8. *Let k be a positive integer. Assume that there appear over E_i exactly k target singular points of type A_1 after the triplet blow-up $\hat{\tau}_i$. Then we have $d_i-k>0$.*

PROOF. By the argument in Lemma 4.7, E_i is one of types $\mathbf{G}_1$, $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{O}$. Especially $l_i \geq 1$.

First assume that E_i is of type $\mathbf{O}$. Since the types of the assistant divisor at any A_1-target singularity on E_i is one of (1)–(3) in Lemma 4.7, it follows that

$$3k+n(E_i\cap\Delta_i')+2n(E_i\cap\mathbf{I})+4n(E_i\cap\mathbf{C})+3n(E_i\cap\mathbf{N})\leq \deg_{P_{i-1}}(\Delta_{i-1})=6l_i\,.$$

Therefore we have

$$d_i-k\geq 3l_i^2+l_i-\frac{1}{2}n(E_i\cap\mathbf{N})-\frac{2}{3}n(E_i\cap\mathbf{C})+\frac{2}{3}n(E_i\cap\mathbf{I})-2n(\mathscr{E}_i\cap\mathbf{I}_B)$$
$$+\frac{1}{3}n(E_i\cap\Delta_i')-n(\mathscr{E}_i\cap\Delta_r'')\,.$$

Since we have $4l_i>2n(E_i\cap\mathbf{N})+(8/3)n(E_i\cap\mathbf{C})+(4/3)n(E_i\cap\mathbf{I})+(2/3)n(E_i\cap\Delta_i')$ by $k>0$, it follows that

$$d_i-k>3l_i^2-3l_i+\frac{3}{2}n(E_i\cap\mathbf{N})+2n(E_i\cap\mathbf{C})+2(n(E_i\cap\mathbf{I})-n(\mathscr{E}_i\cap\mathbf{I}_B))$$
$$+n(E_i\cap\Delta_i')-n(E_i\cap\Delta_r'')\geq 0\,.$$

We next assume that E_i is of type $\mathbf{G}_1$. Since $n_i=3l_i$, we have

$$0<2k\leq 3l_i-2n(E_i\cap\mathbf{N})-2n(E_i\cap\mathbf{C})-n(E_i\cap\mathbf{I})\,.$$

Hence

$$d_i-k\geq 3l_i^2+\frac{3}{2}l_i-\frac{1}{2}n(E_i\cap\mathbf{N})-n(E_i\cap\mathbf{C})+\frac{1}{2}n(E_i\cap\mathbf{I})-2n(\mathscr{E}_i\cap\mathbf{I}_B)$$
$$>3l_i^2-3l_i+\frac{5}{2}n(E_i\cap\mathbf{N})+2n(E_i\cap\mathbf{C})+2(n(E_i\cap\mathbf{I})-n(\mathscr{E}_i\cap\mathbf{I}_B))\geq 0\,.$$

When E_i is of type $\mathbf{H}_1$ or $\mathbf{H}_2$, the argument is similar, and is left to the reader.

q.e.d.

LEMMA 4.9. *Assume that $l_i=0$. If the singular locus of S_{i-1} is not isolated at $\bar{P}_{i-1}$, then P_{i-1} is positively combined.*

PROOF. Since P_{i-1} is a bad target point, the types of local branches $\{C_j\}_{j=1}^t$ of the assistant divisor at P_{i-1} which satisfy the assumption is uniquely determined as follows: One of $\{C_j\}$, say C_1, is of type $\mathbf{N}_2$ and the others $C_2,\dots,C_t(t\geq 2)$ are all of type $\mathbf{G}$.

Then E_i is of type $\mathbf{C}$, and we have $d_i=-11/2$. Set $P_i=E_i\cap C_1'$ and apply the next triplet blow-up $\hat{\tau}_{i+1}$. Then E_{i+1} is of type $\mathbf{I}$, and so $d_{i+1}=-3/2$. Let E_i' be the proper transform of E_i by τ_{i+1}, and set $P_{i+1}=E_{i+1}\cap E_i'$. We apply $\hat{\tau}_{i+2}$. Then E_{i+2} is of type $\mathbf{G}$, and we have $d_{i+2}=4$. Set $P_{i+2}=(\tau_{i+2})^{-1}(E_{i+1}\cap C_1'')$ and apply $\hat{\tau}_{i+3}$. E_{i+3} is of type

G or type **O**. In the former case, we have $d_{i+3}=9/2$, while in the latter case, we have $d_{i+3}\geq 7/2$.

From this, we have $d_i+\cdots+d_{i+3}\geq 1/2$, i.e., P_{i-1} is positively combined. q.e.d.

Next we consider the case $l_i=1$. The following lemma is a consequence of the argument in the proof of Lemma 4.4.

LEMMA 4.10. *Assume that* $l_i=1$. *Then we have* $d_i\leq 0$ *if and only if one of the following conditions is satisfied*:

(1) E_i *is of type* **C** *or* **I** *such that*

(1a) $n(E_i\cap \mathbf{C})=2$ *and* $d_i=-2$, *or*

(1b) $n(E_i\cap \mathbf{C})=n(E_i\cap \mathbf{N})=1$ *and* $d_i=-3/2$, *or*

(1c) $n(E_i\cap \mathbf{N})=2$ *and* $d_i=-1$, *or*

(1d) $n(E_i\cap \mathbf{C})=1$ *and* $d_i=0$.

(2) E_i *is of type* **G** *such that* $n(\mathscr{E}_i\cap \mathbf{I}_{\boldsymbol{B}})=3$ *and* $d_i=0$.

(3) E_i *is of type* **O** *such that* $n(\mathscr{E}_i\cap \Delta_r'')=6$ *and* $d_i=0$.

LEMMA 4.11. *If the condition* (1) *in Lemma* 4.10 *is satisfied, then* P_{i-1} *is positively combined.*

PROOF. Case (1a-**I**), i.e., E_i is of type **I** such that the condition (1a) is satisfied: The branches at P_{i-1} are

$$C_1(\alpha_1, 2),\ C_2(\alpha_2, 2),\ C_3(\delta_3, 0), \ldots, C_t(\delta_t, 0)\,,$$

where $\alpha_j\geq 2$ $(j=1, 2)$ and $\delta_j\geq 1$ $(3\leq j\leq t)$. ($C_3, \ldots$ may not exist.) Put $P_i=E_i\cap C_1'$. After the triplet blow-up $\hat{\tau}_{i+1}$, let $P_{i+2}=(\tau_{i+1})^{-1}(E_i\cap C_2')$. Then E_{i+1} and E_{i+2} are of type **G**, and so $d_{i+1}=d_{i+2}=4$. There is no bad target point on S_{i+2} which is infinitely near to P_{i-1}. Hence we have $\hat{d}(P_{i-1})=6$.

Case (1a-**C**): The branches at P_{i-1} are

$$C_1(\alpha_1, 2),\ C_2(\alpha_2, 2),\ C_3(\gamma, 1),\ C_4(\delta_4, 0), \ldots, C_t(\delta_t, 0)$$

with $\gamma\geq 0$. C_3' does not intersect C_1' or C_2', say C_1'. Then we put $P_i=E_i\cap C_1'$. We have $\hat{d}(P_i)=6$ by (1a-**I**).

Case (1b-**I**): The branches at P_{i-1} are

$$C_1(\alpha, 2),\ C_2(1, 2),\ C_3(\delta_3, 0), \ldots, C_t(\delta_t, 0)\,.$$

Set $P_i=E_i\cap C_1'$. Then E_{i+1} is of type **G**, and so $d_{i+1}=4$, hence $d_i+d_{i+1}>0$.

Case (1b-**C**): Letting C_1 to be the curve of type **C** passing through P_{i-1}, we set $P_i=E_i\cap C_1'$. Then $\hat{\tau}_{i+1}$ satisfies the condition (1a).

Case (1c-**I**): The branches at P_{i-1} are

$$C_1(1, 2),\ C_2(1, 2),\ C_3(\delta_3, 0), \ldots, C_t(\delta_t, 0)\,.$$

Set $P_i=E_i\cap C_1'$. Since E_{i+1} is of type **G** or **O**, we have $d_{i+1}\geq 7/2$.

Case (1c-**C**): Letting C_1 and C_2 to be the curves of type **N** passing through P_{i-1}, we set $P_i = E_i \cap C_1'$ and $P_{i+1} = (\tau_{i+1})^{-1}(E_i \cap C_2')$. Since P_i and P_{i+1} are positively combined by (1b).

Case (1d-**C**): Letting C_1 to be the curve of type **C** passing through P_{i-1}, we set $P_i = E_i \cap C_1'$. Then by (1a), P_i is positively combined.

Case (1d-**I**): The branches at P_{i-1} are one of the following:

(1) $C_1(\alpha, 2)$, $C_2(\gamma_2, 1)$, $C_3(\gamma_3, 1)$, $C_4(\delta_4, 0), \dots, C_t(\delta_t, 0)$,

(2) $C_1(\alpha, 2)$, $C_2(0, 2)$, $C_3(\delta_3, 0), \dots, C_t(\delta_t, 0)$.

In the case (2), we have already shown that the point $P_i = E_i \cap C_1'$ is positively combined.

So we assume that the case (1) occurs. Set $P_i = E_i \cap C_1'$. If both C_2' and C_3', or neither C_2' nor C_3', pass through P_i, then we have already shown that P_i is positively combined. Assume, say C_2', passes through P_i. We have $d_{i+1} = 0$. We repeat triplet blow-ups for sufficiently many times (say k times) so that the proper transform of C_1 does not meet the proper transform of C_2. Then $d_i = \cdots = d_{i+k} = 0$ and P_{i+k} is positively combined.

This completes the proof of Lemma 4.11.

THEOREM 4.12. *Let (V, p) be a germ of a normal 2-dimensional hypersurface singularity of multiplicity 3. Then we have*

$$\mu(V, p) \geq 6p_g(V, p) + 2 .$$

Especially the signature of the Milnor fiber of (V, p) is negative.

Moreover, the equality $\mu(V, p) = 6p_g(V, p) + 2$ holds if and only if (V, p) is a simple elliptic singularity of type $\tilde{E}_6$.

PROOF. By Lemmas 4.3, 4.4, 4.7, 4.8, 4.9, 4.10 and 4.11, the inequality $\mu \geq 6p_g + 2$ is clear. Assume that the equality holds. Then the first triplet blow-up $\hat{\tau}_1$ satisfies the condition (2) or (3) of Lemma 4.10. If the condition (2) is satisfied, then the branches at P_0 are

$$C_1(\gamma_1, 1),\ C_2(\gamma_2, 1),\ C_3(\gamma_3, 1),\ C_4(\delta_4, 0), \dots, C_t(\delta_t, 0)$$

with $\gamma_j \geq 1$ $(1 \leq j \leq 3)$ such that C_1, C_2, and C_3 is nonsingular. In this case, our resolution process is complete already at $\hat{\tau}_1$. The exceptional curve $\bar{E}_1$ is a nonsingular elliptic curve of self-intersection number -3, i.e., this is a simple elliptic singularity of type $\tilde{E}_6$. If the condition (3) is satisfied, we also have the assertion by a similar argument. The negativity of the signature comes from Durfee [D, p. 97].

This completes the proof of Theorem 4.12.

REMARK 4.13. For a normal 2-dimensional hypersurface singularity of multiplicity 3, there is a possibility that we may always have

$$5\mu + \sqrt{2\mu} \geq 36p_g + 8 .$$

From the viewpoint of our method of resolution, the singularities in Example 3.8 seem to be the simplest, and they satisfy $5\mu+\sqrt{2\mu}=36p_g+8$. Moreover, if one lets $l=1$ in Example 3.8, then we obtain $\tilde{E}_6$.

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